University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics

Handout Lectures for Year Two
Chapter Three/ Mass, Bernoulli, and Energy Equations

Course Tutor

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Chapter Three

Mass, Bernoulli, and Energy Equations

1.1 INTRODUCTION

You are already familiar with numerous ***conservation laws*** such as the laws of conservation of mass, conservation of energy, and conservation of momentum. Historically, the conservation laws are first applied to a fixed quantity of matter called a closed system or just a system, and then extended to regions in space called control volumes. The conservation relations are also called balance equations since any conserved quantity must balance during a process. We now give a brief description of the conservation of mass, momentum, and energy relations.

1.2 Conservation of Mass Principle

The conservation of mass principle for a control volume can be expressed as: The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the control volume during Δt. That is,

$$\begin{pmatrix}\text{Total mass entering}\\ \text{the CV during } \Delta t\end{pmatrix} - \begin{pmatrix}\text{Total mass leaving}\\ \text{the CV during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Net change in mass}\\ \text{within the CV during } \Delta t\end{pmatrix}$$

Or, $m_{in} - m_{out} = \Delta m_{CV}$ (kg) (3.1)

It can also be expressed in rate form as,

$\dot{m}_{in} - \dot{m}_{out} = dm_{CV}/dt$ (kg/s) (3.2) where $\dot{m}_{in}$ and m$\dot{m}_{out}$out are the total rates of mass flow into and out of the control volume, and dm_{CV}/dt is the rate of change of mass within the control volume boundaries. Equations 3–1 and 3–2 are often referred to as the ***mass balance*** and are applicable to any control volume undergoing any kind of process.

Consider a control volume of arbitrary shape, as shown in Figure 3–1. The mass of a differential volume dV within the control volume is dm= ρdV. The total mass within the control volume at any instant in time t is determined by integration to be

Total mass within the CV:

$$m_{CV} = \int_{CV} \rho \, dV \qquad (3.3)$$

Then the time rate of change of the amount of mass within the control volume can be expressed as

Rate of change of mass within the CV:

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho \, dV \qquad (3.4)$$

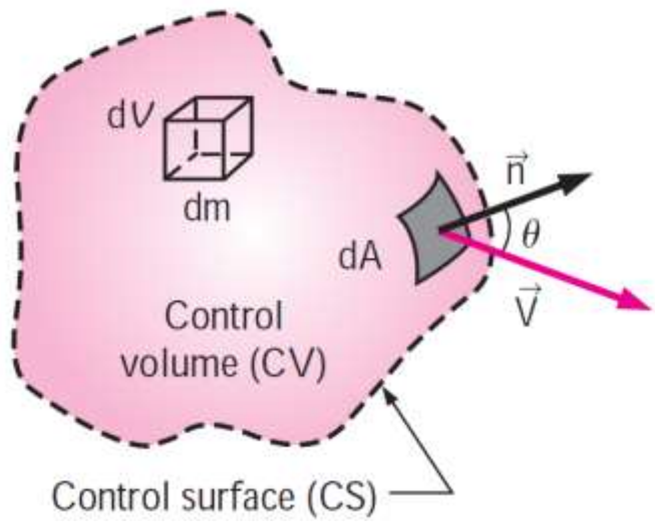


Figure 3.1: The differential control volume dV and the differential control surface dA used in the derivation of the conservation of mass relation.

Using the definition of mass flow rate as,

$$\frac{d}{dt}\int_{CV} \rho \, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m} \quad \text{or} \quad \frac{dm_{CV}}{dt} = \sum_{in} \dot{m} - \sum_{out} \dot{m} \qquad (3.5)$$

There is considerable flexibility in the selection of a control volume when solving a problem. Several control volume choices may be correct, but some are more convenient to work with. A control volume should not introduce any unnecessary complications. The proper choice of a control volume can make the solution of a seemingly complicated problem rather easy. A simple rule in selecting a control volume is to make the control surface normal to flow at all locations where it crosses fluid flow, whenever possible.

1.3 Mass Balance for Steady-Flow Processes

During a steady-flow process, the total amount of mass contained within a control volume does not change with time (m_{CV}= constant). Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it. For a garden hose nozzle in steady operation, for example, the amount of water entering the nozzle per unit time is equal to the amount of water leaving it per unit time. When dealing with steady-flow processes, we are not interested in the amount of mass that flows in or out of a device over time; instead, we are interested in the amount of mass flowing per unit time, that is, the mass flow rate $\dot{m}$. The conservation of mass principle for a general steady-flow system with multiple inlets and outlets can be expressed in rate form as (Figure 3.2)

Steady flow: $\sum_{in} \dot{m} = \sum_{out} \dot{m}$ (kg/s) (3.6)

It states that the total rate of mass entering a control volume is equal to the total rate of mass leaving it.

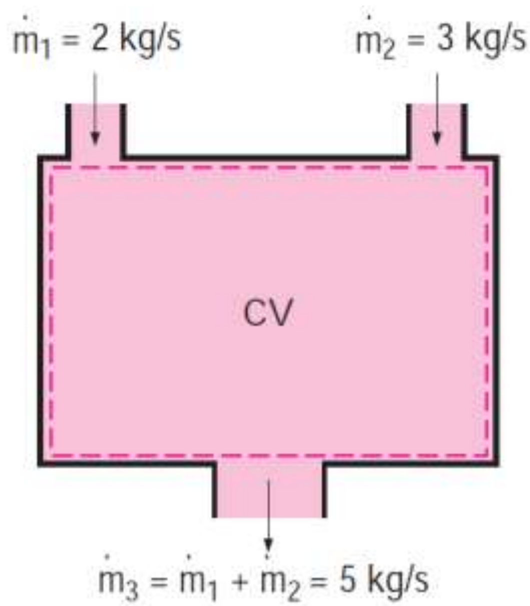


Figure 3.2: Conservation of mass principle for a two-inlet–one-outlet steady-flow system.

Many engineering devices such as nozzles, diffusers, turbines, compressors, and pumps involve a single stream (only one inlet and one outlet). For these cases, we denote the inlet state by the subscript 1 and the outlet state by the subscript 2, and drop the summation signs. Then Eq. 3.6 reduces, for single-stream steady-flow systems, to

Steady flow (single stream): $\dot{m}_1 = \dot{m}_2 \quad \Rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2$ (3.7)

Special Case: Incompressible Flow

The conservation of mass relations can be simplified even further when the fluid is incompressible, which is usually the case for liquids. Canceling the density from both sides of the general steady-flow relation gives

Steady, incompressible flow: $\sum_{in} \dot{V} = \sum_{out} \dot{V}$ (m^3/s) (3.8)

For single-stream steady-flow systems it becomes

Steady, incompressible flow (single stream): $\dot{V}_1 = \dot{V}_2 \Rightarrow V_1 A_1 = V_2 A_2$ (3.9)

It should always be kept in mind that there is no such thing as a "*conservation of volume*" principle. Therefore, the volume flow rates into and out of a steady-flow device may be different. The volume flow rate at the outlet of an air compressor is much less than that at the inlet even though the mass flow rate of air through the compressor is constant (Figure 3.3). This is due to the higher density of air at the compressor exit. For steady flow of liquids, however, the volume flow rates, as well as the mass flow rates, remain constant since liquids are essentially incompressible (constant-density) substances. Water flow through the nozzle of a garden hose is an example of the latter case.

