University of Anbar
College of Engineering
Mechanical Engineering Dept.

Advanced Heat Transfer/ I Conduction and Radiation

Handout Lectures for MSc. / Power Chapter Four/ Steady-State Two-Dimensional Conduction Heat Transfer

Course Tutor

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Chapter Four

Steady-State Two-Dimensional Conduction Heat Transfer

4.1 Introduction

In many cases such problems are grossly oversimplified if a one-dimensional treatment is used, and it is necessary to account for multidimensional effects. In this chapter, we will focus on "*analytical method*" for treating two-dimensional systems under steady-state conditions.

4.2 Boundary-value problems and characteristic-value problems

Consider an ordinary differential equation of second order which may result from the differential formulation of a steady one-dimensional conduction problem. The solution of this equation involves two arbitrary constants which are determined by two conditions, each specified at one boundary of the problem. Problems of this type are called *boundary-value problems* to distinguish them from *initial-value problems*, in which all conditions are specified at one location. Reconsider the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{4-1}$$

Assume that this homogeneous equation involves a parameter "λ" as

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{4-2}$$

And is subject to homogeneous boundary conditions

$y(0) = 0$, and $y(L) = 0$ then the general solution of Equation (4-2) is

$$y = C_1 \sin \lambda x + C_2 \cos \lambda x \tag{4-3}$$

The use of ($y(0) = 0$) results in $C_2 = 0$ and , $y = C_1 \sin \lambda x$

From $(y(L) = 0)$, combined with $y = C_1 \sin \lambda x$, gives $C_1 \sin \lambda L = 0$. The problem has nontrivial solutions only if λ satisfies the $(\sin \lambda L = 0)$. Therefore,

$$\lambda_n = \frac{n\pi}{L}, \qquad \text{where } n = 1, 2, 3, \ldots\ldots, \tag{4-4}$$

And the corresponding solutions of $(y = C_1 \sin \lambda x)$ are,

$$y = C_1 \varphi_n(x)\ , \qquad \varphi_n(x) = \sin(\frac{n\pi}{L})\, x \tag{4-5}$$

Note that no new solutions are obtained when "n" assumes negative integer values. Thus, the foregoing *boundary-value problem* has no solution other than the trivial solution $y = 0$, unless λ assumes one of the characteristic values given by Equation 4-4. Corresponding to each characteristic value of λ_n there exists a characteristic function $\varphi_n(x)$ given by Equation 4-5, such that any constant multiple of this function is a solution of the problem. It is important to note that the *boundary-value problem* given by

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0\ ; \qquad y(0) = 0\ , \quad y(L) = 0$$

has no solution other than the trivial solution $y = 0$ corresponding to $\lambda = 0$. Hence there does not exist any set of characteristic values and characteristic functions for this problem. This illustrates the fact that a *boundary-value problem* may or may not be a *characteristic-value problem*. A *boundary-value problem* is a *characteristic-value problem* when it has particular solutions that are periodic in nature; the period and amplitude of these solutions may or may not be constant.

Therefore, in the next three sections the general properties of characteristic functions are investigated.

4.3 Orthogonality of Characteristic Functions

By definition, two functions $\varphi_n(x)$ and $\varphi_m(x)$ are said to be orthogonal with respect to a weighting function $w(x)$, over a finite interval (a, b), if the integral of the product $w(x)\varphi_n(x)\varphi_m(x)$ over that interval vanishes as

$$\int_a^b w(x)\varphi_n(x)\varphi_m(x)dx = 0\,, \qquad \text{where } m \neq n \tag{4-6}$$

Furthermore, a set of functions is said to be *orthogonal* in (a, b) if all pairs of distinct functions in the set are *orthogonal* in (a, b). The word *orthogonality* comes from vector analysis. Let $\varphi_m(x_i)$ denote a vector in 3D space whose rectangular components are $\varphi_m(x_1)$, $\varphi_m(x_2)$, and $\varphi_m(x_3)$. Two vectors, $\varphi_m(x_i)$ and $\varphi_n(x_i)$, are said to be *orthogonal*, or perpendicular to each other, if

$$\varphi_m(x_i)\,.\,\varphi_n(x_i) = \sum_{i=1}^{3}\varphi_m(x_i)\,.\,\varphi_n(x_i) = 0 \tag{4-7}$$

When the units of length on the coordinate axes vary from one axis to another, the foregoing scalar product assumes the form

$$\varphi_m(x_i)\,.\,\varphi_n(x_i) = \sum_{i=1}^{N} w(x_i)\,\varphi_m(x_i)\,\varphi_n(x_i) = 0 \tag{4-8}$$

Where the weighting numbers $w(x_1)$, $w(x_2)$, and $w(x_3)$ depend upon the units of length used along the three axes. The vectors in an N-Dimensional space having components $\varphi_m(x_i)$, $\varphi_n(x_i)$, i= 1, 2, 3, ……, N are said to be *orthogonal* with respect to the weighting numbers $w(x_i)$.

It will now be shown that the *characteristic functions* of a *characteristic-value problem* are *orthogonal* over a finite interval with respect to a weighting function. To establish this fact, consider the *characteristic-value problem* composed of the linear homogenous second-order differential equation of the general form

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + [f_2(x) + \lambda^2 f_3(x)]y = 0 \tag{4-9}$$

This equation, multiplied through by the factor ($e^{[\int f_1(x)dx]} = p(x)$) and with the functions defined as $f_2(x)p(x) = q(x)$ and $f_3(x)p(x) = w(x)$, may be rearranged in the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda^2 w(x)]y = 0 \tag{4-10}$$

Which is more convenient for the following discussion.

Let λ_m, λ_n be any two distinct characteristic numbers, that is, $m \neq n$ and let $\varphi_m(x)$, $\varphi_n(x)$ be the corresponding characteristic functions. Since $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are solutions of Equation (4-10),

$$\frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) + (q + \lambda_m^2 w)\varphi_m = 0,$$

$$\frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right) + (q + \lambda_n^2 w)\varphi_n = 0.$$

Multiplying the first equation by φ_n and the second by φ_m, then subtracting the second of the resulting equations from the first one gives

$$\varphi_n\,\frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) - \varphi_m\,\frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right) + (\lambda_m^2 - \lambda_n^2)w\varphi_m\varphi_n = 0.$$

Integrating this equation over the finite interval (a, b) yields

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \int_a^b \left[\varphi_n\,\frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) - \varphi_m\,\frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right)\right]dx,$$

and integration by parts for the right-hand member results in

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \left\{p(x)\left[\varphi_n(x)\,\frac{d\varphi_m(x)}{dx} - \varphi_m(x)\,\frac{d\varphi_n(x)}{dx}\right]\right\}\Bigg|_a^b \tag{4-11}$$

Since both $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are particular solutions of Equation 4-10, the right-hand side of Equation 4-11 vanishes when one of the following conditions is prescribed at each end of the interval (a, b):

$$y = 0 \tag{4-12}$$

$$\frac{dy}{dx} = 0 \tag{4-13}$$

$$\frac{dy}{dx} + By = 0 \tag{4-14}$$

Where B is an arbitrary parameter.

The fact that Equation (4-11) vanishes when Equation (4-14) is satisfied may be clarified by rearranging the right-hand member of Equation (4-11) in the form

$$\varphi_n\acute{\varphi}_m - \varphi_m\acute{\varphi}_n = \varphi_n\acute{\varphi}_m - \varphi_m\acute{\varphi}_n \pm B\varphi_m\varphi_n = \varphi_n(\acute{\varphi}_m + B\varphi_m) - \varphi_m(\acute{\varphi}_n + B\varphi_n) \quad (4\text{-}15)$$

Particularly, if $p(x) = 0$ when $x = a$ or $x = b$, the right-hand side of Equation (4-11) vanishes, and the condition given by Equation 4-12, 4-13, or 4-14 satisfied at $x = a$ or $x = b$ can be dropped from the problem provided y and $(\frac{dy}{dx})$ are finite at that point. If $p(b) = p(a)$, the *orthogonality* continues to exist when the boundary conditions are replaced by the conditions $y(b) = y(a)$ and $\acute{y}(b) = \acute{y}(a)$, which are called the periodic boundary conditions.

As an example, reconsider the *characteristic-value problem* given by Equation (4-1). Comparison of Equations (4-1) and (4-10) gives $w(x) = 1$, and the condition of the *orthogonality* for this problem is

$$\int_0^L \varphi_m(x)\,\varphi_n(x)dx = \int_0^L \sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})\,dx = 0 \qquad m \neq n \quad (4\text{-}16)$$

Which can also be verified independently by direct integration.

