University of Anbar
College of Engineering
Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Five
Transient Conduction Heat Transfer

Course Tutor

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Chapter Five

Transient Conduction Heat Transfer

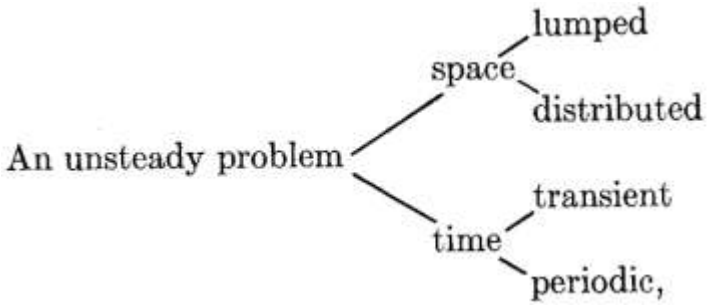
5.1 Introduction

Many heat transfer problems are *time dependent*. Such *unsteady*, or *transient*, problems typically arise when the boundary conditions of a system are changed. For example, if the surface temperature of a system is altered, the temperature at each point in the system will also begin to change. The changes will continue to occur until a steady-state temperature distribution is reached. Consider a hot metal billet that is removed from a furnace and exposed to a cool airstream. Energy is transferred by convection and radiation from its surface to the surroundings. Energy transfer by conduction also occurs from the interior of the metal to the surface, and the temperature at each point in the billet decreases until a steady-state condition is reached. The final properties of the metal will depend significantly on the time-temperature history that results from heat transfer. Controlling the heat transfer is one key to fabricating new materials with enhanced properties.

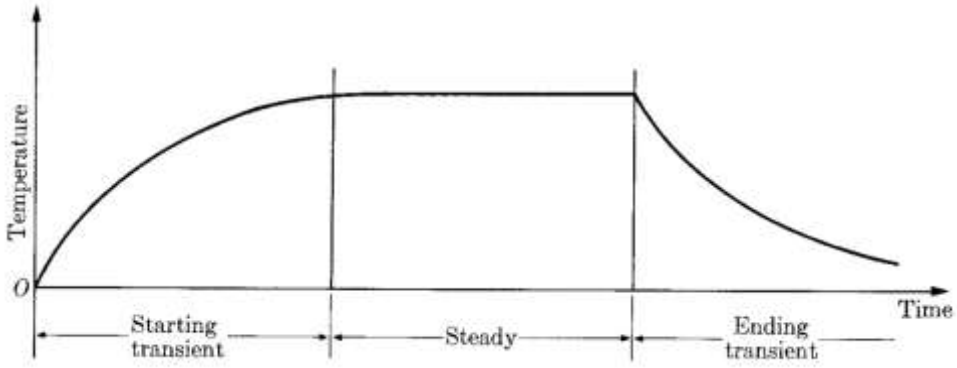
Our objective in this chapter is to develop procedures for determining the time dependence of the temperature distribution within a solid during a transient process, as well as for determining heat transfer between the solid and its surroundings. The nature of the procedure depends on assumptions that may be made for the process. If, for example, temperature gradients within the solid may be neglected, a comparatively simple approach, termed the *lumped capacitance* method, may be used to determine the variation of temperature with time.

Transient problems can be classified with respect to their dependence on space (as lumped or distributed), and since then formulated them accordingly. Also, these

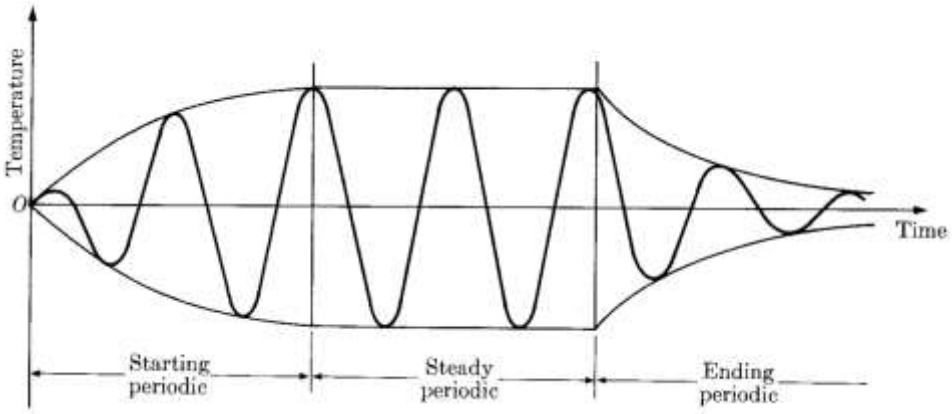
problems may be classified with respect to their dependence on time (as transient or periodic). Thus,



Where, each transient or periodic problem involves a *starting*, a *steady*, and an *ending* time interval as shown in Figure 5.1.



Unsteady Transient



Unsteady Periodic

Figure 5.1: The behavior of unsteady transient and unsteady periodic.

5.2 The Lumped Capacitance Method

A simple transient conduction problem is one for which a solid experiences a sudden change in its thermal environment. Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$ (see Figure 5.2). If the quenching is said to begin at time $t = 0$, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches T_∞. This reduction is due to convection heat transfer at the solid–liquid interface. The essence of the *lumped capacitance method* is the assumption that "*the temperature of the solid is spatially uniform at any instant during the transient process*". This assumption implies that temperature gradients within the solid are negligible.

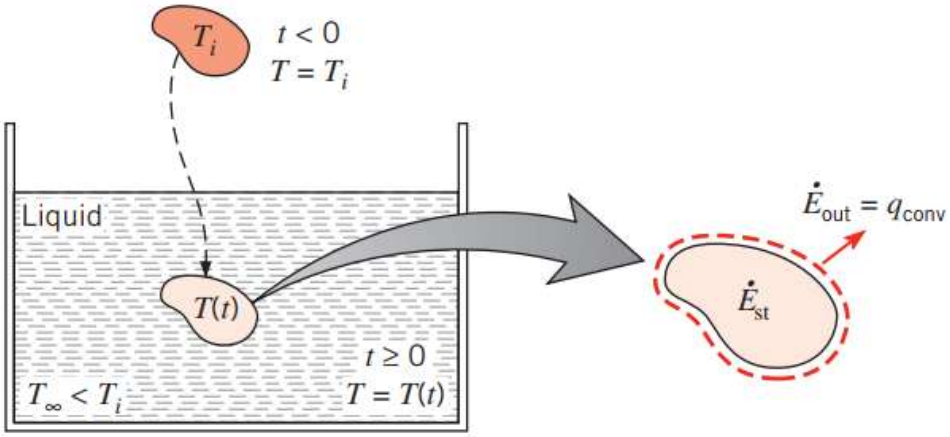


Figure 5.2: Cooling of a hot metal forging.

From *Fourier's law*, heat conduction in the absence of a temperature gradient implies the existence of infinite thermal conductivity. Such a condition is clearly impossible. However, the condition is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings. For now we assume that this is, in fact, the case.

In neglecting temperature gradients within the solid, we can no longer consider the problem from within the framework of the heat equation, since the heat equation is

a differential equation governing the spatial temperature distribution within the solid. Instead, the transient temperature response is determined by formulating an overall *energy balance* on the entire solid. This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy.

$$-E_{out} = E_{st} \tag{5-1}$$

Or, $$-hA_s(T - T_\infty) = \rho VC\frac{dT}{dt} \tag{5-2}$$

Introducing the temperature difference

$$\theta = T - T_\infty \tag{5-3}$$

and recognizing that ($d\theta/dt$) = (dT/dt) if T_∞ is constant, it follows that

$$\frac{\rho VC}{hA_s}\frac{d\theta}{dt} = -\theta \tag{5-4}$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho VC}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt \tag{5-5}$$

Where, $\theta_i = T_i - T_\infty$, thus, evaluating the integrals, it follows that

$$\frac{\rho VC}{hA_s}\ln\frac{\theta_i}{\theta} = t \tag{5-6}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{\left[-\left(\frac{hA_s}{\rho VC}\right)t\right]} \tag{5-7}$$

Equation (5.6) may be used to determine the *time* required for the solid to reach some temperature *T*, or, conversely, Equation 5.7 may be used to compute the *temperature* reached by the solid at some time *t*.

The foregoing results indicate that the difference between the solid and fluid temperatures must decay exponentially to *zero* as *t* approaches infinity. This behavior is shown in Figure 5.3. From Equation 5.7 it is also evident that the quantity ($\rho VC/hA_s$) may be interpreted as a thermal time constant expressed as

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho VC) = R_t C_t \tag{5-8}$$

where, R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment. This behavior is analogous to the voltage decay that occurs when a capacitor is discharged through a resistor in an electrical *RC* circuit.