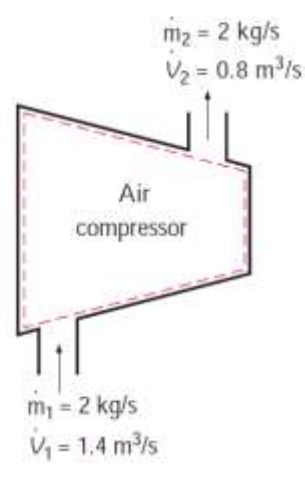


Figure 3.3: During a steady-flow process, volume flow rates are not necessarily conserved although mass flow rates are.

Example 3–1: A garden hose attached with a nozzle is used to fill a 10-gal bucket. The inner diameter of the hose is 2 cm, and it reduces to 0.8 cm at the nozzle exit. If it takes 50 s to fill the bucket with water, determine (***a***) the volume and mass flow rates of water through the hose, and (***b***) the average velocity of water at the nozzle exit.

SOLUTION A garden hose is used to fill a water bucket. The volume and mass flow rates of water and the exit velocity are to be determined.

Assumptions **1** Water is an incompressible substance. **2** Flow through the hose is steady. **3** There is no waste of water by splashing.

Properties We take the density of water to be 1000 kg/m³ = 1 kg/L.

Analysis (*a*) Noting that 10 gal of water are discharged in 50 s, the volume and mass flow rates of water are

$$\dot{V} = \frac{V}{\Delta t} = \frac{10\text{ gal}}{50\text{ s}}\left(\frac{3.7854\text{ L}}{1\text{ gal}}\right) = \mathbf{0.757\ L/s}$$

$$\dot{m} = \rho\dot{V} = (1\text{ kg/L})(0.757\text{ L/s}) = \mathbf{0.757\ kg/s}$$

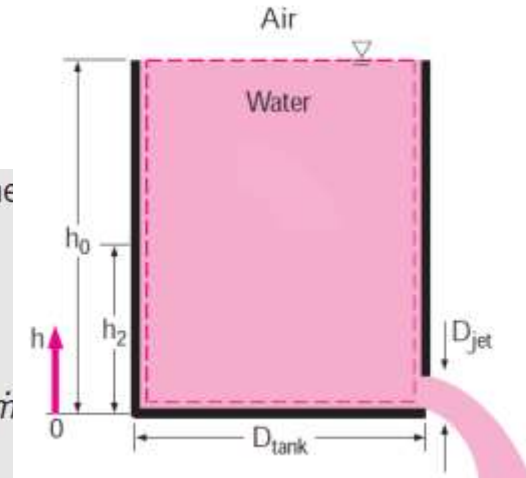
(*b*) The cross-sectional area of the nozzle exit is

$$A_e = \pi r_e^2 = \pi(0.4\text{ cm})^2 = 0.5027\text{ cm}^2 = 0.5027 \times 10^{-4}\text{ m}^2$$

The volume flow rate through the hose and the nozzle is constant. Then the average velocity of water at the nozzle exit becomes

$$V_e = \frac{\dot{V}}{A_e} = \frac{0.757\text{ L/s}}{0.5027 \times 10^{-4}\text{ m}^2}\left(\frac{1\text{ m}^3}{1000\text{ L}}\right) = \mathbf{15.1\ m/s}$$

Example 3–2: A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out (Fig. 3.4). The average velocity of the jet is given by $= \sqrt{2gh}$, where *h* is the height of water in the tank measured from the center of the hole (a variable) and *g* is the gravitational acceleration. Determine how long it will take for the water level in the tank to drop to 2 ft from the bottom.



Solution:

The conservation of mass relation for a control volume process is given in the rate form as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

During this process no mass enters the control volume ($\dot{m}$ mass flow rate of discharged water can be expressed as

$$\dot{m}_{out} = (\rho VA)_{out} = \rho\sqrt{2gh}A_{jet} \qquad \textbf{(2)}$$

where $A_{jet} = \pi D_{jet}^2/4$ is the cross-sectional area of the jet, which is constant. Noting that the density of water is constant, the mass of water in the tank at any time is

$$m_{CV} = \rho V = \rho A_{tank} h \qquad \textbf{(3)}$$

where $A_{tank} = \pi D_{tank}^2/4$ is the base area of the cylindrical tank. Substituting Eqs. 2 and 3 into the mass balance relation (Eq. 1) gives

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank} h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)\,dh}{dt}$$

Canceling the densities and other common terms and separating the variables give

$$dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{dh}{\sqrt{2gh}}$$

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

Substituting, the time of discharge is determined to be

$$t = \frac{\sqrt{4\text{ ft}} - \sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}}\left(\frac{3 \times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \textbf{12.6 min}$$

Therefore, half of the tank will be emptied in 12.6 min after the discharge hole is unplugged.

1.4 The Bernoulli equation

The ***Bernoulli equation*** is *an approximate relation between pressure, velocity, and elevation,* and is valid in *regions of steady, incompressible flow where net frictional forces are negligible* (Fig. 3-4). Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics. In this section, we derive the Bernoulli equation by applying the *conservation of linear momentum principle*, and we demonstrate both its usefulness and its limitations.