We wish to expand an arbitrary function *f*(*x*) into a series of this set as

$$f(x) = b_o\varphi_o(x) + b_1\varphi_1(x) + b_2\varphi_2(x) + \cdots = \sum_{n=0}^{\infty} b_n\varphi_n(x) \quad (4\text{-}17)$$

By multiplying both sides of Equations (4-17) by $w(x)\varphi_m(x)$ and integrating the result over the interval with the assumption that the integral of the infinite sum is equivalent to the sum of the integrals,

$$\int_a^b w(x)f(x)\,\varphi_m(x)dx = \sum_{n=0}^{\infty} A_n \int_a^b w(x)\varphi_n(x)\varphi_m(x)dx \quad (4\text{-}18)$$

All terms in the sum on the right of Equation 4-18 are *zero* except the term corresponding to $n = m$.

$$A_n = \frac{\int_a^b w(x)f(x)\varphi_n(x)dx}{\int_a^b w(x)\varphi^2{}_n(x)dx} \quad (4\text{-}19)$$

4.4 Fourier series

In general, one must somewhere in the problem, express a function (for example $f(x)$) by a series of eigen functions (for example $\sin(n\pi x)$). More generally, if the eigen functions are denoted by $\varphi_n(x)$ the expression is then given by;

$$f(x) = \sum_{n=1}^{\infty} A_n\, \varphi_n(x) \tag{4-20}$$

$\varphi_n(x)$ can usually be expected to be orthogonal with respect to some weighting function $w(x)$. In other words;

$$\int_0^1 w(x)\, \varphi_n(x) dx = \begin{cases} 0 & (for\ m \neq n) \\ C & (for\ m = n) \end{cases} \tag{4-21}$$

Multiply $f(x)$ by $(w(x)\, \varphi_m(x))$ and integrate;

$$\int_0^1 w(x) f(x)\, \varphi_m(x) dx = \int_0^1 \sum_{n=1}^{\infty} A_n\, w(x)\, \varphi_n(x)\varphi_m(x) dx =$$

$$\sum_{n=1}^{\infty} A_n \int_0^1 w(x)\varphi_n(x)\varphi_m(x) dx$$ (Integral = 0 for $m \neq n$, and = C for $m = n$)

Thus, $\int_0^1 w(x) f(x)\, \varphi_m(x) dx = A_m C$

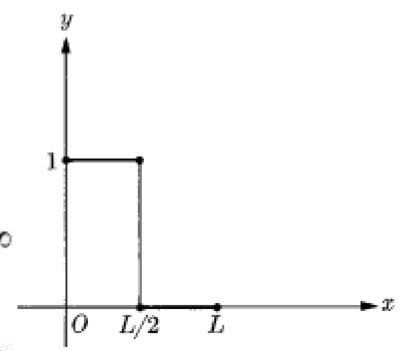
$$\frac{1}{C}\int_0^1 w(x) f(x)\, \varphi_m(x) dx = A_m \tag{4-22}$$

Thus the coefficients have been found since all functions in the integral are known and the integral can be evaluated.

Example 1:

Consider the Fourier *sine* series of the function as;

$$f(x) = \begin{cases} 0, & -\infty < x < 0 \quad \text{and} \quad L/2 < x < \infty \\ 1, & 0 < x < L/2 \end{cases}$$



over the interval $(0, L)$ (Fig. 4–2). The coefficients of the series are

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx = \frac{2}{L}\int_0^{L/2} 1 \cdot \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right);$$

hence the series is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}\right) x.$$

Example 2: Consider now the Fourier cosine series of the previous example, where the coefficients of the series are

Multiply each side by $cos\ (m\pi x)$:

$\int_o^1 x\ cosm\pi xdx = \int_o^1 \sum_{n=o}^{\infty} B_n Cosn\pi x\ Cosm\pi xdx = \sum_{n=o}^{\infty} B_n \int_0^1 Cosn\pi xcosm\pi xdx$

$\int_o^1 xcosm\pi xdx = B_m \int_o^1 cos^2 m\pi xdx \qquad (cos^2 = \frac{1}{2}(1+\cos)\ )$

For $m = o \longrightarrow \int_o^1 xdx = B_0 \int_0^1 dx = B_0 \qquad B_0 = \frac{x^2}{2}]_0^1 = \frac{1}{2}$

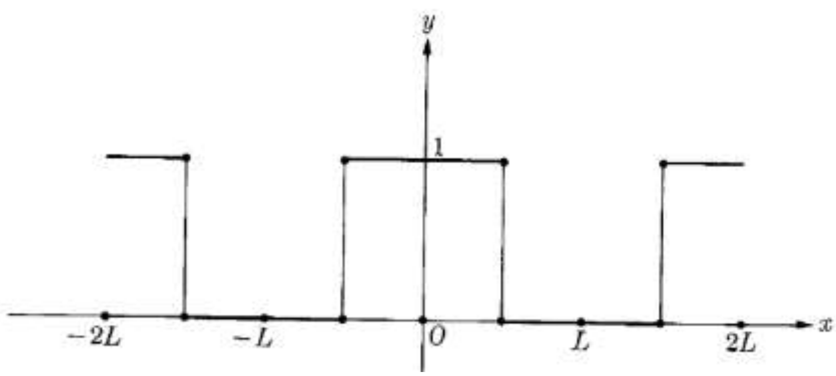
For $m \neq o$; $\int_o^1 xcosm\pi xdx = \frac{B_m}{m\pi}\int_o^1 cos^2 m\pi x\ \ d(m\pi x) = \frac{B_m}{m\pi}(\frac{m\pi x}{2} + \frac{1}{4}$ $\sin 2m\pi x)]_0^1 = \frac{B_m}{2}$

For $m \neq 0 \quad B_m = 2\int_o^x xcosm\pi xdx = \frac{2}{m^2\pi^2}(\cos(m\pi) - 1)$

$g(x) = x = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\ (\cos n\pi - 1)cosn\pi x$

Example 3:

Express a function *f*(*x*), which is piecewise continuous in the interval (-*L*, *L*), in terms of both *sine* and *cosine* having the common period 2*L* (where the function repeats its behavior periodically for all values of *x* as shown in Figure below).



So far, we have seen that any piecewise continuous function can be expressed in the interval $(0, L)$ by a series consisting of sines or cosines with the common period $2L$. When the function is odd, the sine series representation is valid in the interval $(-L, L)$, whereas for an even function the cosine series representation holds in the same interval.

Noting that

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)],$$

where the function in the first brackets is even and that in the second is odd, we arrive at the fact that an arbitrary function can be expressed as the sum of an even function and an odd function. Hence

$$f(x) = f_e(x) + f_o(x).$$

Expressing $f_e(x)$ in terms of cosines and $f_o(x)$ in terms of sines in the interval $(-L, L)$, we have

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi/L)x,$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi/L)x,$$

where

$$a_0 = \frac{1}{L}\int_0^L f_e(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f_e(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$

$$b_n = \frac{2}{L}\int_0^L f_o(x)\sin\left(\frac{n\pi}{L}\right)x\,dx.$$

Since the integrands of these equations are even functions of x, replacing $\int_0^L$ by $\frac{1}{2}\int_{-L}^{L}$ gives

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx.$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi/L)x + b_n \sin(n\pi/L)x], \qquad -L < x < L.$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^{L/2} 1\cdot dx = \tfrac{1}{4},$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{L}\int_{0}^{L/2} 1\cdot\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\sin\frac{n\pi}{2},$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{1}{L}\int_{0}^{L/2} 1\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right).$$

Therefore,

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{L}\right)x + \left(1-\cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)x\right],$$

4.5 Homogeneous Problems

A differential equation is linear if it contains no products of the dependent variable or its derivatives (u^4 or uu_x terms are not permitted).

$(uu_x + vu_y = vu_{yy})$ Nonlinear Equation

A boundary condition is linear if it contains no products of the dependent variable or its derivatives.

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = C\sigma(T^4 - T_\infty^4) \qquad \text{Nonlinear B.C.}$$

A differential equation is homogeneous if when it is satisfied by (u) it is also satisfied by (cu) where(c) is an arbitrary constant (c= o is special case).

$u_{xx} + q''' = u_\theta$ Non Homgeneous Equuation

$cu_{xx} + q''' = cu_\theta$

$or\ \left(u_{xx} + \frac{q'''}{c} = u_\theta\right)$ which is not identical to the original equation.

A boundary condition (different form the initial condition) is homogenous if when satisfied by (u) it is also satisfied by (cu) where (c) is an arbitrary constant;

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h(T - T_\infty) \quad \text{Non Homogenous B.C.}$$

$$-kc\frac{\partial T}{\partial x}\bigg|_{x=L} = h(cT - T_\infty)$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h(T - \frac{T_\infty}{C}) \quad \text{which is not identical to the original B.C.}$$

A homogeneous linear problem will be defined as one in which both the differential equation and its B.Cs. are homogeneous as well as linear. Both these restrictions are essential for separation of variables to be directly applicable

A linear diff. equ. and the B.Cs. are homogeneous when all terms include either the unknown function or one of its derivatives.

4.6 The Method of Separation of Variables: Steady, Two-dimensional Cartesian Geometry

For two-dimensional, steady-state conditions with no generation and constant thermal conductivity, this form is, from Equation 1.17,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{4-1}$$

To appreciate how the method of separation of variables may be used to solve two-dimensional conduction problems, we consider the system of Figure 4.1. Three sides of a thin rectangular plate or a long rectangular rod are maintained at a constant temperature T_1, while the fourth side is maintained at a constant temperature $T_2 \neq T_1$. Assuming negligible heat transfer from the surfaces of the plate or the ends of the rod, temperature gradients normal to the x–y plane may be neglected ($\partial^2 T/\partial z^2 = 0$) and conduction heat transfer is primarily in the *x*- and *y*-directions.