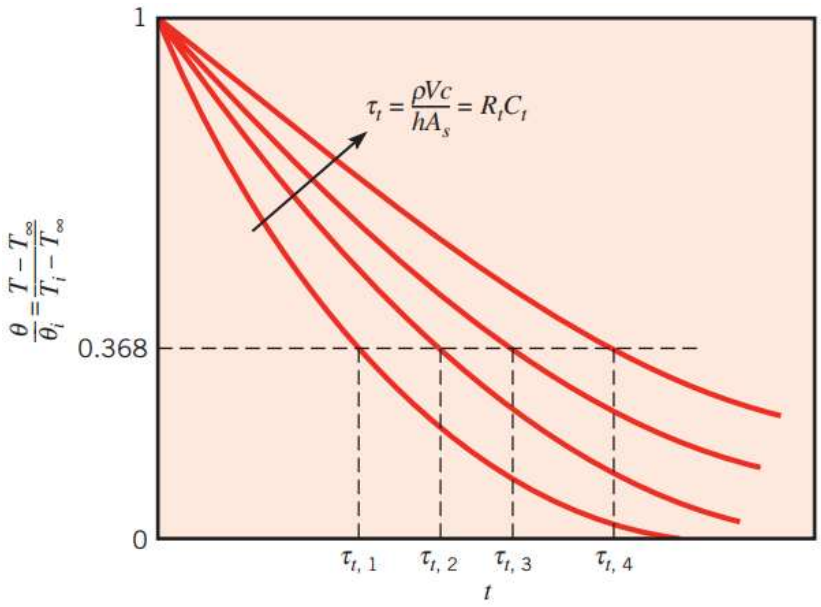


Figure 5.3: Transient temperature response of lumped capacitance solids for different thermal time constants *t.*

To determine the total energy transfer Q occurring up to sometime t, we simply write

$$Q = \int_0^t q\, dt = hA_s \int_0^t \theta\, dt \qquad (5\text{-}9)$$

Substituting for θ from Equation 5.7 and integrating, we obtain

$$Q = (\rho VC)\theta_i \left[1 - e^{\left[-\left(\frac{hA_S}{\rho VC}\right)t\right]}\right] = (\rho VC)\theta_i \left[1 - e^{\left[\frac{-t}{\tau_t}\right]}\right] \qquad (5\text{-}10)$$

The quantity Q is, of course, related to the change in the internal energy of the solid.

$$-Q = \Delta E_{st} \qquad (5\text{-}11)$$

For quenching, Q is positive and the solid experiences a decrease in energy. Equations 5.6, 5.7, and 5.10 also apply to situations where the solid is heated ($\theta < 0$), in which case Q is negative and the internal energy of the solid increases.

5.3 Validity of the Lumped Capacitance Method

From the foregoing results it is easy to see why there is a strong preference for using the lumped capacitance method. It is certainly the simplest and most convenient method that can be used to solve transient heating and cooling problems. Hence it is important to determine under what conditions it may be used with reasonable accuracy.

To develop a suitable criterion considers steady-state conduction through the plane wall of area A (Figure 5.4). Although we are assuming steady-state conditions, the following criterion is readily extended to transient processes. One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence under steady-state conditions the surface energy balance reduces to

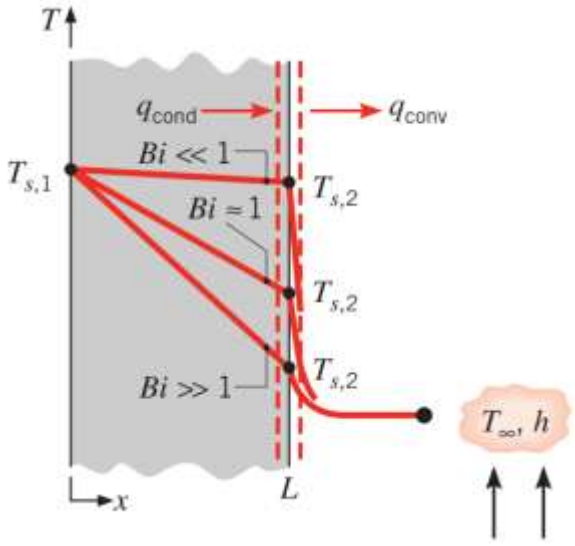


Figure 5.4: Effect of *Biot number* on steady-state temperature distribution in a plane wall with surface convection.

$$\frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = hA_s\left(T_{s,2} - T_\infty\right) \quad (5\text{-}12)$$

where k is the thermal conductivity of the solid. Rearranging, we then obtain

$$\frac{T_{s,1}-T_{s,2}}{T_{s,2}-T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,cond.}}{R_{t,conv.}} = \frac{hL}{k} = Bi \quad (5\text{-}13)$$

The quantity (hL/k) appearing in Equation 5.13 is a dimensionless parameter. It is termed the *Biot number*, and it plays a fundamental role in conduction problems that involve surface convection effects. According to Equation 5.13 and as illustrated in Figure 5.4, the *Biot number* provides a measure of the temperature drop in the solid relative to the temperature difference between the solid's surface and the fluid. From Equation 5.13, it is also evident that the *Biot number* may be interpreted as a ratio of thermal resistances. In particular, if $Bi << 1$, *the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer*. Hence, *the assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small*.

Although we have discussed the *Biot number* in the context of steady-state conditions, we are reconsidering this parameter because of its significance to transient conduction problems. Consider the plane wall of Figure 5.5, which is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$. The problem may be treated as one-dimensional in x, and we are interested in the temperature variation with position and time, $T(x, t)$. This variation is a strong function of the *Biot number*, and three conditions are shown in Figure 5.5. Again, for $Bi << 1$ the temperature gradients in the solid are small and the assumption of a uniform temperature distribution, $T(x, t) \approx T(t)$ is reasonable. Virtually all the temperature difference is between the solid and the fluid, and the solid temperature remains nearly uniform as it decreases to T_∞. For moderate to large values of the *Biot number*, however, the temperature gradients within the solid are significant. Hence $T = T(x, t)$. Note that for $Bi >> 1$, the

temperature difference across the solid is much larger than that between the surface and the fluid.

We conclude this section by emphasizing the importance of the lumped capacitance method. Its inherent simplicity renders it the preferred method for solving transient heating and cooling problems. Hence, *when confronted with such a problem, the very first thing that one should do is calculate the Biot number*. If the following condition is satisfied

$$\frac{hL_c}{k} = Bi < 0.1 \qquad (5\text{-}14)$$

The error associated with using the lumped capacitance method is small. For convenience, it is customary to define the *characteristic length* of Equation 5.14 as the ratio of the solid's

volume to surface area $L_c = V/A_s$. Such a definition facilitates calculation of L_c for solids of complicated shape and reduces to the half-thickness L for a plane wall of thickness $2L$ (Figure 5.5), to $r_o/2$ for a long cylinder, and to $r_o/3$ for a sphere. However, if one wishes to implement the criterion in a conservative fashion, L_c should be associated with the length scale corresponding to the maximum spatial temperature difference. Accordingly, for a symmetrically heated (or cooled) plane wall of thickness $2L$, L_c would remain equal to the half-thickness L. However, for a long cylinder or sphere, L_c would equal the actual radius r_o, rather than $r_o/2$ or $r_o/3$.

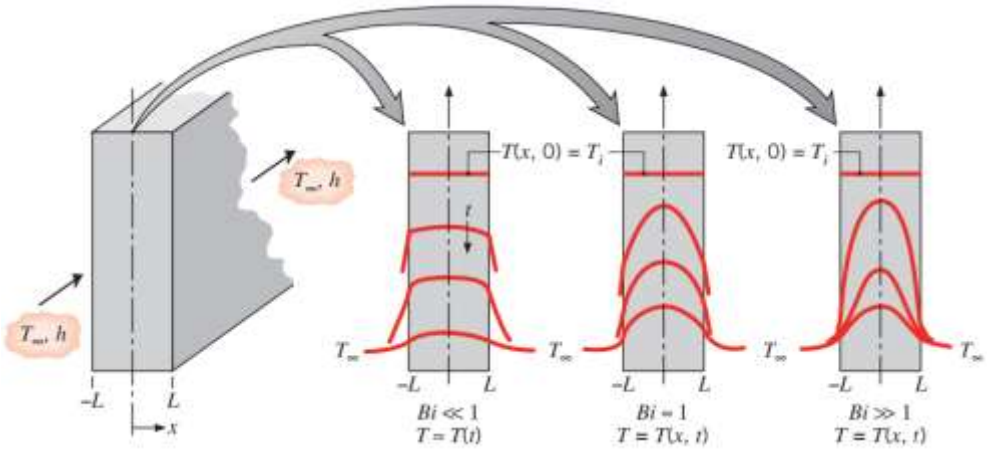


Figure 5.5: Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

Finally, we note that, with $L_c = V/A_s$, the exponent of Equation 5.7 may be expressed as

$$\frac{hA_s\,t}{\rho VC} = \frac{ht}{\rho CL_c} = \frac{hL_c}{k}\frac{k}{\rho C}\frac{t}{{L_c}^2} = \frac{hL_c}{k}\frac{\alpha t}{{L_c}^2} = Bi \times Fo \qquad (5\text{-}15)$$

Example: A steel ball bearing (diameter = 25 mm, ρ_{Steel} = 7833 kg/m^3 and C_{Steel} = 0.465 kJ/kg.°C) is heated in a furnace to a temperature of 750 °C. It is then removed to be quenched in water at 25 °C. If the time required for transferring the ball bearing between the furnace and the water is 8 sec through the atmospheric air environment (air temperature 20 °C). **Determine** by using lumped system analysis, the time it takes the ball bearing to cool to 200 °C. Taking, convection heat transfer coefficients in the air and water are 30 W/m^2.°C and 3000 W/m^2.°C, respectively.