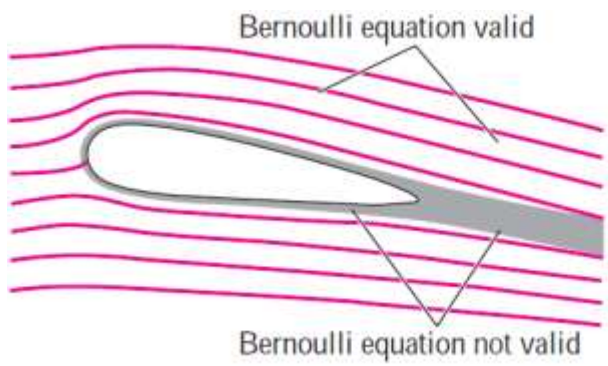


Figure 3.4: The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur

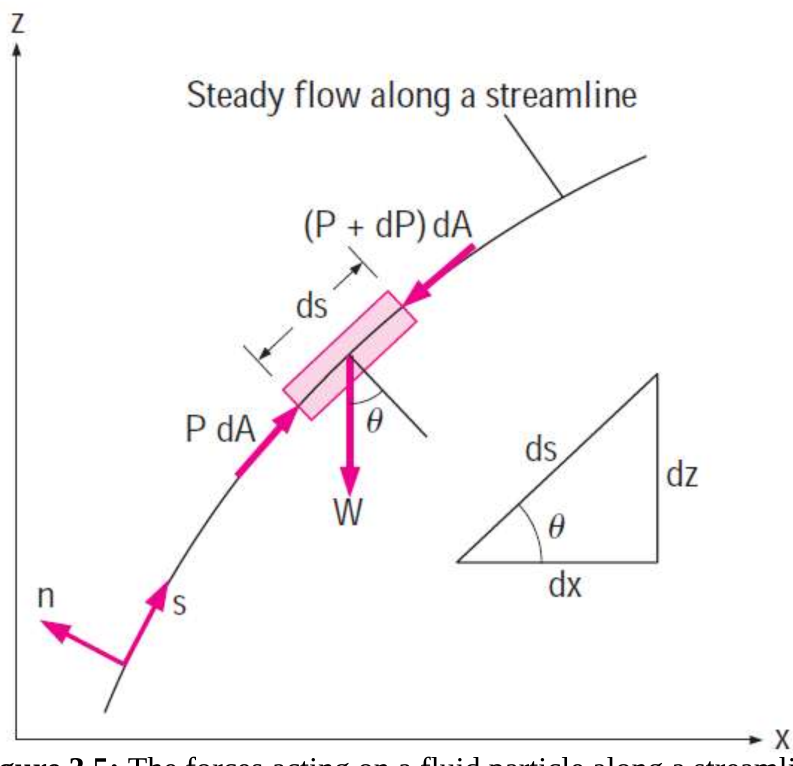


Figure 3.5: The forces acting on a fluid particle along a streamline.

Consider the motion of a fluid particle in a flow field in steady flow described in detail. Applying Newton's second law (which is referred to as the conservation of linear momentum relation in fluid mechanics) in the s-direction on a particle moving along a streamline gives,

$$\sum F_s = ma_s \tag{3.10}$$

In regions of flow where net frictional forces are negligible, the significant forces acting in the ***s***-direction are the pressure (acting on both sides) and the component of the weight of the particle in the ***s***-direction (Figure 3-5). Therefore, Equation 3-10 becomes

$$PdA - (P + dP)dA - W\sin\theta = mV\frac{dV}{ds} \tag{3.11}$$

where θ is the angle between the normal of the streamline and the vertical z-axis at that point, $m = \rho V = \rho dAds$ is the mass, $W = mg = \rho gdAds$ is the weight of the fluid particle, and $\sin\theta = dz/ds$. Substituting,

$$-dPdA - \rho gdA\,ds\,\frac{dz}{ds} = \rho dA\,dsV\frac{dV}{ds} \tag{3.12}$$

Canceling dA from each term and simplifying,

$$-dP - \rho g\,dz = \rho VdV \tag{3.13}$$

Noting that $VdV = 0.5\,d(V^2)$ and dividing each term by ρ gives

$$\frac{dP}{\rho} + \frac{1}{2}d(V^2) + gdz = 0 \tag{3.14}$$

Integrating

Steady flow: $\int\frac{dP}{\rho} + \frac{V^2}{2} + gz = \ constant\ (along\ a\ streamline)$ (3.15)

since the last two terms are exact differentials. In the case of incompressible flow, the first term also becomes an exact differential, and its integration gives

Steady, incompressible flow: $\frac{P}{\rho} + \frac{V^2}{2} + gz = \ constant$ (3.16)

This is the famous ***Bernoulli equation***, which is commonly used in fluid mechanics for steady, incompressible flow along a streamline in inviscid regions of flow. The value of the constant can be evaluated at any point on the streamline where the pressure, density, velocity, and elevation are known. The ***Bernoulli equation*** can also be written between any two points on the same streamline as

Steady, incompressible flow: $\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$ (3.17)

The Bernoulli equation is obtained from the conservation of momentum for a fluid particle moving along a streamline. It can also be obtained from the *first law of thermodynamics* applied to a steady-flow system.

The Bernoulli Equation According to Static, Dynamic, and Stagnation Pressures

The Bernoulli equation states that the sum of the flow, kinetic, and potential energies of a fluid particle along a streamline is constant. Therefore, the kinetic and potential energies of the fluid can be converted to flow energy (and vice versa) during flow, causing the pressure to change. This phenomenon can be made more visible by multiplying the Bernoulli equation by the density ρ,

$P + \rho\frac{V^2}{2} + \rho gz = \text{constant}$ (along a streamline) (3.18)

Each term in this equation has pressure units, and thus each term represents some kind of pressure:

- ✓ P is the ***static pressure*** (it does not incorporate any dynamic effects); it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.
- ✓ $\rho V^2/2$ is the ***dynamic pressure;*** it represents the pressure rise when the fluid in motion is brought to a stop isentropically.
- ✓ ρgz is the ***hydrostatic pressure,*** which is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects, i.e., of fluid weight on pressure.

The sum of the static, dynamic, and hydrostatic pressures is called the **total pressure.** Therefore, the Bernoulli equation states that *the total pressure along a streamline is constant*.

The sum of the static and dynamic pressures is called the **stagnation pressure,** and it is expressed as

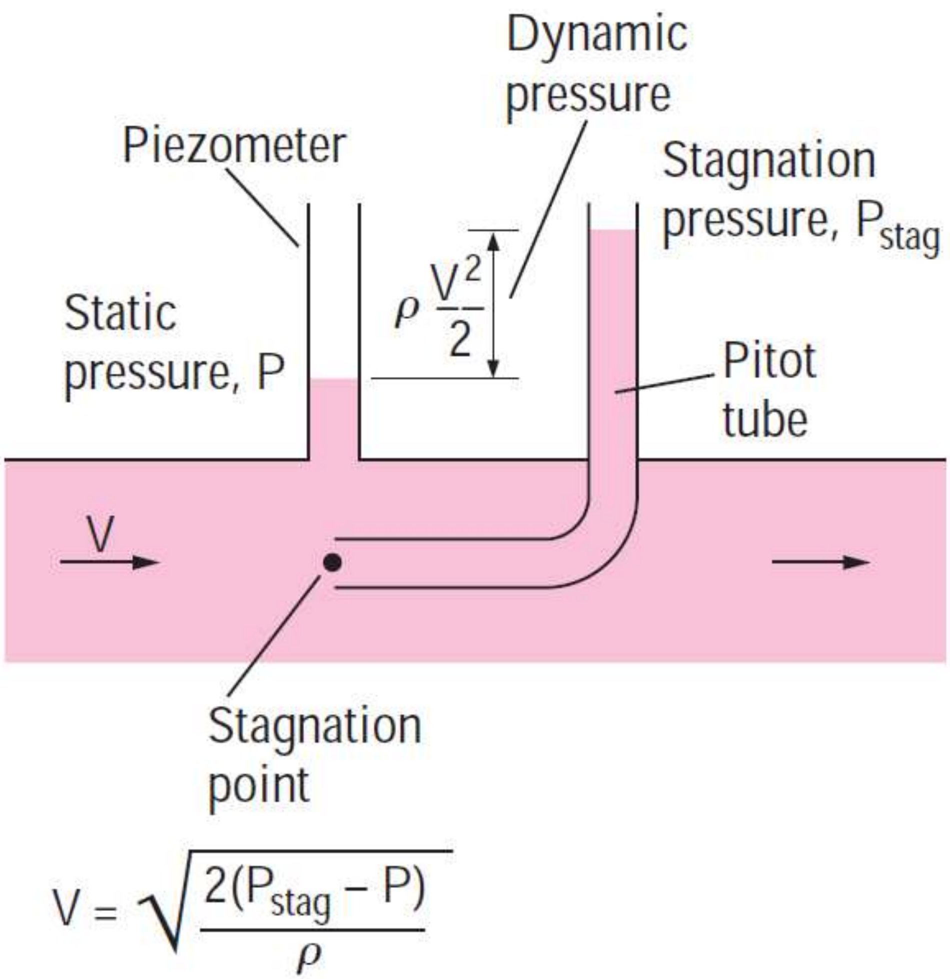


Figure 3.6: The static, dynamic, and stagnation pressures.