We are interested in the temperature distribution $T(x, y)$, but to simplify the solution we introduce the transformation,

$$\theta = \frac{T - T_1}{T_2 - T_1} \quad (4\text{-}2)$$

By Substituting Equation 4.2 into Equation 4.1, the transformed differential equation is then,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \quad (4\text{-}3)$$

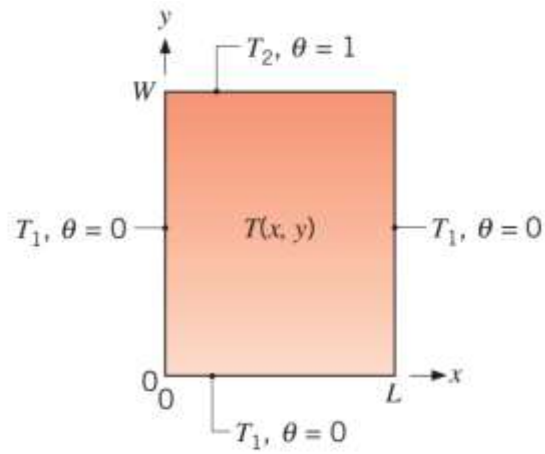


Figure 4.1: Two-dimensional conduction problems, sides of a thin rectangular plate or a long rectangular

Since the equation is second order in both *x* and *y*, two boundary conditions are needed for each of the coordinates. They are,

$\theta(0, y) = 0$, $\theta(x, 0) = 0$, $\theta(L, y) = 0$, $\theta(x, W) = 1$

Note that, through the transformation of Equation 4.2, three of the four boundary conditions are now homogeneous and the value of θ is restricted to the range from 0 to 1. We now apply the separation of variables technique by assuming that the desired solution can be expressed as the product of two functions, one of which depends only on *x* while the other depends only on *y*. The essential features of the method will now be illustrated by means of a steady two- dimensional example. Consider the *second-order* partial differential equation,

$$a_1(x)\frac{\partial^2\theta}{\partial x^2} + a_2(x)\frac{\partial\theta}{\partial x} + a_3(x)\theta + b_1(y)\frac{\partial^2\theta}{\partial y^2} + b_2(y)\frac{\partial\theta}{\partial y} + b_3(y)\theta = 0 \quad (4\text{-}4)$$

A more generalized form of this equation which involves coefficients as functions of both independent variables is not suitable for the separation of variables. That is, assume the existence of a product solution of the form

$$\theta_{(x,y)} = X_{(x)}\, Y_{(y)} \tag{4-5}$$

Where X is a function of x alone and Y is a function y. This assumption becomes meaningful when the two functions X and Y actually satisfy separate differential equations.

Introducing Eq. (4.5) into Eq. (4.4) and dividing the result by XY yields

$$\left[a_1(x)\frac{\partial^2 X}{\partial x^2} + a_2(x)\frac{\partial X}{\partial x} + a_3(x)X\right]\frac{1}{X} = -\left[b_1(y)\frac{\partial^2 Y}{\partial y^2} + b_2(y)\frac{\partial Y}{\partial y} + b_3(y)Y\right]\frac{1}{Y} = 0 \tag{4-6}$$

It is evident that the differential equation is, in fact, separable. That is, the *left-hand* side of the equation depends only on x and the *right-hand* side depends only on y. Hence the equality can apply in general (for any x or y) only if both sides are equal to the same constant. Identifying this, as yet unknown, *separation constant* or *separation parameter* as ($+\lambda^2$) or ($-\lambda^2$), we then have

$$a_1(x)\frac{\partial^2 X}{\partial x^2} + a_2(x)\frac{\partial X}{\partial x} + [a_3(x) \pm \lambda^2]X = 0 \tag{4-7}$$

$$b_1(y)\frac{\partial^2 Y}{\partial y^2} + b_2(y)\frac{\partial Y}{\partial y} + [b_3(y) \pm \lambda^2]Y = 0 \tag{4-8}$$

The method of separation of variables is applicable to steady two-dimensional problems if and when,

i. One of the directions of the problem is expressed by a homogeneous differential equation subject to homogeneous boundary conditions (the homogeneous direction), while the other direction is expressed by a homogeneous differential equation subject to one homogeneous and one nonhomogeneous boundary condition (the nonhomogeneous direction).

ii. The sign of λ^2 is chosen such that the boundary-value problem of the homogeneous direction leads to a characteristic-value problem.

The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \tag{4-9}$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0 \tag{4-10}$$

The partial differential equation has been reduced to two ordinary differential equations. Note that the designation of λ^2 as a positive constant was not arbitrary. If a negative value were selected or a value of $\lambda^2 = 0$ was chosen, it would be impossible to obtain a solution that satisfies the prescribed boundary conditions. The general solutions to Equations 4.9 and 4.10 are, respectively,

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x \tag{4-11}$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{+\lambda y} \tag{4-12}$$

in which case the general form of the two-dimensional solution is

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right) \tag{4-13}$$

The classical method of separation of variables is restricted to linear homogeneous P.D.E.

Example 1: A two-dimensional rectangular plate is subjected to the boundary conditions shown in Figure 4.1. Derive an expression for the steady-state temperature distribution $\theta(x, y)$.

Solution:

The transformed differential equation (applying Eq. 4.3) as,

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

The boundary conditions are needed for each of the coordinates as,

$\theta(0, y) = 0\ , \quad \theta(x, 0) = 0, \quad \theta(L, y) = 0\ , \quad \theta(x, W) = 1$

Assume the existence of a product solution of the form (Eq. 4.5)

$\theta_{(x,y)} = X_{(x)}\, Y_{(y)}$

Where X is a function of x alone and Y is a function y. The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$

$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$

The general form of the two-dimensional solution is,

$$\theta_{(x,y)} = \ (C_1 \cos\lambda x + C_2 \sin\lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right) \qquad (4\text{-}14)$$

Applying the condition that $\theta(0, y)= 0$, it is evident that $C_1= 0$. In addition from the requirement that $\theta(x, 0)= 0$, we obtain

$C_2 \sin\lambda x\ (C_3 + C_4) = 0$

which may only be satisfied if $C_3= -C_4$. Although the requirement could also be satisfied by having $C_2= 0$, this would result in $\theta(x, y)= 0$, which does not satisfy the boundary condition $\theta(x, W)= 1$. If we now invoke the requirement that $\theta(L, y)= 0$, we obtain

$C_2 C_4 \sin\lambda L\left(e^{\lambda y} - e^{-\lambda y}\right) = 0\ ,$

The only way in which this condition may be satisfied (and still have a *nonzero* solution) is by requiring that assume discrete values for which $\sin\lambda L= 0$. These values must then be of the form,

$\lambda = \frac{n\pi}{L} \qquad n = 1, 2, 3, ...$

where the integer $n= 0$ is precluded, since it implies $\theta(x,y)= 0$. The desired solution may now be expressed as

$\theta = C_2 C_4 \sin\frac{n\pi x}{L}\left(e^{\frac{n\pi y}{L}} - e^{\frac{-n\pi y}{L}}\right)$

Combining constants and acknowledging that the new constant may depend on *n*, we obtain

$\theta_{(x,y)} = C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$

where we have also used the fact that $\left(e^{\frac{n\pi y}{L}} - e^{\frac{-n\pi y}{L}}\right) = 2\sinh\left(\frac{n\pi y}{L}\right)$. In this form we have really obtained an infinite number of solutions that satisfy the differential equation and boundary conditions. However, since the problem is linear, a more general solution may be obtained from a superposition of the form

$\theta_{(x,y)} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$

To determine C_n we now apply the remaining boundary condition, which is of the form

$$\theta_{(x,W)} = 1 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L} \qquad (4\text{-}15)$$

Although the above equation would seem to be an extremely complicated relation for evaluating C_n, a standard method is available. It involves writing an infinite series expansion in terms of *orthogonal functions*. An infinite set of functions $g_1(x)$, $g_2(x)$, … , $g_n(x)$, … is said to be orthogonal in the domain $a \le x \le b$ if

$$\int_a^b g_m(x)g_n(x)dx = 0 \qquad m \neq n$$

Many functions exhibit *orthogonality*, including the *trigonometric functions* $sin(n\pi x/L)$ and $cos(n\pi x/L)$ for $0 \le x \le L$. Their utility in the present problem rests with the fact that any function $f(x)$ may be expressed in terms of an infinite series of orthogonal functions

$$f(x) = \sum_{n=1}^{\infty} A_n g_n(x) \qquad (4\text{-}16)$$

The form of the coefficients A_n in this series may be determined by multiplying each side of the equation by $g_m(x)$ and integrating between the limits *a* and *b*.

$$\int_a^b f(x) g_m(x) dx = \int_a^b g_m(x) \sum_{n=1}^{\infty} A_n g_n(x)\, dx$$

However, from above equation it is evident that all but one of the terms on the *right-hand* side of equation 4.16 must be zero, leaving us with

$$\int_a^b f(x) g_m(x) dx = A_m \int_a^b g_m^2\,(x) dx \qquad (4\text{-}17)$$

Hence, solving for A_m, and recognizing that this holds for any A_n by switching *m* to *n*:

$$A_n = \frac{\int_a^b f(x) g_n(x) dx}{\int_a^b g_n^2(x) dx}$$

The properties of orthogonal functions may be used to solve equation 4.15 for C_n by formulating an infinite series for the appropriate form of $f(x)$. From equation 4.16 it is evident that we should choose $f(x)= 1$ and the orthogonal function $g_n(x)= sin(n\pi x/L)$. Substituting into equation 4.17 we obtain,

$$A_n = \frac{\int_0^L \sin\frac{n\pi x}{L} dx}{\int_0^L \sin^2\frac{n\pi x}{L} dx} = \frac{2}{\pi}\ \frac{(-1)^{n+1}+1}{n}$$

Hence from equation 4.16, we have

$$1 = \sum_{n=1}^{\infty} \frac{2}{\pi}\ \frac{(-1)^{n+1}+1}{n}\ \sin\frac{n\pi x}{L} \qquad (4\text{-}18)$$

which is simply the expansion of unity in a *Fourier series*. Comparing equations 4.15 and 4.18 we obtain

$$C_n = \frac{2[(-1)^{n+1}+1]}{n\pi \sinh(\frac{n\pi W}{L})} \qquad n = 1, 2, 3, \ldots. \qquad (4\text{-}19)$$

Substituting equation 4.19 into equation 4.14, we then obtain for the final solution

$$\theta_{(x,y)} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh\frac{n\pi y}{L}}{\sinh\frac{n\pi W}{L}} \qquad (4\text{-}20)$$

The above equation is a convergent series, from which the value of θ may be computed for any x and y. Representative results are shown in the form of isotherms for a schematic of the rectangular plate (see Figure 4.2).