Solution:

$$\frac{T_{(t)}-T_\infty}{T_i-T_\infty} = e^{\frac{-hA_S}{\rho VC}\times t}$$

$$\frac{V}{A} = L_c = \frac{r_o}{3} = \frac{25\times10^{-3}}{2\times3} = 4.2\times10^{-3}\ (m)$$

$$\frac{T_{(8)}-20}{750-20} = e^{\left[\frac{-30\times8}{7833\times0.465\times10^3\times4.2\times10^{-3}}\right]} = 0.98443$$

$$T_{(8)} = 738.6\ °C$$

$$\frac{200-25}{738.6-25} = e^{\left[\frac{-3000\times t}{7833\times0.465\times10^3\times4.2\times10^{-3}}\right]}$$

$$-1.4055 = -0.19611\times t$$

$$t = 7.167\ (sec.)$$

5.4 Distributed Systems Having Stepwise Disturbances

5.4.1 Cartesian Geometry:

For distributed systems having stepwise disturbances, no mathematic beyond that introduced in Chapter 4 is needed. Difficulties arising from nonhomogeneous boundary conditions or differential equations may be eliminated, as with steady problems, by introducing a *change in temperature level* or by *superposition*. Some remarks no non-homogeneities may, however, be helpful. In this connection, let us consider the following four problems.

Therefore, with no *internal generation* and the assumption of *constant thermal conductivity*, the energy equation can then be reduced to,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5\text{-}16)$$

To solve Equation 5.16 for the temperature distribution $T(x, t)$, it is necessary to specify an *initial condition* and *two boundary conditions*. For the typical transient conduction problem of Figure 5.5, the initial condition is

$$T_{(x,0)} = T_i$$

and the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

and,

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T_{(L,t)} - T_\infty\right]$$

($T_{(x,0)} = T_i$) presumes a uniform temperature distribution at time $t = 0$; ($\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$) reflects the symmetry requirement for the midplane of the wall; and ($-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T_{(L,t)} - T_\infty\right]$) describes the surface condition experienced for time $t > 0$. From Equations 5.16 through initial and boundary conditions above, it is

evident that, in addition to depending on x and t, temperatures in the wall also depend on a number of physical parameters. In particular

$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$

Example 1: A plate of thickness $2L$ having the uniform initial temperature T_o is plunged suddenly into a bath at the constant temperature T_∞ as shown in Figure 5.6. The heat transfer coefficient is large. Find the unsteady temperature of the plate.

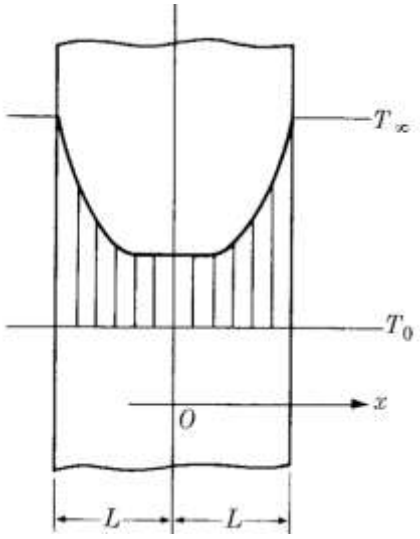


Figure 5.6

Solution:

In terms of ($\theta = T - T_\infty$) and the x-axis of Figure 5.6, the formulation of the problem is,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta_{(x,0)} = \theta_i = T_i - T_\infty$$

$$\left.\frac{\partial\theta_{(0,t)}}{\partial x}\right|_{x=0} = 0$$

$$\theta_{(L,t)} = 0$$

Assume the existence of a product solution of the form (Eq. 5.16)

$$\theta_{(x,t)} = X_{(x)}\,\tau_{(t)}$$

Note that only the x-axis yields a characteristic-value problem; then, with the proper choice of separation constant, the product solution ($\theta_{(x,t)} = X_{(x)}\,\tau_{(t)}$) gives,

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \qquad \rightarrow \qquad \left.\frac{\partial X_{(0)}}{\partial x}\right|_{x=0} = 0 \text{ and } X_{(L)} = 0$$

$$X = C_1 \cos\lambda x + C_2 \sin\lambda x$$

$$X' = -\lambda\, C_1 \sin\lambda x + \lambda\, C_2 \cos\lambda x$$

From B.C. $\left[\left.\frac{\partial X_{(0)}}{\partial x}\right|_{x=0} = 0\right] \rightarrow 0 = -\lambda\, C_1 \sin(\lambda \times 0) + \lambda\, C_2 \cos(\lambda \times 0) \rightarrow C_2 = 0$

From B.C. $[X_{(L)} = 0] \rightarrow 0 = C_1 \cos \lambda L \rightarrow 0 = \cos \lambda L \rightarrow$

$\lambda_n L = (2n+1) \times \frac{\pi}{2}$ $n = 0, 1, 2, 3, \ldots$ (Characteristic values)

$\lambda_n = (2n+1) \times \frac{\pi}{2L}$

$X = A_n \cos \lambda_n x$

$\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0 \rightarrow \frac{d\tau}{\tau} = -\alpha\lambda^2 dt \rightarrow \tau(t) = C_3 e^{-\alpha\lambda^2 t} \rightarrow \tau_n(t) = B_n e^{-\alpha\lambda_n^2 t}$

Therefore, the product solution becomes, $\theta_{(x,t)} = \sum_{n=0}^{\infty} A_n \cos \lambda_n x \; B_n e^{-\alpha\lambda_n^2 t}$

Let $C_n = A_n \times B_n \rightarrow \theta_{(x,t)} = \sum_{n=0}^{\infty} C_n \cos \lambda_n x \; e^{-\alpha\lambda_n^2 t}$

Finally, introducing the initial condition $[\theta_{(x,0)} = \theta_i]$ into transient temperature distribution $(\theta_{(x,t)})$ gives

$\theta_i = \sum_{n=0}^{\infty} C_n \cos \lambda_n x \; e^{-\alpha\lambda_n^2 (0)} \rightarrow \theta_i = \sum_{n=0}^{\infty} C_n \cos \lambda_n x$

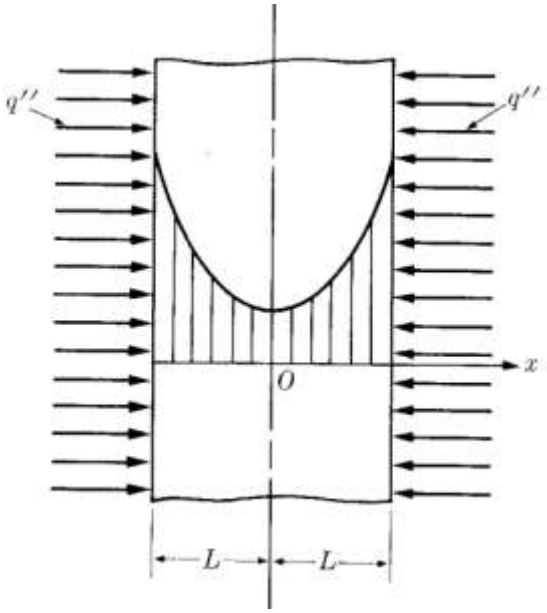
The above equation is the *Fourier cosine series* expansion of θ_i over the interval (0, L). The coefficient (C_n) may be evaluated in the usual manner. The result is

$C_n = (-1)^n \frac{2\theta_i}{\lambda_n L}$

Thus, the unsteady temperature of the plate to be as,

$\frac{\theta_{(x,t)}}{\theta_i} = \frac{T_{(x,t)} - T_\infty}{T_i - T_\infty} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n L} e^{-\alpha\lambda_n^2 t} \cos \lambda_n x$

Example 2: The constant heat flux (q'') is applied to both surfaces of a flat plate of thickness 2L as shown in Figure 5.7.The initial temperature of the plate is T_∞. Find the unsteady temperature of the plate.