$$P_{\text{Stagnation}} = P + \rho\frac{V^2}{2} \qquad \text{(kPa)} \qquad (3.19)$$

The stagnation pressure represents the pressure at a point where the fluid is brought to a complete stop isentropically. The static, dynamic, and stagnation pressures are shown in Figure 3.6. When static and stagnation pressures are measured at a specified location, ***the fluid velocity*** at that location can be calculated from

$$V = \sqrt{\frac{2(P_{\text{Stagnation}} - P)}{\rho}} \qquad \text{(m/s)} \qquad (3.20)$$

Example 1:

Water is flowing from a hose attached to a water main at 400 kPa gage (Figure 3.7). A child places his thumb to cover most of the hose outlet, causing a thin jet of high-speed water to emerge. If the hose is held upward, what is the maximum height that the jet could achieve?

Solution:

The water height will be maximum under the stated assumptions. The velocity inside the hose is relatively low (V_1= 0) and we take the hose outlet as the reference level (z_1= 0). At the top of the water trajectory V_2= 0, and atmospheric pressure pertains. Then the Bernoulli equation simplifies to

$$\frac{P_1}{\rho g} + \cancelto{0}{\frac{V_1^2}{2g}} + \cancelto{0}{z_1} = \frac{P_2}{\rho g} + \cancelto{0}{\frac{V_2^2}{2g}} + z_2 \quad \rightarrow \quad \frac{P_1}{\rho g} = \frac{P_{atm}}{\rho g} + z_2$$

Solving for z_2 and substituting,

$$z_2 = \frac{P_1 - P_{atm}}{\rho g} = \frac{P_{1,\,gage}}{\rho g} = \frac{400 \text{ kPa}}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)}\left(\frac{1000 \text{ N/m}^2}{1 \text{ kPa}}\right)\left(\frac{1 \text{ kg}\cdot\text{m/s}^2}{1 \text{ N}}\right)$$

$$= \mathbf{40.8 \text{ m}}$$

Figure 3.7

Example 2:

A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap (Figure 3.8). A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

Solution:

We take point 1 to be at the free surface of water so that P_1 = *P*atm (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the outlet), and $z_1 = 5$ m and $z_2 = 0$ (we take the reference level at the center of the outlet). Also, P_2 = *P*atm (water discharges into the atmosphere). Then the Bernoulli equation simplifies to

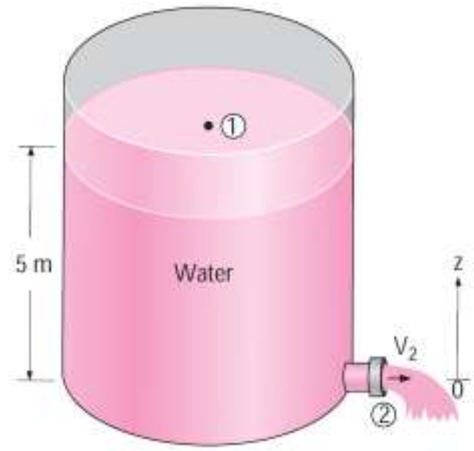


Figure 3.8

$$\cancel{\frac{P_1}{\rho g}} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 = \cancel{\frac{P_2}{\rho g}} + \frac{V_2^2}{2g} + \cancelto{0}{z_2} \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

Solving for V_2 and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81 \text{ m/s}^2)(5 \text{ m})} = \mathbf{9.9 \text{ m/s}}$$

The relation $V = \sqrt{2gz}$ is called the **Toricelli equation.**

Therefore, the water leaves the tank with an initial velocity of 9.9 m/s. This is the same velocity that would manifest if a solid were dropped a distance of 5 m in the absence of air friction drag. (What would the velocity be if the tap were at the bottom of the tank instead of on the side?)

Example 3:

During a trip to the beach (*P*atm = 1 atm = 101.3 kPa), a car runs out of gasoline, and it becomes necessary to siphon gas out of the car of a Good Samaritan (Figure 3.9). The siphon is a small-diameter hose, and to start the siphon it is necessary to insert one siphon end in the full gas tank, fill the hose with gasoline via suction, and then place the other end in a gas can below the level of the gas tank. The difference in pressure between point 1 (at the free surface of the gasoline in the tank) and point 2 (at the outlet of the tube) causes the liquid to flow from the higher to the lower elevation. Point 2 is located 0.75 m below point 1 in this case, and point 3 is located 2 m above point 1. The siphon diameter is 4 mm, and frictional losses in the siphon are to be disregarded. Determine (***a***) the minimum time to withdraw 4 L of gasoline from the tank to the can and (***b***) the pressure at point 3. The density of gasoline is 750 kg/m^3.

S*olution:*

(a) We take point 1 to be at the free surface of gasoline in the tank so that $P_1 = P$atm (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the tube diameter), and $z_2 = 0$ (point 2 is taken as the reference level). Also, $P_2 = P$atm (gasoline

$$\frac{\cancel{P_1}}{\cancel{\rho g}}+\frac{\cancelto{0}{V_1^2}}{2g}+z_1=\frac{\cancel{P_2}}{\cancel{\rho g}}+\frac{V_2^2}{2g}+\cancelto{0}{z_2} \quad\rightarrow\quad z_1=\frac{V_2^2}{2g}$$

Solving for V_2 and substituting,

$$V_2=\sqrt{2gz_1}=\sqrt{2(9.81\text{ m/s}^2)(0.75\text{ m})}=3.84\text{ m/s}$$

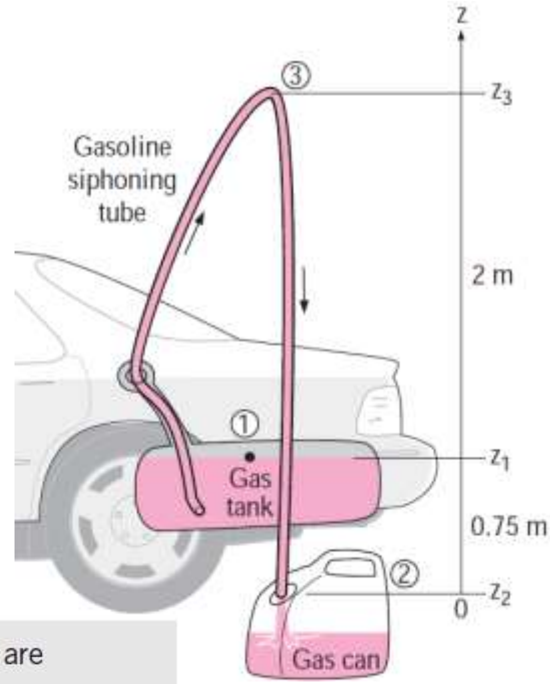


Figure 3.9

The cross-sectional area of the tube and the flow rate of gasoline are

$$A=\pi D^2/4=\pi(5\times10^{-3}\text{ m})^2/4=1.96\times10^{-5}\text{ m}^2$$

$$\dot{V}=V_2A=(3.84\text{ m/s})(1.96\times10^{-5}\text{ m}^2)=7.53\times10^{-5}\text{ m}^3\text{/s}=0.0753\text{ L/s}$$