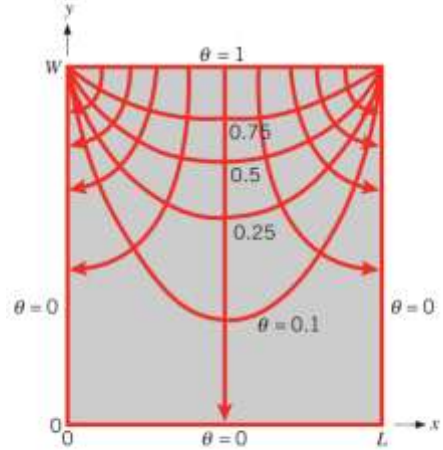


Figure 4.2: Isotherms and heat flow lines for two-dimensional conduction in a rectangular plate.

Example 2:

Derive an expression for the steady-state temperature distribution $\theta(x, y)$ of the extended surface as shown in Figure (4-3) for a finite heat transfer coefficient (h).

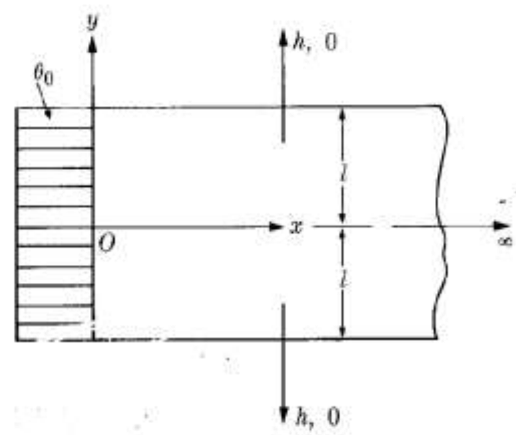


Figure 4.2: 2D extended surface.

<u>Solution:</u>

The transformed differential equation (applying Eq. 4.3) as,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

The boundary conditions are needed for each of the coordinates as,

$$\theta(0,y) = \theta_0\,, \quad \theta(\infty,y) = 0, \quad \frac{\partial\theta(x,0)}{\partial y} = 0\,, \quad -k\frac{\partial\theta(x,l)}{\partial y} = h\theta(x,l)$$

Assume the existence of a product solution of the form (Eq. 4.5)

$$\theta_{(x,y)} = X_{(x)}\,Y_{(y)}$$

Where X is a function of x alone and Y is a function y. The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0 \qquad , \theta(0, y) = \theta_0 , \quad \theta(\infty, y) = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 Y = 0 \qquad , \quad \frac{\partial \theta(x,0)}{\partial y} = 0 , \quad -k\frac{\partial \theta(x,l)}{\partial y} = h\theta(x, l)$$

The general form of the two-dimensional solution is,

$$\theta_{(x,y)} = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(C_3 \cos \lambda y + C_4 \sin \lambda y) \qquad (4\text{-}21)$$

Applying the condition that $\frac{\partial \theta(x,0)}{\partial y} = 0$, So

$$\frac{\partial \theta_{(x,y)}}{\partial y} = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin \lambda y - \lambda\, C_4 \cos \lambda y) \qquad (4\text{-}22)$$

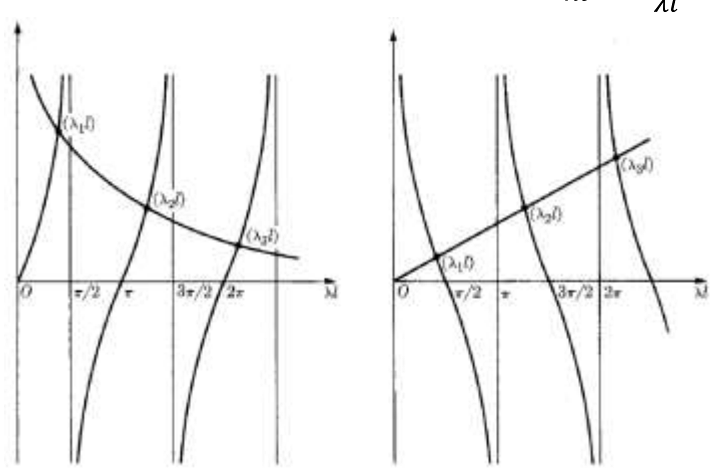
$$\frac{\partial \theta_{(x,0)}}{\partial y} = 0 = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin(\lambda \times 0) - \lambda\, C_4 \cos(\lambda \times 0))$$

From this condition, it is evident that $C_4 = 0$. Therefore, the characteristic function is $(\cos \lambda y)$. In addition from the requirement that $-k\frac{\partial \theta(x,l)}{\partial y} = h\theta(x, l)$, we obtain

$$-k\left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin(\lambda l)) = h\left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(C_3 \cos \lambda l)$$

$\tan \lambda y = \frac{h}{k} \times \frac{1}{\lambda} \times \frac{l}{l} = \frac{1}{\lambda l} Bi$ or $\cot \lambda y = \frac{\lambda l}{Bi}$, where *Bi* is *Biot number* ($\frac{hl}{k} = Bi$)

The characteristic values are the roots of [$\tan \lambda_n y = \frac{Bi}{\lambda l}$ or $\cot \lambda_n y = \frac{\lambda_n l}{Bi}$] as shown,



Now, by applying the condition in x-axis ($\theta(\infty, y) = 0$) as

$\theta_{(\infty,y)} = \left(C_1 e^{-\lambda_n \infty} + C_2 e^{\lambda_n \infty}\right)(C_3 \cos \lambda_n y) = 0$, From this condition, it is evident that $C_2 = 0$.

$$\theta_{(x,y)} = \sum_{n=1}^{\infty} A_n\, e^{-\lambda_n x} \cos \lambda_n y \qquad (4\text{-}23)$$

For last condition ($\theta(0, y) = \theta_0$),

$$\theta_0 = \sum_{n=1}^{\infty} A_n\, e^{(-\lambda_n \times 0)} \cos \lambda_n y$$

$$\theta_0 = \sum_{n=1}^{\infty} A_n \cos \lambda_n y \qquad (4\text{-}24)$$

Although the above equation (4-24) would seem to be an extremely complicated relation for evaluating A_n, a standard method is available. It involves writing an infinite series expansion in terms of *orthogonal functions*. An infinite set of functions $g_1(x)$, $g_2(x)$, … , $g_n(x)$, … is said to be orthogonal in the domain $a \leq x \leq b$ if, $\int_a^b g_m(x) g_n(x) dx = 0 \qquad m \neq n$

The form of the coefficients A_n in this series may be determined by multiplying each side of the equation by $g_m(x)$ and integrating between the limits a and b.

$$\int_a^b f(x) g_m(x) dx = \int_a^b g_m(x) \sum_{n=1}^{\infty} A_n g_n(x)\, dx$$

However, from above equation it is evident that all but one of the terms on the *right-hand* side of equation 4.16 must be zero, leaving us with

$$\int_a^b f(x) g_m(x) dx = A_m \int_a^b g_m^2\,(x) dx \qquad (m = n)$$

Therefore, $\theta_0 \int_0^l \cos \lambda_n y\, dy = A_n \int_0^l \cos^2 \lambda_n y dy$

$$\frac{\theta_0}{\lambda_n} [\sin \lambda_n y]_0^l = \frac{A_n}{\lambda_n} \left[\frac{\lambda_n y}{2} + \frac{1}{4} \sin 2\lambda_n y\right]_0^l$$

$$2\theta_0 \sin \lambda_n l = A_n[\lambda_n l + \sin \lambda_n l \cos \lambda_n l]$$

$$A_n = \frac{2\theta_0 \sin \lambda_n l}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l}$$

Hence, the steady-state temperature distribution $\theta(x, y)$ of the extended surface is,

$$\frac{\theta_{(x,y)}}{\theta_0} = 2 \sum_{n=1}^{\infty} \left(\frac{\sin \lambda_n l}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l}\right) e^{-\lambda_n x} \cos \lambda_n y$$

4.7 Nonhomogeneous Problems

There are many engineering problems in which the P.D.Es. and/ or the B.Cs. are not homogeneous. An example would be a plane-wall, nuclear reactor fuel element which is suddenly turned on. The P.D.E. describing this problem is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4-25}$$

The generation term $(\frac{\dot{g}}{k})$ makes the above equation non-homogenous. Another example would be a plane wall subjected to an ambient fluid whose temperature is fluctuating with time, defined as;

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}; \qquad \frac{\partial T(0,t)}{\partial x} = 0 \quad , T(L,t) = A\sin(\omega t)$$

In this case, the P.D.E. is homogeneous as is the B.C. at (x= 0). The B.C. at (x= L) is non-homogeneous (but linear), and consequently the problem is non-homogeneous, (but linear). The above examples cannot be made homogeneous by simply subtracting a constant from (T) $[\theta = T - Const.]$ as can be done in the lumped heat capacity problem. This section will discuss two methods of handing these more complicated problems;

- ✓ Partial Solutions.
- ✓ Variation of Parameters.

✓ Partial Solutions

A non-homogeneous problem can often be converted into a homogeneous one by the use of "*Partial Solution*" to the nonhomogeneous problem. A partial solution is one that satisfies only a part of the original problem. In a transient problem the most common partial solution would be the steady state solution. It satisfies the B.Cs. but not the initial condition. In addition, it deletes the time derivative from the P.D.E.

The steps in obtaining solutions to non-homogeneous problems by using the steady-state solution are;

1. Let $\theta = \theta_h + \theta_p$ where θ the general solution, θ_h is homogenous solution and θ_p is partial solution.
2. Determine the steady-state solutionθ_h.
3. Introduce ($\theta_p = \theta - \theta_h$) to make the problem homogeneous.
4. Solve for (θ_p) in the usual manner (*Separation of Variables*).
5. The complete solution is ($\theta = \theta_h + \theta_p$).