Solution:

In terms of ($\theta = T - T_\infty$) and the x-axis of Figure 5.7, the formulation of the problem is,

$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial\theta}{\partial t}$

$\theta_{(x,0)} = 0$

Figure 5.7

$\left.\frac{\partial \theta_{(0,t)}}{\partial x}\right|_{x=0} = 0$

$k\frac{\partial \theta_{(L,t)}}{\partial x} = q"$

Let us try to solve the problem by the assumption of below equation as,

$\theta_{(x,t)} = \Psi_{(x,t)} + \Phi_{(x)}$

The separation constant that is selected to make the *x*-axis of $\Psi_{(x,t)}$ a *characteristic-value* problem forces $\Psi_{(x,t)}$ to be an exponentially decaying function in time, hence in the limit $\Psi_{(x,t)} \to 0$ and $\theta_{(x,t)} \to \Phi_{(x)}$ as $t \to \infty$. This result violates the physics of the problem, the temperature of the plate should increase without limit as $t \to \infty$. To satisfy this condition, we modify the above equation by the addition of the term $\varphi_{(t)}$ such that $\varphi_{(t)} \to \infty$ as $t \to \infty$. Thus the proper assumption is,

$\theta_{(x,t)} = \Psi_{(x,t)} + \Phi_{(x)} + \varphi_{(t)}$

Now in terms of above equation, the formulation of the problem becomes

$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Psi}{\partial t}$, $\quad \Psi_{(x,0)} = -\Phi_{(x)} - \varphi_{(0)}$, $\quad \left.\frac{\partial \Psi_{(0,t)}}{\partial x}\right|_{x=0} = 0$, $\quad \left.\frac{\partial \Psi_{(L,t)}}{\partial x}\right|_{x=L} = 0$

$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}$, $\quad \left.\frac{d\Phi_{(0)}}{dx}\right|_{x=0} = 0$, $\quad +k\frac{d\Phi_{(L)}}{dx} = q"$

Since $\varphi_{(t)}$ and $\Phi_{(x)}$ can vary independently, ($\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}$) holds when it is equal to a constant, say *C*. Then the general solution of ($\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}$) is obtained in the form

$\varphi_{(t)} = \alpha C t + C_1$

$\Phi_{(x)} = \frac{1}{2}Cx^2 + C_2 x + C_3$

Here C_2 and C may readily be evaluated by introducing ($\Phi_{(x)}$) equation into ($+k\frac{d\Phi_{(L)}}{dx} = q"$) equation. The result is $C_2 = 0$, $C = q"/kL$. Hence, $\varphi_{(t)} =$

$q"t/\rho CL + C_1$ and $\Phi_{(x)} = q"x^2/2kL + C_3$, where C_1 and C_3 are the remaining constants. However, noting that the solution of $\Psi_{(x,t)}$ depends on $\varphi_{(t)}$ and $\Phi_{(x)}$, may arbitrarily set these constants equal to *zero*. Thus,

$\varphi_{(t)} = \frac{q"t}{\rho CL}$ and $\Phi_{(x)} = \frac{q"x^2}{2kL}$

On the other hand, the product solution $\Psi_{(x,t)} = X_{(x)}\tau_{(t)}$ applied to $(\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t})$ results in

$\frac{d^2X}{dx^2} + \lambda^2 X = 0$, $\frac{dX_{(0)}}{dx} = 0$, $\frac{dX_{(L)}}{dx} = 0$

The solution of above equation is,

$X_n(x) = A_n \cos\lambda_n x$ Characteristic functions

$\lambda_n = \frac{n\pi}{L}$ where, n= 0, 1, 2, 3, ….. Characteristic values

$\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$

The solution of above equation is,

$\tau_n = C_n e^{-\alpha\lambda_n^2 t}$

Thus the product solution of $\Psi_{(x,t)}$ yields

$\Psi_{(x,t)} = a_o + \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$

Where, $a_o = A_o C_o$ and $a_n = A_n C_n$

Finally, the initial value of above equation, which is equal to $-\Phi_{(x)}$, gives

$-\frac{q"x^2}{2kL} = a_o + \sum_{n=1}^{\infty} a_n \cos\lambda_n x$

The coefficients a_o and a_n are,

$a_o = -\frac{q"L}{6k}$ and $a_n = -(-1)^n \frac{2q"L}{k(\lambda_n L)^2}$

Therefore, the unsteady temperature disturibution of the plate is,

$\frac{\theta_{(x,t)}}{q"L/k} = \frac{\alpha t}{L^2} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(\lambda_n L)^2} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$

5.4.2 Cylindrical Geometry

Example 3: An infinitely long rod of radius R having the uniform initial temperature T_o is plunged suddenly into a bath at temperature T_∞ as shown in Figure 5.8. The heat transfer coefficient is large. Find the expression of unsteady temperature disturibution of the rod.

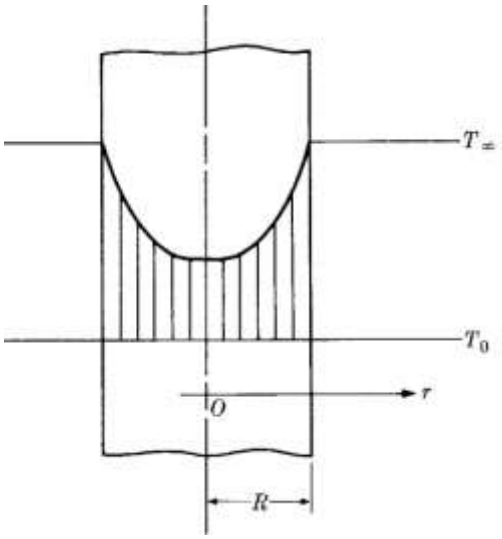


Figure 5.8

Solution:

In terms of ($\theta = T - T_\infty$) and the r-axis of Figure 5.8, the formulation of the problem is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta_{(r,0)} = \theta_o = T_o - T_\infty$$

$$\theta_{(R,t)} = 0$$

$$\left.\frac{\partial\theta_{(0,t)}}{\partial r}\right|_{r=0} = 0$$

The product solution $\theta_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ introduced into formulation equation results in

$$\frac{d}{dr}\left(r\,\frac{d\mathcal{R}}{dr}\right) + \lambda^2 r\mathcal{R} = 0 \;; \qquad \mathcal{R}_{(R)} = 0 \;, \qquad \left.\frac{d\mathcal{R}_{(0)}}{dr}\right|_{r=0} = 0$$

The solution of above equation is,

$\mathcal{R}_{n_{(r)}} = A_n J_o(\lambda_n r)$ Characteristic functions

And the zeros of ($J_o(\lambda_n R) = 0$) are the characteristic values. The solution of equation ($\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$) is,

$$\tau_{n_{(t)}} = C_n e^{-\alpha\lambda_n^2 t}$$

Hence the product solution becomes

$$\theta_{(r,t)} = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t} J_o(\lambda_n r)$$

Where , $a_n = A_n C_n$

The initial value of above equation is,

$\theta_o = \sum_{n=1}^{\infty} a_n J_o(\lambda_n r)$

The above equation is the *Fourier-Bessel series* expansion of θ_o. Here the coefficient a_n may be calculated in the usual manner. The result is,

$a_n = \frac{2\theta_o}{(\lambda_n R)\ J_1(\lambda_n R)}$

Thus, introducing the a_n equation into $\theta_{(r,t)}$ equation, the unsteady temperature disturibution of the rod is,

$\theta_{(r,t)} = \frac{T_{(r,t)} - T_\infty}{T_o - T_\infty} = 2\sum_{n=1}^{\infty} \frac{e^{-\alpha\lambda_n^2 t} J_o(\lambda_n r)}{(\lambda_n R)\ J_1(\lambda_n R)}$

5.4.3 Spherical Geometry

Example 4: A solid sphere of radius *R* having the uniform initial temperature T_o is plunged suddenly into a bath at temperature T_∞ as shown in Figure 5.9. The heat transfer coefficie temperature disturibution of th

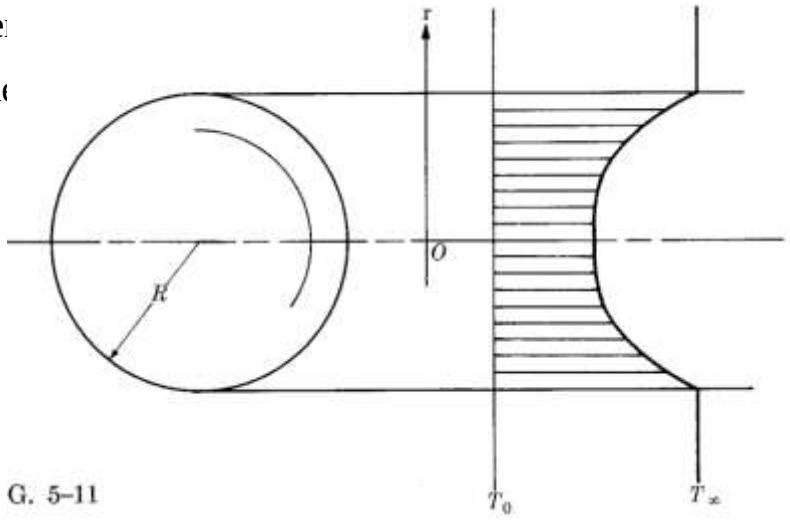


Figure 5.9

Solution:

The formulation of the problem in terms of ($\theta = T - T_\infty$) is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta_{(r,0)} = \theta_o = T_o - T_\infty$$

$$\theta_{(R,t)} = 0$$

$$\left.\frac{\partial\theta_{(0,t)}}{\partial r}\right|_{r=0} = 0 \quad \text{or} \quad \theta_{(0,t)} = finite$$

The product solution $\theta_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ introduced into formulation equation yields

$$\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) + \lambda^2 r^2\mathcal{R} = 0 \; ; \qquad \mathcal{R}_{(R)} = 0 \; , \qquad \left.\frac{d\mathcal{R}_{(0)}}{dr}\right|_{r=0} = 0$$

Particular solutions corresponding to the differential equation of above equation are,

$$J_{1/2}(\lambda r)/r^{1/2} \qquad \text{and} \qquad J_{-1/2}(\lambda r)/r^{1/2}$$

Furthermore, noting from *Bessel series* properties that

$$J_{1/2}(\lambda r) \sim \sin\lambda r/r^{1/2} \qquad \text{and} \qquad J_{-1/2}(\lambda r) \sim \cos\lambda r/r^{1/2}$$

The solutions of above equations may be rearranged to give,

$$\sin\lambda r/r \qquad \text{and} \qquad \cos\lambda r/r$$

This result explains the use of the well-known transformation

$$\theta_{(r,t)} = \Psi_{(r,t)}/r$$

$$\frac{\partial^2\Psi}{\partial r^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t}$$

$$\Psi_{(r,0)} = r\theta_o = r(T_o - T_\infty)$$

$$\Psi_{(R,t)} = 0$$

$$\Psi_{(0,t)} = 0$$

Hence, the problem is reduced to a problem of Cartesian geometry.