Then the time needed to siphon 4 L of gasoline becomes

$$\Delta t=\frac{V}{\dot{V}}=\frac{4\text{ L}}{0.0753\text{ L/s}}=\mathbf{53.1\ s}$$

(*b*) The pressure at point 3 can be determined by writing the Bernoulli equation between points 2 and 3. Noting that $V_2 = V_3$ (conservation of mass), $z_2 = 0$, and $P_2 = P_{atm}$,

$$\frac{P_2}{\rho g}+\frac{\cancel{V_2^2}}{\cancel{2g}}+\cancelto{0}{z_2}=\frac{P_3}{\rho g}+\frac{\cancel{V_3^2}}{\cancel{2g}}+z_3 \quad\rightarrow\quad \frac{P_{atm}}{\rho g}=\frac{P_3}{\rho g}+z_3$$

Solving for P_3 and substituting,

$$P_3=P_{atm}-\rho g z_3$$

$$=101.3\text{ kPa}-(750\text{ kg/m}^3)(9.81\text{ m/s}^2)(2.75\text{ m})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right)\left(\frac{1\text{ kPa}}{1000\text{ N/m}^2}\right)$$

$$=\mathbf{81.1\ kPa}$$

Example 4:

A piezometer and a ***Pitot tube*** are tapped into a horizontal water pipe, as shown in Figure 3-10, to measure static and stagnation (static + dynamic) pressures. For the indicated water column heights, determine the velocity at the c enter of the pipe.

Solution:

We take points 1 and 2 along the centerline of the pipe, with point 1 directly under the piezometer and point 2 at the tip of the Pitot tube. This is a steady flow with straight and parallel streamlines, and the gage pressures at points 1 and 2 can be expressed as

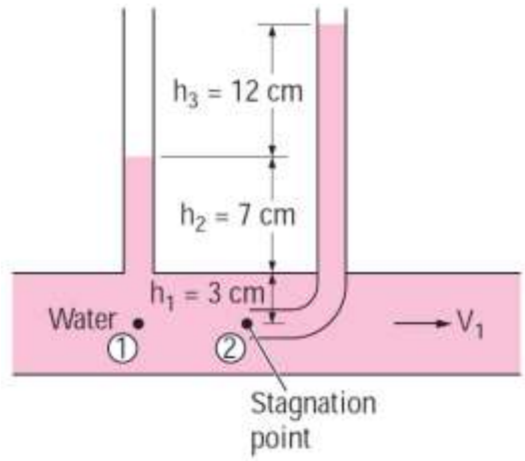


Figure 3.10, Schematic for Example

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

Noting that point 2 is a stagnation point and thus $V_2 = 0$ and $z_1 = z_2$, the application of the Bernoulli equation between points 1 and 2 gives

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}^{\nearrow 0} + \cancel{z_2} \quad \rightarrow \quad \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Substituting the P_1 and P_2 expressions gives

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

Solving for V_1 and substituting,

$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81 \text{ m/s}^2)(0.12 \text{ m})} = \mathbf{1.53 \text{ m/s}}$$

1.5 Mechanical energy and efficiency

The *mechanical energy* can be defined as the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine. Kinetic and potential energies are the familiar forms of mechanical energy. Thermal energy is not mechanical energy, however, since it cannot be converted to work directly and completely (the second law of thermodynamics).

A ***pump*** transfers mechanical energy to a fluid by raising its pressure, and a ***turbine*** extracts mechanical energy from a fluid by dropping its pressure. Therefore, the pressure of a flowing fluid is also associated with its mechanical energy.

The steady-flow energy equation on a unit-mass basis can be written conveniently as a mechanical energy balance as,

$$w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + e_{\text{mech, loss}}$$

Noting that $W_{\text{shaft, net in}} = W_{\text{shaft, in}} - W_{\text{shaft, out}} = W_{\text{pump}} - W_{\text{turbine}}$, the mechanical energy balance can be written more explicitly as,

$$\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 + w_{\text{pump}} = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + w_{\text{turbine}} + e_{\text{mech, loss}}$$

where W_{pump} is the mechanical work input (due to the presence of a pump, fan, compressor, etc.) and W_{turbine} is the mechanical work output. When the flow is incompressible, either absolute or gage pressure can be used for P since P_{atm}/ρ would appear on both sides and would cancel out. $e_{\text{mech, loss}}$ is the *total* mechanical power loss, which consists of pump and turbine losses as well as the frictional losses in the piping network. Multiplying above Equation by the mass flow rate $\dot{m}$ gives:

$$\dot{m}\left(\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right) + \dot{W}_{\text{pump}} = \dot{m}\left(\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech, loss}}$$

By convention, irreversible pump and turbine losses are treated separately from irreversible losses due to other components of the piping system. Thus the energy equation can be expressed in its most common form in terms of heads as,

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{\text{pump, u}} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{\text{turbine, e}} + h_L$$

where $h_{\text{pump, u}} = \frac{w_{\text{pump, u}}}{g} = \frac{\dot{W}_{\text{pump, u}}}{\dot{m}g} = \frac{\eta_{\text{pump}}\dot{W}_{\text{pump}}}{\dot{m}g}$ is the useful head delivered to the fluid by the pump. Because of irreversible losses in the pump, $h_{\text{pump, u}}$ is less than $\dot{W}_{\text{pump}}/\dot{m}g$ by the factor η_{pump}. Similarly, $h_{\text{turbine, e}} = \frac{w_{\text{turbine, e}}}{g} = \frac{\dot{W}_{\text{turbine, e}}}{\dot{m}g} = \frac{\dot{W}_{\text{turbine}}}{\eta_{\text{turbine}}\dot{m}g}$ is the extracted head removed from the fluid by the turbine. Because of irreversible losses in the turbine, $h_{\text{turbine, e}}$ is greater than $\dot{W}_{\text{turbine}}/\dot{m}g$ by the factor η_{turbine}. Finally, $h_L = \frac{e_{\text{mech loss, piping}}}{g} = \frac{\dot{E}_{\text{mech loss, piping}}}{\dot{m}g}$ is the irreversible head loss between 1 and 2 due to all components of the piping system other than the pump or turbine.

Example 5:

The pump of a water distribution system is powered by a 15-kW electric motor whose efficiency is 90 percent (**Figure 3.11**). The water flow rate through the pump is 50 L/s. The diameters of the inlet and outlet pipes are the same, and the elevation difference across the pump is negligible. If the pressures at the inlet and outlet of the pump are measured to be 100 kPa and 300 kPa (absolute), respectively, determine (**a**) the mechanical efficiency of the pump and (**b**) the temperature rise of water as it flows through the pump due to the mechanical inefficiency.