Example 3:

Consider an electric heater made from a solid rod of rectangular cross section (2L×2l) and designed according to one of the forms shown in **Figure 4.3**. The temperature variation along the rod can be neglected. The internal energy generation ($\dot{g}$) in the heater is uniform. The heat transfer coefficient (h) is large. Find the steady-state temperature of the electric heater.

Solution:

The formulation of the problem in Figure 4.3 is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\dot{g}}{k} = 0 \qquad (4\text{-}26)$$

The Boundary conditions are,

$$\frac{\partial\theta(0,y)}{\partial x} = 0 \qquad \frac{\partial\theta(x,0)}{\partial y} = 0$$

$$\theta(L,y) = 0 \qquad \theta(x,l) = 0$$

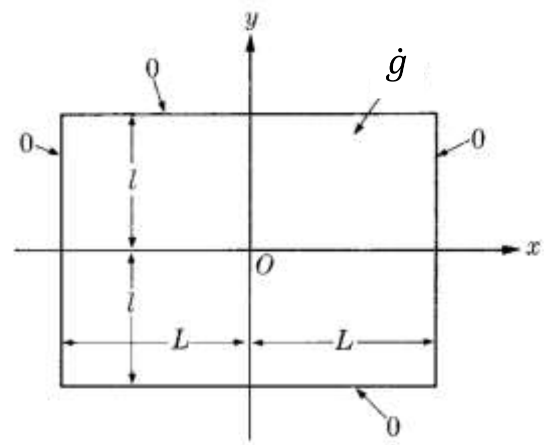


Figure 4.3: electric heater with rectangular cross section area.

The above partial differential equation, being *Non-homogenous*, is not separable.

Therefore, the general solution of the problem is now assumed to be,

$$\theta(x,y) = \theta_h(x,y) + \theta_p(x) \qquad (4\text{-}27)$$

Or, $\theta(x,y) = \theta_h(x,y) + \theta_p(y)$ (4-28)

$\theta_h(x,y)$ is solution of homogeneous part of the partial differential equation $[\nabla^2\theta_h = 0]$ with neglecting $(\dot{g})$.

$\theta_p(x)$ or $\theta_p(y)$ is partial solution part of the partial differential equation $[\frac{\partial^2\theta_p}{\partial x^2} + \frac{\dot{g}}{k} = 0]$ or $[\frac{\partial^2\theta_p}{\partial y^2} + \frac{\dot{g}}{k} = 0]$.

With the inclusion of the internal energy generation $(\dot{g})$ in the formulation of the One-dimensional problem, $\theta_p(x)$ or $\theta_p(y)$, the differential equation to be satisfied by the Two-dimensional problem, $\theta_h(x,y)$, can be made homogeneous.

Partial solution part:

$$\frac{\partial^2\theta_p}{\partial x^2} + \frac{\dot{g}}{k} = 0 \qquad \longrightarrow \qquad \theta_p = \frac{\dot{g}}{2k}x^2 + Ax + B$$

B.C.1: $\frac{\partial\theta_p(x=0)}{\partial x} = 0 \qquad \longrightarrow \qquad A = 0$

B.C.2: $\theta(x = L) = 0 \qquad \longrightarrow \qquad B = \frac{\dot{g}}{2k}L^2$

$$\theta_p = \frac{\dot{g}}{2k}L^2\left(1 - \left(\frac{x}{L}\right)^2\right) \tag{4-29}$$

The homogeneous solution part:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{4-30}$$

B.C.1: $\frac{\partial\theta_h(0,y)}{\partial x} = 0$

B.C.2: $\theta_h(L,y) = 0$

B.C.3: $\frac{\partial\theta_h(x,0)}{\partial y} = 0$

B.C.4: $\theta_h(x,l) = -\theta_p(x)$

From Eq. 4.27, $\theta(x,y) = \theta_h(x,y) + \theta_p(x)$,

$$\theta_h(x,y) = \theta(x,y) - \theta_p(x) \qquad \longrightarrow \qquad \frac{\partial\theta_h(x,y)}{\partial y} = \frac{\partial\theta(x,y)}{\partial y} - \frac{\partial\theta_p(x)}{\partial y} = 0$$

Now let, $\theta_h(x,y) = X(x).Y(y)$ (4-31)

Applying Eq. 4.31 into Eq. 4.30,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2$$

Therefore, $\theta_h(x,y) = (C_1 \cos\lambda x + C_2 \sin\lambda x)\,(C_3 \cosh\lambda y + C_4 \sinh\lambda y)$ (4-32)

From B.C.1, $C_2 = 0$ & From B.C.3, $C_4 = 0$

Thus, $\theta_h(x,y) = C_1 \cos\lambda x\, C_3 \cosh\lambda y$ (4-33)

From B.C.2: $0 = C_1 \cos\lambda L\, C_3 \cosh\lambda y$ $\quad 0 = C \cos\lambda L \cosh\lambda y$

Since $\cosh\lambda y \neq 0$ $\quad \cos\lambda L = 0$ $\quad \lambda L = \frac{\pi}{2}(2n+1)$ (4-34)

$$\lambda_n = \frac{\pi}{2}\frac{(2n+1)}{L}$$

Thus, $\theta_h(x,y) = \sum_{n=0}^{\infty} C_n \cos\lambda_n x \cosh\lambda_n y$ (4-35)

From B.C.4: $\sum_{n=0}^{\infty} C_n \cos\lambda_n x \cosh\lambda_n l = -\frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)$ $\quad \times (\cos\lambda_n x)$

$$C_n \cosh\lambda_n l \int_0^L \cos^2\lambda_n x\, dx = -\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

$$C_n \cosh\lambda_n l\,\frac{L}{2} = -\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

$$C_n = \frac{-1}{\cosh\lambda_n l}\frac{2}{L}\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx \quad \rightarrow$$

$$C_n = \frac{-1}{\cosh\lambda_n l}\frac{\dot{g}L}{k}\int_0^L \left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

Now, taken: $\int_0^L \left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx = \int_0^L \cos\lambda_n x\, dx - \frac{1}{b^2}\int_0^L x^2 \cos\lambda_n x\, dx$

Integrating left hand side, $\int_0^L \cos\lambda_n x\, dx = \frac{1}{\lambda_n}\sin\lambda_n L = \frac{1}{\lambda_n}\sin\frac{2n+1}{L}\frac{\pi}{2}L = \frac{1}{\lambda_n}(-1)^n$

Integrating right hand side, $\int_0^L \underbrace{x^2}_{u}\underbrace{\cos\lambda_n x\, dx}_{dv} = x^2\frac{\sin\lambda_n x}{\lambda_n}\Big]_0^L - 2\int_0^L \frac{\sin\lambda_n x dx}{\lambda_n}$

$$= \frac{x^2\sin\lambda_n x}{\lambda_n}\Big]_0^L - \frac{2}{\lambda_n}\left(x\frac{-\cos\lambda_n x}{\lambda_n} + \int_0^L \frac{\cos\lambda_n x dx}{\lambda_n}\right)$$

$$= \frac{x^2\sin\lambda_n x}{\lambda_n}\Big]_0^L + \frac{2}{\lambda_n}x\cos\lambda_n x\Big|_0^L - \frac{2}{\lambda_n{}^3}[\sin\lambda_n x]_0^L$$

$= \frac{x^2 \sin \lambda_n x}{\lambda_n}\Big|_0^L + \frac{2}{{\lambda_n}^2} x\cos \lambda_n x|_0^L - \frac{2}{{\lambda_n}^3} \sin \lambda_n x|_0^L$

$= \frac{L^2 \sin \lambda_n L}{\lambda_n} + \frac{2}{{\lambda_n}^2} L \cos \lambda_n L - \frac{2}{{\lambda_n}^3} \sin \lambda_n L = \frac{L^2}{\lambda_n}(-1)^n + \frac{2L}{{\lambda_n}^2}(0) - \frac{2}{{\lambda_n}^3}(-1)^n$

$= (-1)^n (\frac{L^2}{\lambda_n} - \frac{2}{{\lambda_n}^3})$

So, $\int_0^L \left(1 - \left(\frac{x}{L}\right)^2\right) \cos \lambda_n x \, dx = \frac{(-1)^n}{\lambda_n} - \frac{(-1)^n}{L^2}(\frac{L^2}{\lambda_n} - \frac{2}{{\lambda_n}^3}) = \frac{2}{L^2 {\lambda_n}^3}(-1)^n$

$C_n = \frac{-1}{\cosh \lambda_n l} \frac{\dot{g}L}{k} \frac{2}{L^2 {\lambda_n}^3}(-1)^n = \frac{-2\dot{g}}{kL^2 {\lambda_n}^3 \cosh \lambda_n l}(-1)^n$

$$\theta(x,y) = \sum_{n=0}^{\infty} \frac{-2\dot{g}}{kL^2 {\lambda_n}^3 \cosh \lambda_n l}(-1)^n \cos \lambda_n x \cosh \lambda_n y + \frac{\dot{g}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

$\frac{\theta(x,y)}{\frac{\dot{g}L^2}{2k}} = \frac{1}{2}\left(1 - \left(\frac{x}{L}\right)^2\right) - 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} \frac{\cosh \lambda_n y}{\cosh \lambda_n l} \cos \lambda_n x$

✓ **Variation of Parameters**

In some cases there may not be steady-state solution or you may not be able to find it. In these cases; variation of parameters method may be used. The procedure is outlined as follows;

1. Set up a problem corresponding to the original one by simply setting all the non-homogeneous terms equal to zero.

2. Determine the *eigen-functions* and *eigen-condition* for the "corresponding homogeneous problem".

3. Construct a solution to the original non-homogeneous problem of the form;

$\theta(x,t) = \sum_n A_n(t)\emptyset_n(x)$

Where, the $\emptyset_n(x)$ is the eigen-functions you have obtained above from the corresponding homogeneous problem.