The product solution $\Psi_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ applied to ($\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t}$) yields

$\frac{d^2\mathcal{R}}{dr^2} + \lambda^2\mathcal{R} = 0$, $\mathcal{R}_{(0)} = 0$ and $\mathcal{R}_{(R)} = 0$

The solution of above equation is,

$\mathcal{R}_n(r) = A_n \sin \lambda_n r$ Characteristic functions

$\lambda_n = \frac{n\pi}{R}$ where, *n*= 0, 1, 2, 3, ….. Characteristic values

The solution of equation ($\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$) is,

$\tau_{n_{(t)}} = C_n e^{-\alpha\lambda_n^2 t}$

Hence the product solution becomes

$\Psi_{(r,t)} = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t} \sin(\lambda_n r)$

Where , $a_n = A_n C_n$

$r\theta_o = \sum_{n=1}^{\infty} a_n \sin(\lambda_n r)$

The coefficient a_n is

$a_n = (-1)^{n+1} \frac{2\theta_o}{\lambda_n}$

Finally, the unsteady temperature disturibution of the sphere is found to be,

$\theta_{(r,t)} = \frac{T_{(r,t)} - T_\infty}{T_o - T_\infty} = 2\sum_{n=1}^{\infty}(-1)^{n+1} e^{-\alpha\lambda_n^2 t} \frac{\sin \lambda_n r}{\lambda_n r}$

5.5 The Numerical Method of Transient Heat Conduction

Analytical solutions to transient problems are restricted to simple geometries and boundary conditions, such as the one-dimensional cases considered in the preceding sections. For some simple *two-* and *three-dimensional* geometries, analytical solutions are still possible. However, in many cases the geometry and/or boundary conditions preclude the use of analytical techniques, and recourse must be made to *finite-difference* (or *finite-element*) methods.

5.1.1 Discretization of the Heat Equation:

✓ **The Explicit Method**

Once again consider the two-dimensional system of Figure 5.10. Under transient conditions with constant properties and no internal generation, the appropriate form of the heat equation, Equation 1-18-b, is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5\text{-}17)$$

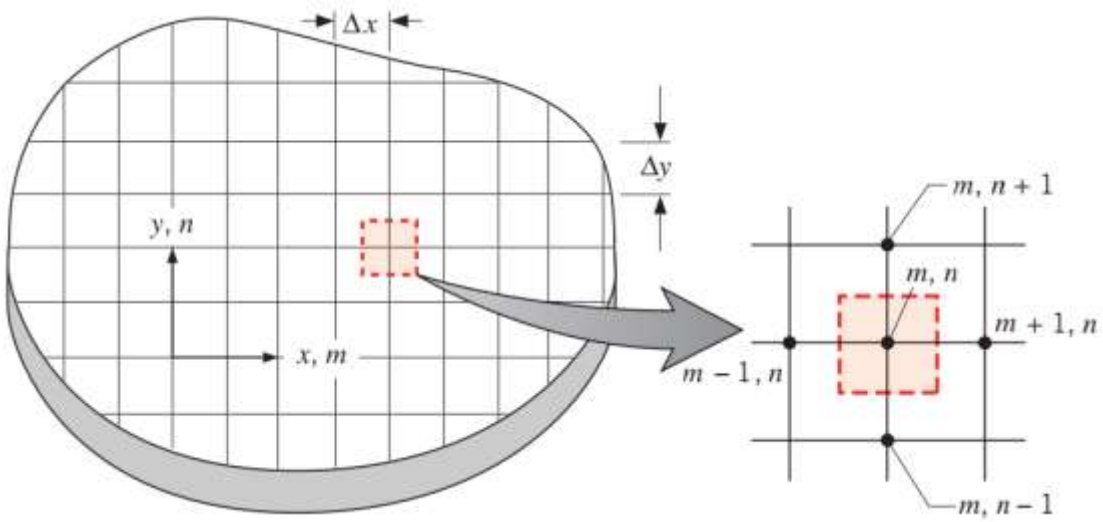


Figure 5.10: Two-dimensional conduction, Nodal network.

To obtain the *finite-difference* form of this equation, we may use the *central-difference* approximations to the spatial derivatives prescribed as,

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n}-T_{m,n}}{\Delta x} \qquad \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n}-T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n}}{\Delta x}$$

Therefore,

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2} \qquad (5\text{-}18)$$

Proceeding in a similar fashion, it is readily shown that

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} = \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} \qquad (5\text{-}19)$$

Once again the *m* and *n* subscripts may be used to designate the *x*- and *y*-locations of discrete nodal points. However, in addition to being discretized in space, the problem must be discretized in time. The integer *p* is introduced for this purpose, where, $t = p\,\Delta t$

and the *finite-difference* approximation to the time derivative in Equation 5.17 is expressed as

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} = \frac{T^{p+1}{}_{m,n}-T^{p}{}_{m,n}}{\Delta t} \qquad (5\text{-}20)$$

The superscript *p* is used to denote the time dependence of *T*, and the time derivative is expressed in terms of the difference in temperatures associated with the *new* (*p*+1) and *previous* (or *old*) (*p*) times. Hence calculations must be performed at successive times separated by the interval Δt, and just as a *finite-difference* solution restricts temperature determination to discrete points in space, it also restricts it to discrete points in time.

If Equations 5.18, 5.19 and 5.20 is substituted into Equation 5.17, the nature of the *finite-difference* solution will depend on the specific time at which temperatures are evaluated in the *finite-difference* approximations to the spatial derivatives. In the explicit method of solution, these temperatures are evaluated at the *previous* (*p*)

time. Hence Equation 5.20 is considered to be a *forward-difference* approximation to the time derivative. Evaluating terms on the *right-hand* side of Equations 5.17 at *p* and substituting into Equation 5.17, the explicit form of the *finite-difference* equation for the interior node (*m*, *n*) is

$$\frac{T^P{}_{m+1,n}+T^P{}_{m-1,n}-2T^P{}_{m,n}}{(\Delta x)^2}+\frac{T^P{}_{m,n+1}+T^P{}_{m,n-1}-2T^P{}_{m,n}}{(\Delta y)^2}=\frac{1}{\alpha}\frac{T^{p+1}{}_{m,n}-T^p{}_{m,n}}{\Delta t} \qquad (5\text{-}21)$$

Solving for the nodal temperature at the *new* (*p*+1) time and assuming that $\Delta x = \Delta y$, it follows that

$$T^{p+1}{}_{m,n}=Fo\left(T^P{}_{m+1,n}+T^P{}_{m-1,n}+T^P{}_{m,n+1}+T^P{}_{m,n-1}\right)+(1-4Fo)T^p{}_{m,n} \qquad (5\text{-}22)$$

where *Fo* is a *finite-difference* form of the *Fourier number*, which is the ratio of diffusive or conductive transport rate to the quantity storage rate.

$$Fo=\frac{\alpha\,\Delta t}{(\Delta x)^2} \qquad (5\text{-}23)$$

This approach can easily be extended to *one-* or *three-dimensional* systems. If the system is *one-dimensional* in *x*, the explicit form of the *finite-difference* equation for an interior node *m* reduces to

$$T^{p+1}{}_m=Fo\,(T^P{}_{m+1}+T^P{}_{m-1})+(1-2Fo)T^p{}_m \qquad (5\text{-}24)$$

Equations 5.22 and 5.24 are *explicit* because *unknown* nodal temperatures for the new time are determined exclusively by *known* nodal temperatures at the previous time. Hence calculation of the unknown temperatures is straightforward. Since the temperature of each interior node is known at $t = 0$ ($p = 0$) from prescribed initial conditions, the calculations begin at $t = \Delta t$ ($p = 1$), where Equation 5.22 or 5.24 is applied to each interior node to determine its temperature. With

With temperatures known for $t = \Delta t$, the appropriate finite-difference equation is then applied at each node to determine its temperature at $t = 2\Delta t$ ($p = 2$). In this way,

the transient temperature distribution is obtained by *marching out in time*, using intervals of Δt.

The accuracy of the *finite-difference* solution may be improved by decreasing the values of Δx and Δt. Of course, the number of interior nodal points that must be considered increases with decreasing Δx, and the number of time intervals required to carry the solution to a prescribed final time increases with decreasing Δt. Hence the computation time increases with decreasing Δx and Δt. The choice of Δx is typically based on a compromise between accuracy and computational requirements. Once this selection has been made, however, the value of Δt may not be chosen independently. It is, instead, determined by stability requirements.