Solution:

1 The flow is steady and incompressible.
2 The pump is driven by an external motor so that the heat generated by the motor is dissipated to the atmosphere.
3 The elevation difference between the inlet and outlet of the pump is negligible, $z_1 \approx z_2$.
4 The inlet and outlet diameters are the same and thus the inlet and outlet velocities and kinetic energy correction factors are equal, $V_1 = V_2$.

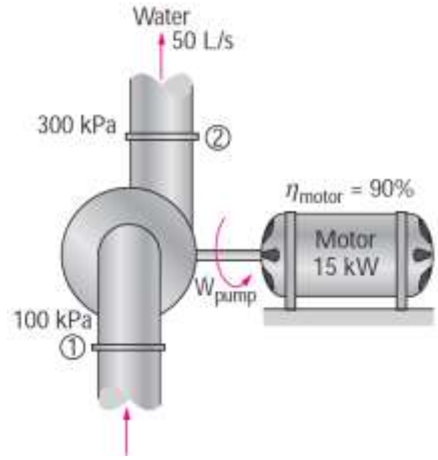


Figure 3.11

(**a**) The mass flow rate of water through the pump is

$$\dot{m} = \rho\dot{V} = (1\text{ kg/L})(50\text{ L/s}) = 50\text{ kg/s}$$

The motor draws 15 kW of power and is 90 percent efficient. Thus the mechanical (shaft) power it delivers to the pump is

$$\dot{W}_{\text{pump, shaft}} = \eta_{\text{motor}}\dot{W}_{\text{electric}} = (0.90)(15\text{ kW}) = 13.5\text{ kW}$$

To determine the mechanical efficiency of the pump, we need to know the increase in the mechanical energy of the fluid as it flows through the pump, which is

$$\Delta\dot{E}_{\text{mech, fluid}} = \dot{E}_{\text{mech, out}} - \dot{E}_{\text{mech, in}} = \dot{m}\left(\frac{P_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) - \dot{m}\left(\frac{P_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right)$$

Where α is the kinetic energy correction factor.

Simplifying it for this case and substituting the given values,

$$\Delta\dot{E}_{\text{mech, fluid}} = \dot{m}\left(\frac{P_2 - P_1}{\rho}\right) = (50\text{ kg/s})\left(\frac{(300 - 100)\text{ kPa}}{1000\text{ kg/m}^3}\right)\left(\frac{1\text{ kJ}}{1\text{ kPa}\cdot\text{m}^3}\right) = 10\text{ kW}$$

Then the mechanical efficiency of the pump becomes

$$\eta_{\text{pump}} = \frac{\dot{W}_{\text{pump, u}}}{\dot{W}_{\text{pump, shaft}}} = \frac{\Delta\dot{E}_{\text{mech, fluid}}}{\dot{W}_{\text{pump, shaft}}} = \frac{10\text{ kW}}{13.5\text{ kW}} = \mathbf{0.741} \quad \text{or} \quad \mathbf{74.1\%}$$

(**b**) Of the 13.5-kW mechanical power supplied by the pump, only 10 kW is imparted to the fluid as mechanical energy. The remaining 3.5 kW is converted to thermal energy due to frictional effects, and this "lost" mechanical energy manifests itself as a heating effect in the fluid,

$$\dot{E}_{\text{mech, loss}} = \dot{W}_{\text{pump, shaft}} - \Delta\dot{E}_{\text{mech, fluid}} = 13.5 - 10 = 3.5 \text{ kW}$$

The temperature rise of water due to this mechanical inefficiency is determined from the thermal energy balance, $\dot{E}_{\text{mech, loss}} = \dot{m}(u_2 - u_1) = \dot{m}c\Delta T.$

$$\Delta T = \frac{\dot{E}_{\text{mech, loss}}}{\dot{m}c} = \frac{3.5 \text{ kW}}{(50 \text{ kg/s})(4.18 \text{ kJ/ kg}\cdot{}^\circ\text{C})} = \mathbf{0.017^\circ C}$$

Example 6:

In a hydroelectric power plant, 100 m^3/s of water flows from an elevation of 120 m to a turbine, where electric power is generated (Figure 3-12). The total irreversible head loss in the piping system from point 1 to point 2 (excluding the turbine unit) is determined to be 35 m. If the overall efficiency of the turbine–generator is 80 percent, estimate the electric power output.

Solution The mass flow rate of water through the turbine is

$$\dot{m} = \rho\dot{V} = (1000 \text{ kg/m}^3)(100 \text{ m}^3\text{/s}) = 10^5 \text{ kg/s}$$

We take point 2 as the reference level, and thus $z_2 = 0$. Also, both points 1 and 2 are open to the atmosphere ($P_1 = P_2 = P_{atm}$) and the flow velocities are negligible at both points ($V_1 = V_2 = 0$). Then the energy equation for steady, incompressible flow reduces to

$$\cancel{\frac{P_1}{\rho g}} + \alpha_1\cancel{\frac{V_1^2}{2g}} + z_1 + h_{\text{pump, u}}^{\nearrow 0} = \cancel{\frac{P_2}{\rho g}} + \alpha_2\cancel{\frac{V_2^2}{2g}} + z_2^{\nearrow 0} + h_{\text{turbine, e}} + h_L \rightarrow$$

$$h_{\text{turbine, e}} = z_1 - h_L$$

Substituting, the extracted turbine head and the corresponding turbine power are

$$h_{\text{turbine, e}} = z_1 - h_L = 120 - 35 = 85 \text{ m}$$

$$\dot{W}_{\text{turbine, e}} = \dot{m}gh_{\text{turbine, e}} = (10^5 \text{ kg/s})(9.81 \text{ m/s}^2)(85 \text{ m})\left(\frac{1 \text{ kJ/kg}}{1000 \text{ m}^2\text{/s}^2}\right) = 83{,}400 \text{ kW}$$

Therefore, a perfect turbine–generator would generate 83,400 kW of electricity from this resource. The electric power generated by the actual unit is

$$\dot{W}_{electric} = \eta_{turbine-gen}\dot{W}_{turbine,\,e} = (0.80)(83.4\ \text{MW}) = \mathbf{66.7\ MW}$$

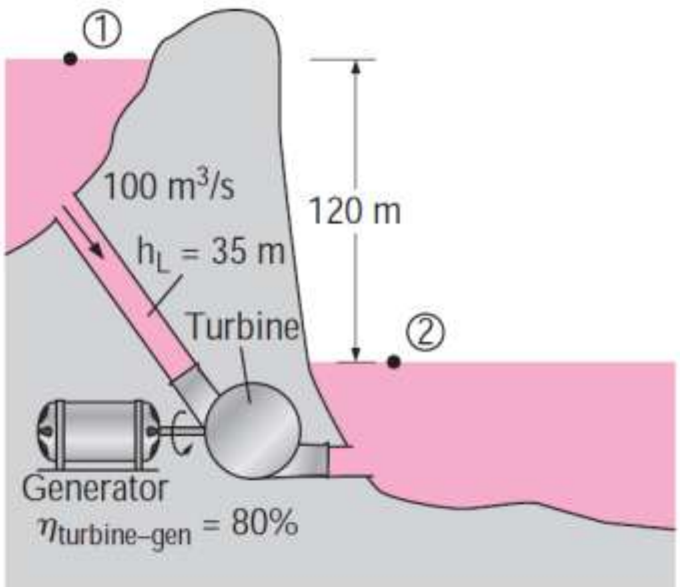


Figure 3.12

Example 7:

Water is pumped from a lower reservoir to a higher reservoir by a pump that provides 20 kW of useful mechanical power to the water (Figure 3.13). The free surface of the upper reservoir is 45 m higher than the surface of the lower reservoir. If the flow rate of water is measured to be 0.03 m^3/s, determine the irreversible head loss of the system and the lost mechanical power during this process.