4. Evaluate $A_n(t)$ in the usual manner making use of the orthogonally of the $\emptyset_n(x)$.

That is; $\int_0^1 \theta(x,t)\, \varphi_m(x) dx = \sum_n^{\infty} A_n\,(t) \int_0^1 \varphi_n(x)\varphi_m(x) dx = CA_m(t)$

Or, $A_m(t) = \frac{1}{C}\int_0^1 \theta(x,t)\,\varphi_m(x)dx$

Here θ is still unknown.

5. Set up an O.D.E. and B.Cs. for $A_m(t)$.

6. Solve for $A_m(t)$.

7. Complete the solution; $\theta(x,t) = \sum_n A_n(t)\emptyset_n(x)$

Example 4:

Consider a plane wall whose initial normalized temperature is zero. The face (x= 0) is suddenly changed to a normalized temperature of unity at zero time, while the face at (x= 1) is maintained at the initial temperature. Determine the temperature distribution.

Solution:

(1) Set every non-homogeneous term equal to zero, thus the problem will be;

$$\frac{\partial^2\theta_h}{\partial x^2} = \frac{\partial\theta_h}{\partial t} \quad \ldots\ldots\ldots(1)$$

$$\left.\begin{array}{ll} \text{I.C.} & \theta_h(x,0) = 0 \\ \text{B.C.1} & \theta_h(0,t) = 0 \\ \text{B.C.2} & \theta_h(1,t) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(2)$$

(2) Determine the eigen-functions.

$$\text{Let } \theta_n = X(x)\tau(t) \qquad \ldots\ldots(3)$$

$$\text{Thus;} \quad \frac{X''}{x} = \frac{\tau'}{\tau} = -\lambda^2$$

$$\left.\begin{array}{l} X'' + \lambda^2 X = o \\ X(0) = 0 \\ X(1) = 0 \end{array}\right\} \begin{array}{l} X = A\sin\lambda x + B\cos\lambda x \\ X(0) = 0 \quad \rightarrow B = o \\ X(1) = 0 \quad \rightarrow \sin\lambda = o\,, \lambda_n = n\pi \end{array}$$

Therefore;

$\Phi_n(\mathrm{x}) = \sin(\mathrm{n}\pi x) \quad \ldots\ldots(4)$

(eigen-function corresponds to the homogenous problem)

3- Construct a solution;

$$\theta(\mathrm{x,t}) = \sum_{n=1}^{\infty} A_n(\mathrm{t})\,\Phi_n(\mathrm{x}) = \sum_{n=1}^{\infty} A_n(\mathrm{t}) \sin(\mathrm{n}\pi x) \ldots\ldots(5)$$

4- $\int_o^1 \theta(x,t) sinm\pi x dx = \sum_{n=1}^{\infty} A_n(\mathrm{t}) \int_o^1 sinn\pi x\ sinm\pi x dx = \frac{1}{2} A_m(\mathrm{t})$

$$where; \quad A_m(\mathrm{t}) = 2\int_o^1 \theta(x,t)\sin(m\pi x)\,dx \ldots\ldots(6)$$

5- Set up an O.D.E. for $A_m(t)$ by differentiating with respect to (t). Thus;

$$\frac{dA_m(\mathrm{t})}{dt} = 2\int_o^1 \frac{\partial\theta(x,t)}{\partial t} \sin(\mathrm{m}\pi x)dx$$

Now, since; $\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t} \Longrightarrow \frac{\partial\theta(x,t)}{\partial t} = \frac{\partial^2\theta(x,t)}{\partial x^2}$, hence;

$$\frac{dA_m(\mathrm{t})}{dt} = 2\int_o^1 \frac{\partial^2\theta}{\partial x^2} \sin(\mathrm{m}\pi x)dx = 2\left\{\left(\frac{\partial\theta}{\partial x}\sin m\pi x\right)\Big|_o^1 - m\pi\int_o^1 \frac{\partial\theta}{\partial x}\cos m\pi x dx\right\}$$

Noting that; $\frac{\partial\theta}{\partial x}(1,\mathrm{t})$ is unknown, but $\sin \mathrm{m}\pi 1 = o\ for x = 1$.

Similarly $\frac{\partial\theta}{\partial x}(0,\mathrm{t})$ is unknown, but $\sin \mathrm{m}\pi 0 = 0\ for\ x = 0$.

Thus; $\frac{dA_m(\mathrm{t})}{dt} = -2m\pi\int_o^1 \frac{\partial\theta}{\partial x}\cos m\pi x dx$

$$= -2\mathrm{m}\pi\left\{[\theta\cos m\pi]_0^1 + m\pi\int_0^1 \theta\ sinm\pi x dx\right\}$$

$$\frac{dA_m(\mathrm{t})}{dt} = -2m\pi\left\{\theta(1,t)\cos m\pi - \theta(o,t)\ 1 + m\pi\frac{A_m(\mathrm{t})}{2}\right\}$$

$$\frac{dA_m(\mathrm{t})}{dt} = 2m\pi - m^2\pi^2 A_m(t)$$

$$\frac{dA_m}{dt} + m^2\pi^2 A_m = 2m\pi \quad \ldots\ldots(7)$$

$$\mathrm{I.C.} \quad A_m(0) = 2\int_o^1 \theta(x,0)\sin m\pi x dx \rightarrow A_m(0) = 0$$

Equation (7) can be solved using integrating function (I);

$$\mathrm{I} = \exp \int m^2 \pi^2 \ dt = e^{m^2\pi^2 t}$$

Then; $\frac{d}{dt}(A_m e^{m^2\pi^2 t}) = 2\mathrm{m}\pi e^{m^2\pi^2 t}$

$$A_m e^{m^2\pi^2 t} = \frac{2m\pi}{m^2\pi^2} \ e^{m^2\pi^2 t} \ + C_1$$

$$A_m(0) = 0 \ \rightarrow C_1 = -\frac{2}{m\pi}$$

Thus; $A_m(\mathrm{t}) \ = \frac{2}{m\pi} \ - \frac{2}{m\pi} \ e^{-m^2\pi^2 t}$

Hence, from (5) $\theta(x,t) = \sum_{n=1}^{\infty}(\frac{2}{n\pi} - \frac{2}{n\pi} \ e^{-n^2\pi^2 t})\sin \mathrm{n}\pi x$

Or; $\theta(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \pi x}{n} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin\pi x e^{-n^2\pi^2 t}}{n}$

4.8 Cylindrical Geometry

The inherent nature of cylindrical coordinates implies three types of 2-D problems in the form $T(r,\varphi)$, $T(r,z)$ and $T(\varphi,z)$, see Figure 4.4.

Since $T(\varphi,z)$ has no physical significance (except in thin–walled tubes, which can be investigated in terms of Cartesian Coordinates), it is not considered here. $T(r,z)$ may depend on the expansion of an arbitrary function into a series in terms of cylindrical (*Bessel*) functions. Problems of the type $T(r,\varphi)$, on the other hand, require no further mathematical background than that needed for Cartesian geometry.

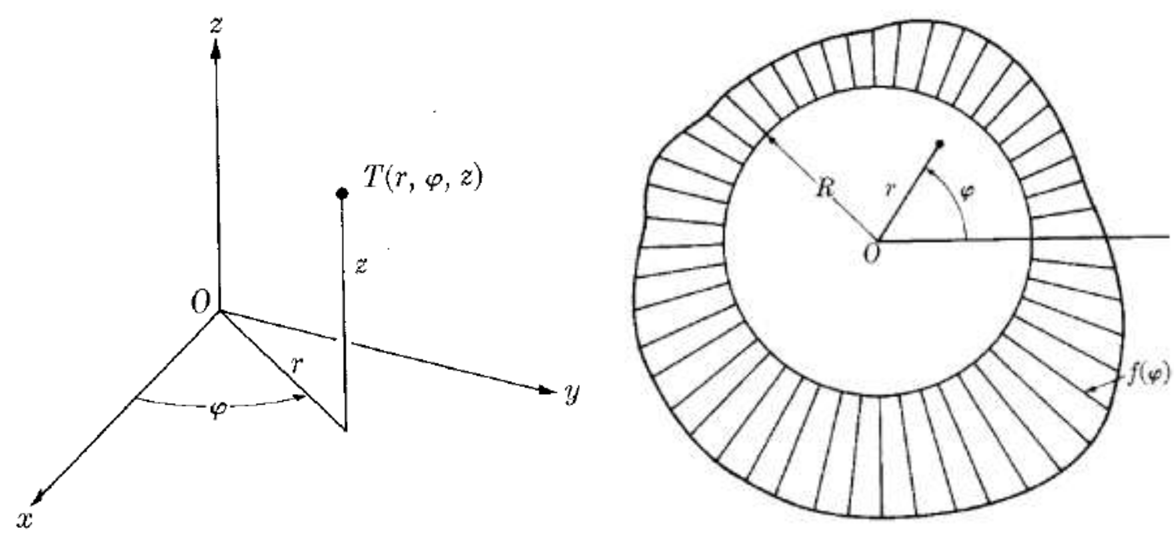


Figure 4.4: Infinitely long rod with cylindrical coordinates.

When a problem of type $T(r,z)$ is orthogonal in the (r-direction), it can be solved by the proper choice of the separation constant leading to a 2^{nd} order O.D.E. in (z) satisfied by hyperbolic function, and to Bessel equation in (r). If the z-direction is orthogonal, the problem do not need additional mathematics and can be solved by using circular functions in Z and the *modified Bessel* functions in (r).