An undesirable feature of the *explicit* method is that it is not unconditionally *stable*. In a transient problem, the solution for the nodal temperatures should continuously approach final (steady-state) values with increasing time. However, with the explicit method, this solution may be characterized by numerically induced oscillations, which are physically impossible. The oscillations may become *unstable*, causing the solution to diverge from the actual steady-state conditions. To prevent such erroneous results, the prescribed value of Δt must be maintained below a certain limit, which depends on Δx and other parameters of the system. This dependence is termed a stability criterion, which may be obtained mathematically or demonstrated from a thermodynamic argument. For the problems of interest in this text, *the criterion is determined by requiring that the coefficient associated with the node of interest at the previous time is greater than or equal to zero*.

In general, this is done by collecting all terms involving $T^{p}{}_{m,n}$ to obtain the form of the coefficient. This result is then used to obtain a limiting relation involving *Fo*, from which the maximum allowable value of Δt may be determined. For

example, with Equations 5.22 and 5.24 already expressed in the desired form, it follows that the stability criterion for a *one-dimensional* interior node is $(1 - 2Fo) \geq 0$, or $Fo \leq \frac{1}{2}$ and for a *two-dimensional* node, it is $(1 - 4Fo) \geq 0$, or $o \leq \frac{1}{4}$.

For prescribed values of Δx and α, these criteria may be used to determine upper limits to the value of Δt.

Equations 5.22 and 5.24 may also be derived by applying the *energy balance* method to a control volume about the interior node. Accounting for changes in thermal energy storage, a general form of the *energy balance* equation may be expressed as

$$E_{in} + E_g = E_{st} \qquad (5\text{-}25)$$

In the interest of adopting a consistent methodology, it is again assumed that all heat flow is *into* the node.

Example 5: Consider the surface node of the *one-dimensional* system shown in Figure 5.11. To more accurately determine thermal conditions near the surface, this node has been assigned a thickness that is one-half that of the interior nodes. Assuming convection transfer from an adjoining fluid and no generation, it follows from Equation 5.25 that

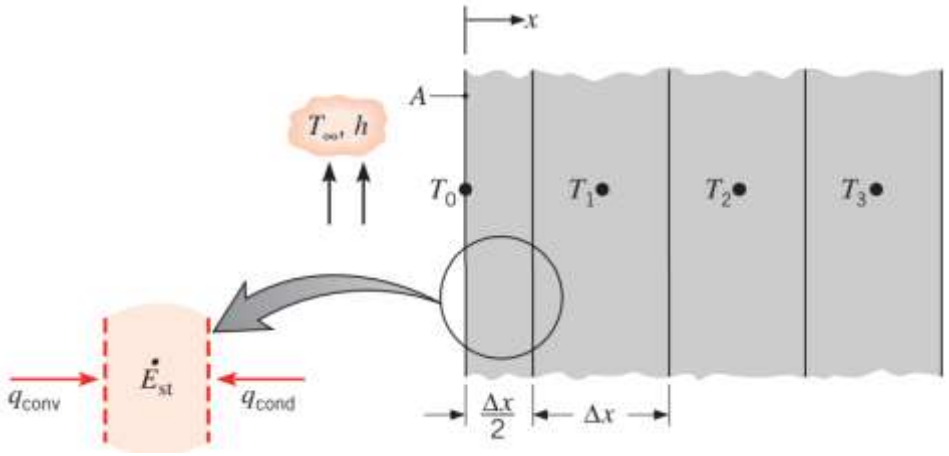


Figure 5.11: Surface node with convection and one-dimensional transient conduction.

$$hA\left(T_\infty - T_o^p\right) + \frac{kA}{\Delta x}\left(T_1^p - T_o^p\right) = \rho CA\frac{\Delta x}{2}\frac{\left(T_o^{p+1} - T_o^p\right)}{\Delta t} \quad (5\text{-}26)$$

or, solving for the surface temperature at $t + \Delta t$,

$$T_o^{p+1} = \frac{2h\Delta t}{\rho C\Delta x}\left(T_\infty - T_o^p\right) + \frac{2\alpha\Delta x}{\Delta x^2}\left(T_1^p - T_o^p\right) + T_o^p \quad (5\text{-}27)$$

Recognizing that $(2h\Delta t/\rho C\Delta x) = 2(h\Delta x/k)(\alpha\Delta t/\Delta x^2) = 2BiFo$ and grouping terms involving T_o^p, it follows that

$$T_o^{p+1} = 2Fo\left(T_1^p - BiT_\infty\right) + (1 - 2Fo - 2BiFo)T_o^p \quad (5\text{-}28)$$

The *finite-difference* form of the *Biot number* is

$$Bi = \frac{h\Delta x}{k}$$

Recalling the procedure for determining the stability criterion, we require that the coefficient for be greater than or equal to zero. Hence

$$1 - 2Fo - 2BiFo \geq 0 \quad (5\text{-}29)$$

Or, $Fo(1 + Bi) \leq \frac{1}{2}$

Since the complete finite-difference solution requires the use of Equation 5.24 for the interior nodes, as well as Equation 5.28 for the surface node, Equation 5.29 must be contrasted with ($Fo \leq \frac{1}{2}$) to determine which requirement is more stringent. Since $Bi \geq 0$, it is apparent that the limiting value of *Fo* for Equation 5.29 is less than that for Equation 5.82. To ensure stability for all nodes, Equation 5.29 should therefore be used to select the maximum allowable value of *Fo*, and hence Δt, to be used in the calculations.

Example 6: A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L$= 20 mm as shown in Figure 5.12 and is convectively cooled at both surfaces, with h= 1100 W/m^{2}K and T_∞= 250 °C. At normal operating power, heat is generated uniformly within the element at a volumetric rate of q_1= 10^7 W/m^3. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to q_2= 2×10^7 W/m^3, and use the *explicit finite-difference* method to determine the fuel element temperature distribution after 1.5 s. The fuel element thermal properties are k= 30 W/mK and α= 5×10^{-6} m^2/s.

Solution:

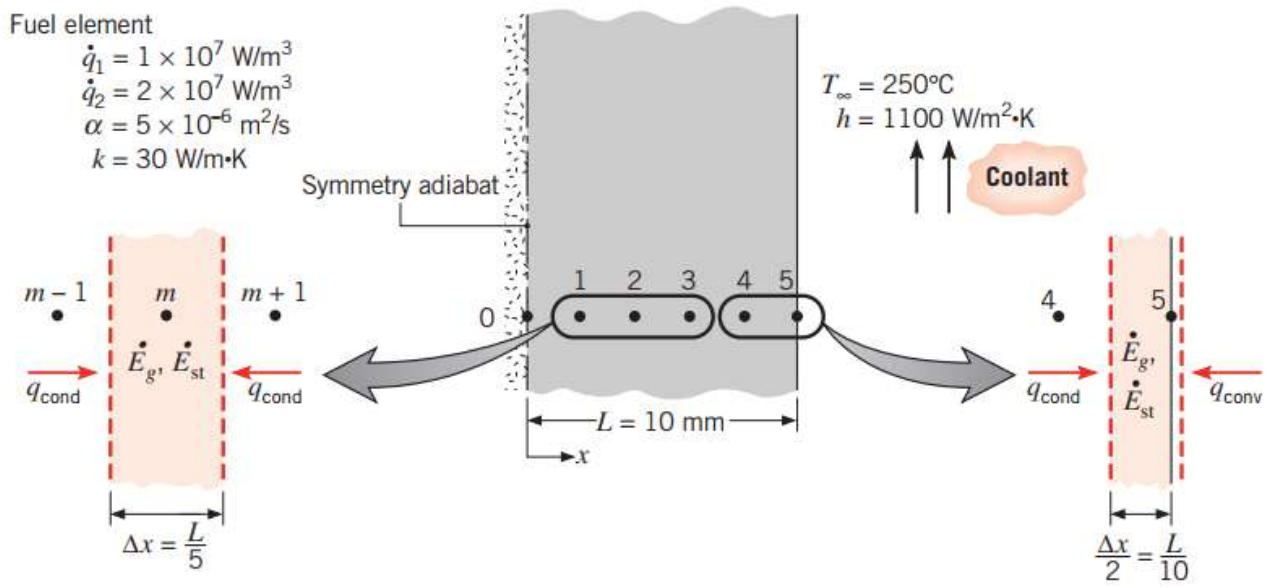


Figure 5.12: A rectangular fuel element with surface cooling.