Solution:

The mass flow rate of water through the system is

$$\dot{m} = \rho\dot{V} = (1000\ \text{kg/m}^3)(0.03\ \text{m}^3/\text{s}) = 30\ \text{kg/s}$$

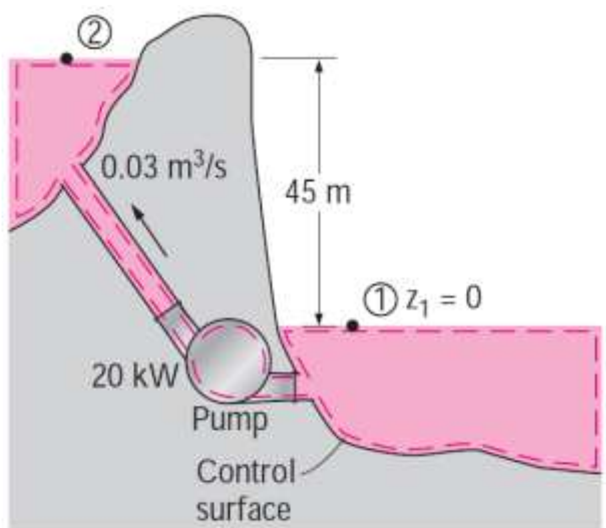


Figure 3.13

We choose points 1 and 2 at the free surfaces of the lower and upper reservoirs, respectively, and take the surface of the lower reservoir as the reference level (z_1 = 0). Both points are open to the atmosphere ($P_1 = P_2 = P_{atm}$) and the velocities at both locations are negligible ($V_1 = V_2 = 0$). Then the energy equation for steady incompressible flow for a control volume between 1 and 2 reduces to

$$\dot{m}\left(\cancel{\frac{P_1}{\rho}} + \alpha_1 \frac{V_1^2}{2}^{\nearrow 0} + gz_1^{\nearrow 0}\right) + \dot{W}_{pump} = \dot{m}\left(\cancel{\frac{P_2}{\rho}} + \alpha_2 \frac{V_2^2}{2}^{\nearrow 0} + gz_2\right) + \dot{W}_{turbine}^{\nearrow 0} + \dot{E}_{mech,\ loss}$$

$$\dot{W}_{pump} = \dot{m}gz_2 + \dot{E}_{mech,\ loss} \quad \rightarrow \quad \dot{E}_{mech,\ loss} = \dot{W}_{pump} - \dot{m}gz_2$$

Substituting, the lost mechanical power and head loss are determined to be

$$\dot{E}_{mech,\ loss} = 20\text{ kW} - (30\text{ kg/s})(9.81\text{ m/s}^2)(45\text{ m})\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right)\left(\frac{1\text{ kW}}{1000\text{ N}\cdot\text{m/s}}\right)$$

$$= \mathbf{6.76\ kW}$$

Noting that the entire mechanical losses are due to frictional losses in piping and thus $\dot{E}_{mech,\ loss} = \dot{E}_{mech,\ loss,\ piping}$, the irreversible head loss is determined to be

$$h_L = \frac{\dot{E}_{mech\ loss,\ piping}}{\dot{m}g} = \frac{6.76\text{ kW}}{(30\text{ kg/s})(9.81\text{ m/s}^2)}\left(\frac{1\text{ kg}\cdot\text{m/s}^2}{1\text{ N}}\right)\left(\frac{1000\text{ N}\cdot\text{m/s}}{1\text{ kW}}\right) = \mathbf{23.0\ m}$$

Consider a container of height h filled with water, as shown in Figure 3-14, with the reference level selected at the bottom surface. The gage pressure and the potential energy per unit mass are, respectively, $P_A = 0$ and $pe_A = gh$ at point A at the free surface, and $P_B = \rho gh$ and $pe_B = 0$ at point B at the bottom of the container. An ideal hydraulic turbine would produce the same work per unit mass $w_{turbine} = gh$ whether it receives water (or any other fluid with constant density) from the top or from the bottom of the container. Note that we are also assuming ideal flow (no irreversible losses) through the pipe leading from the tank to the turbine. Therefore, the total mechanical energy of water at the bottom is equivalent to that at the top.

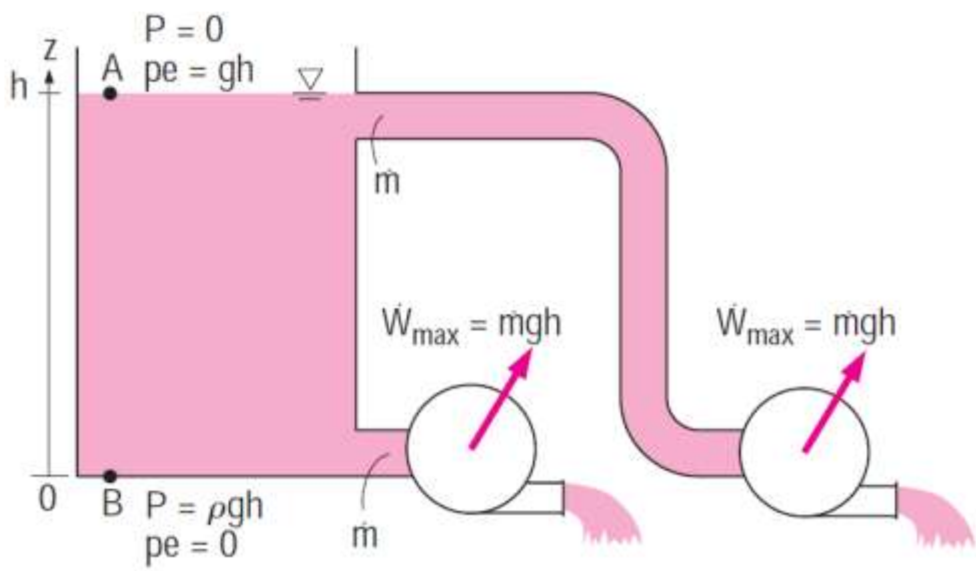


Figure 3.14: The mechanical energy of water at the bottom of a container is equal to the mechanical energy at any depth including the free surface of the container.