Example 5:

The surface temperature of an infinitely long solid rod of radius R is specified as $f(\varphi)$, see Figure 4.4. Find the steady-state temperature of the rod.

Solution:

For steady-state $\frac{\partial}{\partial \mathrm{t}} = 0$, No heat generation $\dot{g} = 0$, 2D $\frac{\partial}{\partial \mathrm{z}} = 0$

Thus: $\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\left(\frac{\partial^2 \mathrm{T}}{\partial \phi^2}\right) = 0$ (4-36)

$$\left(\frac{\partial^2 T}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right) = 0$$

And the problem boundary condition as:

$T(0,\varphi) = finite$, $T(R,\varphi) = f(\varphi)$,

$T(r,\varphi) = T(r,\varphi + 2\pi)$, $\frac{\partial T(r,\varphi)}{r\partial\varphi} = \frac{\partial T(r,\varphi+2\pi)}{r\partial\varphi}$

The r-direction cannot be made orthogonal by any transformation. This leaves (φ) as the only possible orthogonal direction. Hence the product solution $T(r,\varphi) = R(r)\phi(\varphi)$ with the proper choice of the separation constant yields

$T(r,\varphi) = R(r)\phi(\varphi)$ (4-37)

By substituting Equation 4.37 into Equation 4.36 to obtain that

$\left(\frac{\partial^2 R(r)}{\partial r^2}\phi(\varphi)\right) + \frac{1}{r}\left(\frac{\partial R(r)}{\partial r}\phi(\varphi)\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}R(r)\right) = 0$ Dividing by $(R(r)\phi(\varphi))$

$$\left(\frac{\partial^2 R(r)}{\partial r^2}/R(r)\right) + \frac{1}{r}\left(\frac{\partial R(r)}{\partial r}/R(r)\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}/\phi(\varphi)\right) = 0$$

$$r^2\left(\frac{\partial^2 R(r)}{\partial r^2}/R(r)\right) + r\left(\frac{\partial R(r)}{\partial r}/R(r)\right) = -\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}/\phi(\varphi)\right) = \lambda^2$$

$\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2} + \lambda^2\phi(\varphi) = 0$ with B.Cs. $T(r,\varphi) = T(r,\varphi + 2\pi)$ & $\frac{\partial T(r,\varphi)}{r\partial\varphi} = \frac{\partial T(r,\varphi+2\pi)}{r\partial\varphi}$

$r^2\left(\frac{\partial^2 R(r)}{\partial r^2}\right) + r\left(\frac{\partial R(r)}{\partial r}\right) - \lambda^2 R(r) = 0$ with B.Cs. $T(0,\varphi) = finite$

$\phi(\varphi) = A\cos\lambda\varphi + B\sin\lambda\varphi$ (4-38)

$$A\sin\lambda\varphi + B\cos\lambda\varphi = A\sin\lambda(\varphi + 2\pi) + B\cos\lambda(\varphi + 2\pi)$$

But, $\sin\lambda\varphi = \sin\lambda(\varphi + 2\pi) = \sin\lambda\varphi\cos 2\lambda\pi + \cos\lambda\varphi\sin 2\lambda\pi$

$\cos\lambda\varphi = \cos\lambda(\varphi + 2\pi)$ where $\lambda = n,$ $n = 0, 1, 2, \ldots..$

$$\phi(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi) \tag{4-39}$$

Sol. of (5) $r^2R'' + rR' - \lambda^2 R = 0$

Let; $r = e^z \rightarrow z = \ln r$

$$R' = \frac{dR}{dr} = \frac{dR}{dz}\frac{dz}{dr} = \frac{1}{r}\frac{dR}{dz}$$

$$\frac{d^2R}{dr^2} = R'' = \frac{d}{dr}\left(\frac{dR}{dz}\right) = -\frac{1}{r^2}\frac{dR}{dz} + \frac{1}{r^2}\frac{d^2R}{dz^2}$$

Thus; $\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} - \lambda^2 R = 0$

Or; $\frac{d^2R}{dz^2} - \lambda^2 R = 0$

Thus; $R = c\, e^{\lambda z} + De^{-\lambda z} = ce^{\lambda \ln r} + De^{-\lambda \ln r}$

Or; $R = c\, r^{\lambda} + Dr^{-\lambda}$ ……. (7)

Or, $(\lambda = n)$;

$R = c_n r^n + D_n r^{-n}$

B.C.1 R(0) = finite $\rightarrow D_n = 0$

Hence; $R_n = C_n r^n$ …… (8)

Thus; $T(r,\phi) = \sum_{n=o}^{\infty} R_n\,\Phi_n = R_0\,\Phi_0 + \sum_{n=1}^{\infty} R_n\,\phi_n$

Hence; $T(r,\phi) = a_0 + \sum_{n=1}^{\infty} r^n\,(a_n \cos(n\phi) + b_n \sin(n\phi))$ ..(9)

Where; $a_0 = A_0\,C_0$, $a_n = A_n\,C_n$, $b_n = B_n\,C_n$

Using B.C.2;

$f(\phi) = a_0 + \sum_{n=1}^{\infty} R^n\,(a_n \cos n\phi + b_n \sin n\phi)$ ……. (10)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi$$

$$a_n\,R_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi$$

$$b_n\,R^n = \frac{1}{\pi}\int_o^{2\pi} f(\phi) sin n\phi d\phi$$

For the problem defined by;

$$\begin{aligned} T(R,\phi) &= T_o \quad 0<\phi<\pi \\ &= 0 \quad \pi<\phi<2\pi \end{aligned}$$

Equ.(11) gives;

$$\left.\begin{aligned} a_0 &= \frac{1}{2}T_0 \\ a_n &= 0 \\ b_n R^n &= 2T_0/n\pi \end{aligned}\right\} n=1,3,5....$$

Thus;

$$\frac{T(r,\phi)}{T_0} = \frac{1}{2} + 2\sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\frac{r}{R}\right)^n \sin(n\phi)$$

4.9 Spherical Geometry

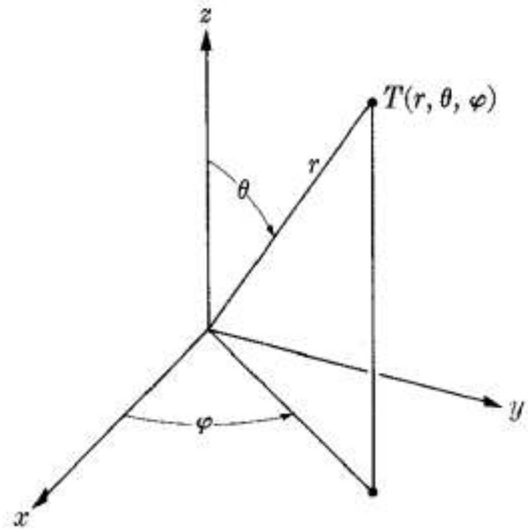


Figure 4.5: Spherical coordinates.

When a spherical problem depends on the cone angle (θ), see the figure, its solution can be reduced to the expansion of arbitrary function into series of "*Legendre Polynomial*". The linear 2nd order differential equation with variable coefficients.

The linear second-order differential equation with variable coefficients

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad (4\text{-}40)$$

Is known as "*Legendre's Equation*", and its solutions are known as "*Legendre*'s *Functions*". In particular, when (n= 0) or a positive integer, the solutions of (4-40) are called "*Legendre Polynomial*". The solution of (4-40) may be obtained by the method of power series as;

$$y(x) = a_0 P_n(x) + a_1 Q_n(x) \qquad (4\text{-}41)$$

Where,

$P_n(x)$ is *Legendre Polynomial* of degree (n) of the first kind.

$Q_n(x)$ is *Legendre Polynomial* of degree (n) of the second kind.

Hence the *Legendre Polynomials* $P_n(x)$ are the characteristic functions of the characteristic value problem stated by [$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n-1)y = 0$].

These polynomials form an orthogonal set with respect to the weighting function $w(x) = 1$ over the interval (-1, 1); that is

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad \text{if} \qquad m \neq n. \qquad (4\text{-}42)$$

Now the expansion of an arbitrary function $f(x)$ in terms of appropriate Legendre polynomials, the *Fourier-Legendre series*, may be written in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad -1 < x < 1$$

Here the coefficient a_n again follows

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx} \tag{4-43}$$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n\, dx \tag{4-44}$$

If $f(x)$ and its first n derivatives are continuous in the interval, integrating the right-hand side of the above equation n times by parts gives

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2 - 1)^n \frac{d^n f(x)}{dx^n}\, dx \tag{4-45}$$

Now, replacing by in the above equation, and employing the n^{th} derivative of equation [$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. $\qquad \frac{d^n P_n(x)}{dx^n} = \frac{(2n)!}{2^n n!}$

we find the denominator to be

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - x^2)^n\, dx \tag{4-46}$$

The right-hand side of the above equation integrated n times by parts yields

$$\int_{-1}^{1} (1 - x^2)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n + 1)!} \tag{4-47}$$

Introducing Equation (4-47) into Equation (4-46), obtain

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n + 1} \tag{4-48}$$

Hence the coefficient (a_n) becomes,

$$a_n = \begin{cases} \dfrac{2n + 1}{2} \displaystyle\int_{-1}^{1} f(x) P_n(x)\, dx, \\[2ex] \dfrac{2n + 1}{2^{n+1} n!} \displaystyle\int_{-1}^{1} (1 - x^2)^n \frac{d^n f(x)}{dx^n}\, dx. \end{cases} \tag{4-49}$$

The second form of Equation (4.49) can be used only if $f(x)$ and its first n derivatives are continuous in (-1, 1).