A numerical solution will be obtained using a space increment of Δx= 2 mm. Since there is symmetry about the midplane, the nodal network yields six unknown nodal temperatures. Using the *energy balance* method, Equation 5.25, an *explicit finite-difference* equation may be derived for any interior node m.

$$kA\frac{T_{m-1}^{p}-T_{m}^{p}}{\Delta x}+kA\frac{T_{m+1}^{p}-T_{m}^{p}}{\Delta x}+qA\,\Delta x=\rho A\,\Delta x\,c\frac{T_{m}^{p+1}-T_{m}^{p}}{\Delta t}$$

Solving for T_m^{p+1} and rearranging,

$$T_m^{p+1}=Fo\left[T_{m-1}^{p}+T_{m+1}^{p}+\frac{q(\Delta x)^2}{k}\right]+(1-2Fo)T_m^{p} \qquad (1)$$

This equation may be used for node 0, with $T_{m-1}^{p}=T_{m+1}^{p}$, as well as for nodes 1, 2, 3, and 4. Applying energy conservation to a control volume about node 5,

$$hA(T_\infty-T_5^{p})+kA\frac{T_4^{p}-T_5^{p}}{\Delta x}+qA\frac{\Delta x}{2}=\rho A\frac{\Delta x}{2}c\frac{T_5^{p+1}-T_5^{p}}{\Delta t}$$

$$T_5^{p+1}=2Fo\left[T_4^{p}+Bi\,T_\infty+\frac{q(\Delta x)^2}{2k}\right]+(1-2Fo-2Bi\,Fo)T_5^{p} \qquad (2)$$

Since the most restrictive stability criterion is associated with Equation 2, we select *Fo* from the requirement that

$$Fo(1+Bi)\le\frac{1}{2}$$

$$Bi=\frac{h\,\Delta x}{k}=\frac{1100\ \text{W/m}^2\cdot\text{K}\ (0.002\ \text{m})}{30\ \text{W/m}\cdot\text{K}}=0.0733 \qquad Fo\le 0.466$$

$$\Delta t=\frac{Fo(\Delta x)^2}{\alpha}\le\frac{0.466(2\times10^{-3}\ \text{m})^2}{5\times10^{-6}\ \text{m}^2/\text{s}}\le 0.373\ \text{s}$$

To be well within the stability limit, we select Δt= 0.3 s, which corresponds to

$$Fo=\frac{5\times10^{-6}\ \text{m}^2/\text{s}(0.3\ \text{s})}{(2\times10^{-3}\ \text{m})^2}=0.375$$

Substituting numerical values, including q_2= 2×10^7 W/m^3, the nodal equations become

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$

$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_1^p$$

$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_2^p$$

$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_3^p$$

$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_4^p$$

$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.195T_5^p$$

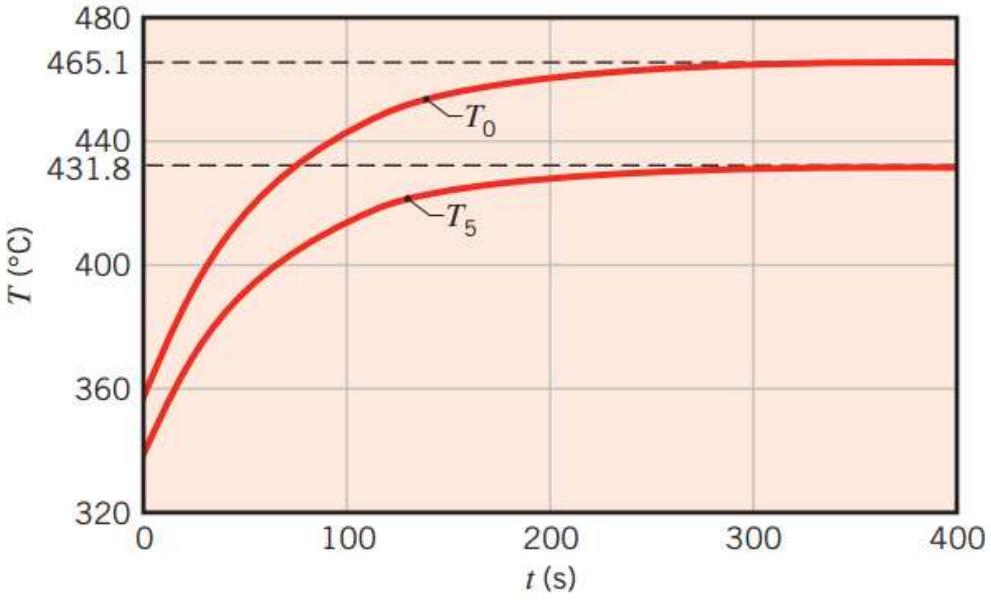
$$T_5 = T_\infty + \frac{qL}{h} = 250°\text{C} + \frac{10^7\ \text{W/m}^3 \times 0.01\ \text{m}}{1100\ \text{W/m}^2 \cdot \text{K}} = 340.91°\text{C}$$

$$T(x) = 16.67\left(1 - \frac{x^2}{L^2}\right) + 340.91°\text{C}$$

Computed temperatures for the nodal points of interest are shown in the first row of the accompanying table. Using the finite-difference equations, the nodal temperatures may be sequentially calculated with a time increment of 0.3 s until the desired final time is reached. The results are illustrated in rows 2 through 6 of the table and may be contrasted with the new steady-state condition (row 7)

Tabulated Nodal Temperatures

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82



✓ **The Implicit Method**

In the *explicit finite-difference* scheme, the temperature of any node at $t+\Delta t$ may be calculated from knowledge of temperatures at the same and neighboring nodes for the *preceding time t*. Hence determination of a nodal temperature at some time is *independent* of temperatures at other nodes for the *same time*. Although the method offers computational convenience, it suffers from limitations on the selection of Δt. For a given space increment, the time interval must be compatible with stability requirements. Frequently, this dictates the use of extremely small values of Δt, and a very large number of time intervals may be necessary to obtain a solution.

A reduction in the amount of computation time may often be realized by employing an *implicit*, rather than *explicit, finite-difference* scheme. The *implicit* form of a finite-difference equation may be derived by using Equation 5.20 to approximate the time derivative, while evaluating all other temperatures at the *new* (p+1) time, instead of the *previous* (p) time. Equation 5.20 is then considered to provide a *backward-difference* approximation to the time derivative. In contrast to

Equation 5.21, the *implicit* form of the *finite-difference* equation for the interior node of a *two-dimensional* system is then

$$\frac{T^{P+1}{}_{m+1,n}+T^{P+1}{}_{m-1,n}-2T^{P+1}{}_{m,n}}{(\Delta x)^2}+\frac{T^{P+1}{}_{m,n+1}+T^{P+1}{}_{m,n-1}-2T^{P+1}{}_{m,n}}{(\Delta y)^2}=\frac{1}{\alpha}\frac{T^{p+1}{}_{m,n}-T^{p}{}_{m,n}}{\Delta t}$$

(5-30)

Rearranging and assuming $\Delta x = \Delta y$, it follows that

$$T^{p}{}_{m,n}=(1+4Fo)T^{p+1}_{m,n}-Fo\left(T^{p+1}_{m+1,n}+T^{p+1}_{m-1,n}+T^{p+1}_{m,n+1}+T^{p+1}_{m,n-1}\right) \qquad (5\text{-}31)$$

From Equation 5.31 it is evident that the new temperature of the (m, n) node depends on the *new* temperatures of its adjoining nodes, which are, in general, unknown. Hence, to determine the unknown nodal temperatures at $t+\Delta t$, the corresponding nodal equations must be *solved simultaneously*. Such a solution may be affected by using *Gauss–Seidel* iteration or matrix inversion. The *marching* solution would then involve *simultaneously solving* the nodal equations at each time $t= \Delta t$, $t= 2\Delta t$, ….., until the desired final time was reached.

Relative to the *explicit* method, the *implicit* formulation has the important advantage of being *unconditionally stable*. That is, the solution remains stable for all space and time intervals, in which case there are no restrictions on Δx and Δt. Since larger values of Δt may therefore be used with an implicit method, computation times may often be reduced, with little loss of accuracy. Nevertheless, to maximize accuracy, Δt should be sufficiently small to ensure that the results are independent of further reductions in its value.

The *implicit* form of a *finite-difference* equation may also be derived from the *energy balance* method. For the surface node of Figure 5.11, it is readily shown that

$$T^{p}_{o}+2FoBiT_{\infty}=(1+2Fo+2FoBi)T^{p+1}_{o}-2FoT^{p+1}_{1} \qquad (5\text{-}32)$$

For any interior node of Figure 5.11, it may also be shown that

$$(1+2Fo)T^{p+1}_{m}-Fo\left(T^{p+1}_{m-1}+T^{p+1}_{m+1}\right)=T^{p}_{m} \qquad (5\text{-}33)$$

Forms of the implicit finite-difference equation for other common geometries are presented in Table below. Each equation may be derived by applying the *energy balance* method.

Conguration	(a) Explicit Method: Finite-Difference Equation	Stability Criterion	(b) Implicit Method
	$T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4Fo)T_{m,n}^{p}$ (5.79)	$Fo \le \frac{1}{4}$ (5.83)	$(1 + 4Fo)T_{m,n}^{p+1} - Fo(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p}$ (5.95)
1. Interior node			
	$T_{m,n}^{p+1} = \frac{2}{3}Fo(T_{m+1,n}^{p} + 2T_{m-1,n}^{p} + 2T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_\infty) + (1 - 4Fo - \frac{4}{3}Bi\,Fo)T_{m,n}^{p}$ (5.88)	$Fo(3 + Bi) \le \frac{3}{4}$ (5.89)	$(1 + 4Fo(1 + \frac{1}{3}Bi))T_{m,n}^{p+1} - \frac{2}{3}Fo \cdot (T_{m+1,n}^{p+1} + 2T_{m-1,n}^{p+1} + 2T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + \frac{4}{3}Bi\,Fo\,T_\infty$ (5.98)
2. Node at interior corner with convection			
	$T_{m,n}^{p+1} = Fo(2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_\infty) + (1 - 4Fo - 2Bi\,Fo)T_{m,n}^{p}$ (5.90)	$Fo(2 + Bi) \le \frac{1}{2}$ (5.91)	$(1 + 2Fo(2 + Bi))T_{m,n}^{p+1} - Fo(2T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 2Bi\,Fo\,T_\infty$ (5.99)
3. Node at plane surface with convection[a]			
	$T_{m,n}^{p+1} = 2Fo(T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2Bi\,T_\infty) + (1 - 4Fo - 4Bi\,Fo)T_{m,n}^{p}$ (5.92)	$Fo(1 + Bi) \le \frac{1}{4}$ (5.93)	$(1 + 4Fo(1 + Bi))T_{m,n}^{p+1} - 2Fo(T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 4Bi\,Fo\,T_\infty$ (5.100)
4. Node at exterior corner with convection			

[a] To obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set Bi equal to zero.