Furthermore, noting that $P_n(x)$ is an even function of x when n is ***even***, and an odd function when n is ***odd***, we have

For an ***Even*** function $f(x)$,

$$a_n = \begin{cases} (2n+1)\int_0^1 f(x)P_n(x)\,dx, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{4-50}$$

For an ***Odd*** function $f(x)$

$$a_n = \begin{cases} (2n+1)\int_0^1 f(x)P_n(x)\,dx, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \tag{4-51}$$

<u>Example 6:</u>

The surface temperature of a sphere of radius R is specified in the form $f(\theta)$. Find the steady-state temperature distribution of the sphere.

<u>Solution:</u>

The formulation of the problem

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = 0,$$

$$T(0,\theta) = \text{finite},$$

$$T(R,\theta) = f(\theta).$$

The missing boundaries in the θ-direction will be discussed later.

Since θ is the only possible orthogonal direction, with the appropriate choice of separation constant the product solution $[T(r,\theta) = \mathcal{R}(r)\vartheta(\theta)]$ yields

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\vartheta}{d\theta}\right) + \lambda\vartheta = 0, \quad \text{and} \quad r^2\frac{d^2\mathcal{R}}{dr^2} + 2r\frac{d\mathcal{R}}{dr} - \lambda\mathcal{R} = 0$$

First rearranged in the form
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1-\cos^2\theta}{\sin\theta}\frac{d\vartheta}{d\theta}\right)+\lambda\vartheta=0,$$

Then transformed with $x = \cos\theta \rightarrow [\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx}]$ and $[\frac{d\vartheta}{d\theta} = \frac{d\vartheta}{dx}\frac{dx}{d\theta} = \sin\theta\frac{d\vartheta}{dx}]$, may be written as,
$$\frac{d}{dx}\left[(1-x^2)\frac{d\vartheta}{dx}\right]+n(n+1)\vartheta=0,$$

where $n(n+1)=\lambda$. This is a Legendre equation. Its particular solution, finite at $x=\pm1(\theta=0,\pi)$, is

$$\vartheta_n = A_n\psi_n(\theta), \qquad \psi_n(\theta)=P_n(\cos\theta), \quad \text{characteristic functions,}$$

$$n = 0, 1, 2, 3, \ldots, \quad \text{characteristic values.}$$

Here the condition of finiteness specifying the characteristic functions and characteristic values takes care of the two missing boundary conditions in the θ-direction. The general solution of the equidimensional equation given by:

$$\mathcal{R}_n(r) = C_n r^n + D_n r^{-(n+1)} \qquad \text{where } n = -\tfrac{1}{2}+(\lambda+\tfrac{1}{4})^{1/2}.$$

Thus, $\mathcal{R}_n(r) = C_n r^n.$

Thus the product solution of the problem is

$$T(r,\theta)=\sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta), \qquad \text{where } a_n = A_nC_n.$$

The use of $T(R,\theta)=f(\theta)$ and reduced the above equation to

$$f(\theta)=\sum_{n=0}^{\infty} a_n R^n P_n(\cos\theta),$$

which is the expansion of $f(\theta)$ into a Fourier-Legendre series. Here the coefficient a_n is readily obtained

$$a_n R^n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta$$

In particular, if the surface temperature is specified† as

$$T(R, \theta) = f(\theta) = \begin{cases} T_0, & 0 < \theta < \pi/2, \\ 0\ , & \pi/2 < \theta < \pi, \end{cases}$$

$$a_n R^n = T_0 \left(\frac{2n+1}{2}\right) \int_0^1 P_n(x)\, dx$$

$$a_0 = \tfrac{1}{2} T_0 \int_0^1 dx = \tfrac{1}{2} T_0,$$

$$a_1 R = \tfrac{3}{2} T_0 \int_0^1 x\, dx = \tfrac{3}{4} T_0,$$

$$a_2 R^2 = 0,$$

$$a_3 R^3 = \tfrac{7}{2} \cdot \tfrac{1}{2} T_0 \int_0^1 (5x^3 - 3x)\, dx = -\tfrac{7}{16} T_0,$$

$$a_4 R^4 = 0,$$

$$a_5 R^5 = \tfrac{11}{2} \cdot \tfrac{1}{8} T_0 \int_0^1 (63x^5 - 70x^3 + 15x)\, dx = \tfrac{11}{32} T_0,$$

$$\vdots$$

Hence the solution of the problem is

$$\frac{T(r, \theta)}{T_0} = \frac{1}{2} + \frac{3}{4}\left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{11}{32}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \cdots$$

4.10 Heterogeneous Solids (Variable Thermal Conductivity)

Heterogeneous solids are becoming increasingly important because of the large ranges of temperature involved in problems of technology, as in reactor fuel elements, space vehicle components, solidification of castings …etc. The equation of heat conduction for heterogeneous solids;

Cartesian coordinates

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

If k, C_p, $\dot{g}$ are functions of space only, the above three equations become a linear differential equation with variable coefficients. If (k & C_p) are dependent on temperature but independent of space, however, the above equations become nonlinear and difficult to solve. Usually numerical methods have to be employed. A number of analytical methods are also available. One of these, *Kirchhoff's method*, is to a large extent general.

Above equations may be reduced to a linear differential equation by introducing a new temperature θ related to the temperature T of the problem by the *Kirchhoff transformation*,

$$\theta=\frac{1}{k_R}\int_{T_R}^{T}k(T)\,dT \qquad (4\text{-}52)$$

where T_R denotes a convenient reference temperature, and $k_R = k(T_R)$. T_R and k_R are introduced merely to give θ the dimensions of temperature and a definite value. It follows from Eq. (4-52) that

$\frac{d\theta}{dt} = \frac{k}{k_R}\frac{dT}{dt}$ (4-53)

$\nabla\theta = \frac{k}{k_R}\,\nabla T$ (4-54)

Inserting Eqs. (4-53) and (4-54) into energy equations, we have

$\frac{d\theta}{dt} = a\nabla^2\theta + \left(\frac{a}{k_R}\right)\dot{g}$ (4-55)

where a and $\dot{g}$ are expressed as functions of the new variable θ. For many solids, however, the temperature dependence of a can be neglected compared to that of k. In such cases, if $\dot{g}$ is independent of T, Eq. (4-55) becomes identical to Eq. (4-52) except for the different but constant coefficient of $\dot{g}$. Thus the solutions obtained for homogeneous solids may be readily utilized for heterogeneous solids by replacing T by θ and ρC_p by k_R/a, provided that the boundary conditions prescribe T or $\frac{\partial T}{\partial n}$. This remark does not hold if the boundary conditions involve the convective term $h(T_o - T\infty)$. The following one-dimensional example illustrates the use of the method.

Example 7:

A liquid is boiled by a flat electric heater plate of thickness $2L$. The internal energy $\dot{g}$ generated electrically may be assumed to be uniform. The boiling temperature of the liquid, corresponding to a specified pressure, is Too (see Fig. 4.6). Find the steady-state temperature of the plate for

(i) $k = k(T)$; (ii) $k = k_R\,(1 + (\beta T)$.

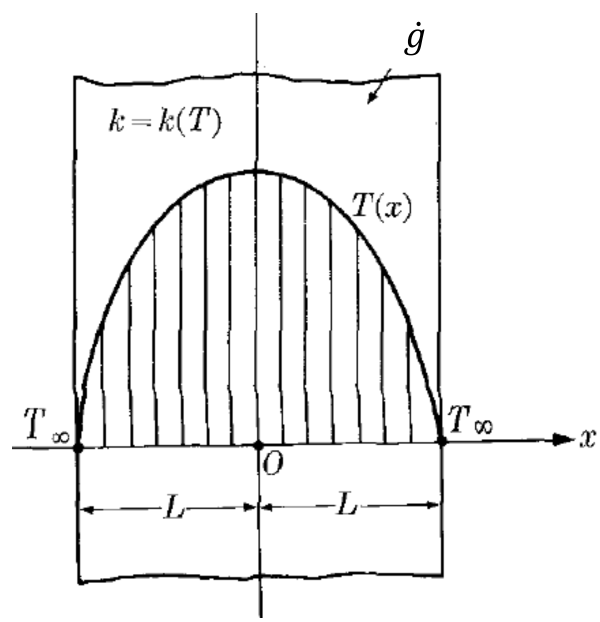


Figure 4.6: Details for Example 7.

Solution:

The formulation of the problem is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{g}=0 \qquad (4\text{-}56)$$

$$\frac{dT(0)}{dx}=0 \qquad \text{and} \qquad T(L)=T_\infty$$

Employing the one-dimensional form of Eq. (4-53), $\frac{d\theta}{dt}=\frac{k}{k_R}\frac{dT}{dt}$

We may transform Eq. (4-56) to

$$\frac{d^2\theta}{dx^2}+\frac{\dot{g}}{k_R}=0 \qquad \frac{d\theta(0)}{dx}=0 \qquad \text{and} \qquad \theta(L)=\theta_\infty$$

where, according to Eq. (4-52), $\theta=\frac{1}{k_R}\int_{T_R}^{T_\infty}k(T)\,dT$

The solution of Eq. (4-56) is $\frac{\theta(x)-\theta_\infty}{\dot{g}L^2/2k_R}=1-\left(\frac{x}{L}\right)^2$ (4-57)

Introducing Eqs. (θ) and (θ_∞) into Eq. (4-57), we obtain the temperature of the plate in terms of T as follows:

$\frac{\frac{1}{k_R}\int_{T_\infty}^{T}k(T)dT}{\dot{g}L^2/2k_R}=1-\left(\frac{x}{L}\right)^2$ For the special case $k = k_R$ (l +(βT), the equation becomes

$$\frac{[T(x)-T_\infty]+\left(\frac{\beta}{2}\right)[T^2(x)-T^2{}_\infty]}{\dot{g}L^2/2k_R}=1-\left(\frac{x}{L}\right)^2$$