Example 7: A thick slab of copper initially at a uniform temperature of 20°C is suddenly exposed to radiation at one surface such that the net heat flux is maintained at a constant value of 3×10^5 W/m^2 (see Figure 5.13). Using the explicit and implicit finite-difference techniques with a space increment of Δx = 75 mm, determine the temperature at the irradiated surface and at an interior point that is 150 mm from the surface after 2 min have elapsed. Compare the results with those obtained from an appropriate analytical solution.

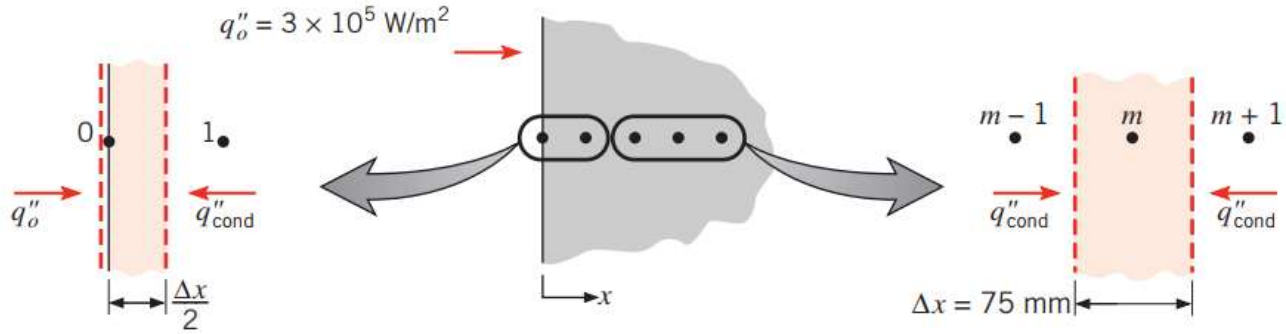


Figure 5.13: A Thick slab of copper, initially at a uniform temperature.

Solution:

Properties: Copper (300 K): k= 401 W/mK, α= 117×10^{-6} m^2/s.

An *explicit* form of the *finite-difference* equation for the surface node may be obtained by applying an *energy balance* to a control volume about the node.

$$q''_o A + kA\frac{T_1^p - T_0^p}{\Delta x} = \rho A\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

$$T_0^{p+1} = 2Fo\left(\frac{q''_0\Delta x}{k} + T_1^p\right) + (1 - 2Fo)T_0^p$$

The *finite-difference* equation for any interior node is given by Equation 5.29. Both the surface and interior nodes are governed by the *stability criterion*

$$Fo \leq \frac{1}{2}$$

Noting that the *finite-difference* equations are simplified by choosing the maximum allowable value of *Fo*, we select *Fo*= 0.5. Hence

$$\Delta t = Fo\frac{(\Delta x)^2}{\alpha} = \frac{1}{2}\frac{(0.075\ \text{m})^2}{117 \times 10^{-6}\ \text{m}^2/\text{s}} = 24\ \text{s}$$

$$\frac{q_o'' \Delta x}{k} = \frac{3 \times 10^5\ \text{W/m}^2\,(0.075\ \text{m})}{401\ \text{W/m} \cdot \text{K}} = 56.1°\text{C}$$

the *finite-difference* equations become

$$T_0^{p+1} = 56.1°\text{C} + T_1^p \qquad \text{and} \qquad T_m^{p+1} = \frac{T_{m+1}^p + T_{m-1}^p}{2}$$

for the surface and interior nodes, respectively. Performing the calculations, the results are tabulated as follows:

Explicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4
0	0	20	20	20	20	20
1	24	76.1	20	20	20	20
2	48	76.1	48.1	20	20	20
3	72	104.2	48.1	34.0	20	20
4	96	104.2	69.1	34.0	27.0	20
5	120	125.2	69.1	48.1	27.0	23.5

After 2 min, the surface temperature and the desired interior temperature are T_o 125.2°C and T_2= 48.1°C.

To determine the extent to which the accuracy may be improved by reducing *Fo*, let us redo the calculations for *Fo* = 1/4 (Δt = 12 s). The *finite-difference* equations are then of the form

$$T_0^{p+1} = \frac{1}{2}(56.1°C + T_1^p) + \frac{1}{2}T_0^p$$

$$T_m^{p+1} = \frac{1}{4}(T_{m+1}^p + T_{m-1}^p) + \frac{1}{2}T_m^p$$

and the results of the calculations are tabulated as follows:

Explicit Finite-Difference Solution for $Fo = \frac{1}{4}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20	20	20	20	20	20	20	20	20
1	12	48.1	20	20	20	20	20	20	20	20
2	24	62.1	27.0	20	20	20	20	20	20	20
3	36	72.6	34.0	21.8	20	20	20	20	20	20
4	48	81.4	40.6	24.4	20.4	20	20	20	20	20
5	60	89.0	46.7	27.5	21.3	20.1	20	20	20	20
6	72	95.9	52.5	30.7	22.5	20.4	20.0	20	20	20
7	84	102.3	57.9	34.1	24.1	20.8	20.1	20.0	20	20
8	96	108.1	63.1	37.6	25.8	21.5	20.3	20.0	20.0	20
9	108	113.6	67.9	41.0	27.6	22.2	20.5	20.1	20.0	20.0
10	120	118.8	72.6	44.4	29.6	23.2	20.8	20.2	20.0	20.0

After 2 min, the desired temperatures are T_o= 118.8°C and T_2= 44.4°C. Comparing the above results with those obtained for , it is clear that by reducing *Fo* we have diminished the problem of recurring temperatures. We have also predicted greater thermal penetration (to node 6 instead of node 3). An assessment of the improvement in accuracy will be given later, by comparison with an exact solution. In the absence of an exact solution, the value of *Fo* could be successively reduced until the results became essentially independent of *Fo*.

Performing an *energy balance* on a control volume about the surface node, the implicit form of the finite-difference equation is

$$q_o'' + k\frac{T_1^{p+1} - T_0^{p+1}}{\Delta x} = \rho\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^{p}}{\Delta t}$$

$$(1 + 2Fo)T_0^{p+1} - 2FoT_1^{p+1} = \frac{2\alpha q_o''\,\Delta t}{k\,\Delta x} + T_0^{p}$$

Arbitrarily choosing $Fo = \frac{1}{2}$ ($\Delta t = 24$ s), it follows that

$$2T_0^{p+1} - T_1^{p+1} = 56.1 + T_0^{p}$$

$$-T_{m-1}^{p+1} + 4T_m^{p+1} - T_{m+1}^{p+1} = 2T_m^{p}$$

In contrast to the *explicit* method, the *implicit* method requires the simultaneous solution of the nodal equations for all nodes at time p+1. Hence, the number of nodes under consideration must be limited to some finite number, and a boundary condition must be applied at the last node. The number of nodes may be limited to those that are affected significantly by the change in boundary condition for the time of interest. From the results of the explicit method, it is evident that we are safe in choosing nine nodes corresponding to T_o, T_1, . . . , T_8. We are thereby assuming that, at t= 120 s, there has been no change in T_9, and the boundary condition is implemented numerically as T_9= 20°C.

We now have a set of nine equations that must be solved simultaneously for each time increment. We can express the equations in the form [A][T]= [C],

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 56.1 + T_0^p \\ 2T_1^p \\ 2T_2^p \\ 2T_3^p \\ 2T_4^p \\ 2T_5^p \\ 2T_6^p \\ 2T_7^p \\ 2T_8^p + T_9^{p+1} \end{bmatrix}$$

Note that numerical values for the components of [*C*] are determined from previous values of the nodal temperatures. Note also how the finite-difference equation for node 8 appears in matrices [*A*] and [*C*], with $T^{p+1}_9 = 20°C$, as indicated previously. A table of nodal temperatures may be compiled, beginning with the first row ($p = 0$) corresponding to the prescribed initial condition. To obtain nodal temperatures for subsequent times, the matrix equation must be solved. At each time step $p+1$, [*C*] is updated using the previous time step (p) values. The process

is carried out five times to determine the nodal temperatures at 120 s. The desired temperatures are T_0= 114.7°C and T_2= 44.2°C.

Implicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
1	24	52.4	28.7	22.3	20.6	20.2	20.0	20.0	20.0	20.0
2	48	74.0	39.5	26.6	22.1	20.7	20.2	20.1	20.0	20.0
3	72	90.2	50.3	32.0	24.4	21.6	20.6	20.2	20.1	20.0
4	96	103.4	60.5	38.0	27.4	22.9	21.1	20.4	20.2	20.1
5	120	114.7	70.0	44.2	30.9	24.7	21.9	20.8	20.3	20.1

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