The transfer of mechanical energy is usually accomplished by a rotating shaft, and thus mechanical work is often referred to as shaft work. A pump or a fan receives shaft work (usually from an electric motor) and transfers it to the fluid as mechanical energy (less frictional losses). A turbine, on the other hand, converts the mechanical energy of a fluid to shaft work. In the absence of any irreversibilities such as friction, mechanical energy can be converted entirely from one mechanical form to another, and the ***mechanical efficiency*** of a device or process can be defined as,

$$\eta_{\text{mech}} = \frac{\text{Mechanical energy output}}{\text{Mechanical energy input}} = \frac{E_{\text{mech, out}}}{E_{\text{mech, in}}} = 1 - \frac{E_{\text{mech, loss}}}{E_{\text{mech, in}}}$$

A conversion efficiency of less than 100 percent indicates that conversion is less than perfect and some losses have occurred during conversion. A mechanical efficiency of 97 percent indicates that 3 percent of the mechanical energy input is converted to thermal energy as a result of frictional heating, and this will manifest itself as a slight rise in the temperature of the fluid.

The degree of perfection of the conversion process between the mechanical work supplied or extracted and the mechanical energy of the fluid is expressed by the ***pump efficiency*** and ***turbine efficiency***, defined as

$$\eta_{\text{pump}} = \frac{\text{Mechanical energy increase of the fluid}}{\text{Mechanical energy input}} = \frac{\Delta \dot{E}_{\text{mech, fluid}}}{\dot{W}_{\text{shaft, in}}} = \frac{\dot{W}_{\text{pump, u}}}{\dot{W}_{\text{pump}}}$$

where $\Delta E_{\text{mech,fluid}} = E_{\text{mech,out}} - E_{\text{mech, in}}$ is the rate of increase in the mechanical energy of the fluid, which is equivalent to the ***useful pumping power*** $W_{\text{pump, }u}$ supplied to the fluid, and

$$\eta_{\text{turbine}} = \frac{\text{Mechanical energy output}}{\text{Mechanical energy decrease of the fluid}} = \frac{W_{\text{shaft, out}}}{|\Delta \dot{E}_{\text{mech, fluid}}|} = \frac{W_{\text{turbine}}}{\dot{W}_{\text{turbine, e}}}$$

where $\Delta E_{\text{mech, fluid}} = E_{\text{mech, in}} - E_{\text{mech, out}}$ is the rate of decrease in the mechanical energy of the fluid, which is equivalent to the mechanical power extracted from the fluid by the turbine *W* turbine, *e*, and we use the absolute value sign to avoid negative values for efficiencies. A pump or turbine efficiency of 100 percent indicates perfect conversion between the shaft work and the mechanical energy of the fluid, and this value can be approached (but never attained) as the frictional effects are minimized.

Example 8: The water in a large lake is to be used to generate electricity by the installation of a hydraulic turbine–generator at a location where the depth of the water is 50 m (Figure **3.15**). Water is to be supplied at a rate of 5000 kg/s. If the electric power generated is measured to be 1862 kW and the generator efficiency is 95 percent, determine (***a***) the overall efficiency of the turbine– generator, (***b***) the mechanical efficiency of the turbine, and (***c***) the shaft power supplied by the turbine to the generator.

Solution:

(***a***) We take the bottom of the lake as the reference level for convenience. Then kinetic and potential energies of water are zero, and the change in its mechanical energy per unit mass becomes

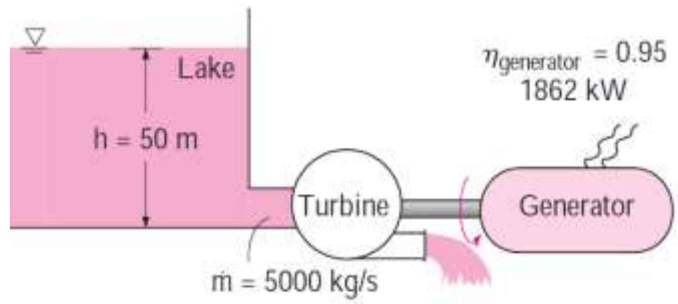


Figure 3.15: Schematic for Example 8.

$$e_{mech,\,in} - e_{mech,\,out} = \frac{P}{\rho} - 0 = gh = (9.81\ \text{m/s}^2)(50\ \text{m})\left(\frac{1\ \text{kJ/kg}}{1000\ \text{m}^2/\text{s}^2}\right) = 0.491\ \text{kJ/kg}$$

Then the rate at which mechanical energy is supplied to the turbine by the fluid and the overall efficiency become

$$|\Delta \dot{E}_{mech,\,fluid}| = \dot{m}(e_{mech,\,in} - e_{mech,\,out}) = (5000\ \text{kg/s})(0.491\ \text{kJ/kg}) = 2455\ \text{kW}$$

$$\eta_{overall} = \eta_{turbine-gen} = \frac{\dot{W}_{elect,\,out}}{|\Delta \dot{E}_{mech,\,fluid}|} = \frac{1862\ \text{kW}}{2455\ \text{kW}} = \mathbf{0.76}$$

(***b***) Knowing the overall and generator efficiencies, the mechanical efficiency of the turbine is determined from

$$\eta_{turbine-gen} = \eta_{turbine}\,\eta_{generator} \rightarrow \eta_{turbine} = \frac{\eta_{turbine-gen}}{\eta_{generator}} = \frac{0.76}{0.95} = \mathbf{0.80}$$

(***c***) The shaft power output is determined from the definition of mechanical efficiency,

$$\dot{W}_{shaft,\,out} = \eta_{turbine}|\Delta \dot{E}_{mech,\,fluid}| = (0.80)(2455\ \text{kW}) = \mathbf{1964\ kW}$$

1.6 The linear momentum equation

Newton's second law for a system of mass m subjected to a net force $\vec{F}$ is expressed as

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V})$$

Where $m\vec{V}$ is the linear momentum of the system. Noting that both the density and velocity may change from point to point within the system, Newton's second law can be expressed more generally as

$$\sum \vec{F} = \frac{d}{dt}\int_{sys} \rho\vec{V}\, dV$$

where $\delta m = \rho\, dv$ is the mass of a differential volume element dv, and is its momentum. Therefore, Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called an *inertial coordinate system* or *inertial reference frame*. Accelerating systems such as aircraft during takeoff are best analyzed using non-inertial (or accelerating) coordinate systems fixed to the aircraft. Note that the above equation is a vector relation, and thus the quantities $\vec{F}$ and $\vec{V}$ have direction as well as magnitude.

The general form of the linear momentum equation that applies to fixed, moving, or deforming control volumes is obtained to be

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

In General:

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, dV + \int_{CS} \rho\vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

Note that the momentum equation is a *vector equation*, and thus each term should be treated as a vector. Also, the components of this equation can be resolved along orthogonal coordinates (such as *x*, *y*, and *z* in the Cartesian coordinate system) for convenience.

The above equation is exact for fixed control volumes, it is not always convenient when solving practical engineering problems because of the integrals. Instead, as we did for conservation of mass, we would like to rewrite the above equation in terms of average velocities and mass flow rates through inlets and outlets. In other words, our desire is to rewrite the equation in *algebraic* rather than *integral* form. In many practical applications, fluid crosses the boundaries of the control volume at one or more inlets and one or more outlets, and carries with it some momentum into or out of the control volume. For simplicity, we always draw our control surface such that it slices normal to the inflow or outflow velocity at each such inlet or outlet (Figure 3.16). The mass flow rate $\dot{m}$ into or out of the control volume across an inlet or outlet at which ρ is nearly constant is

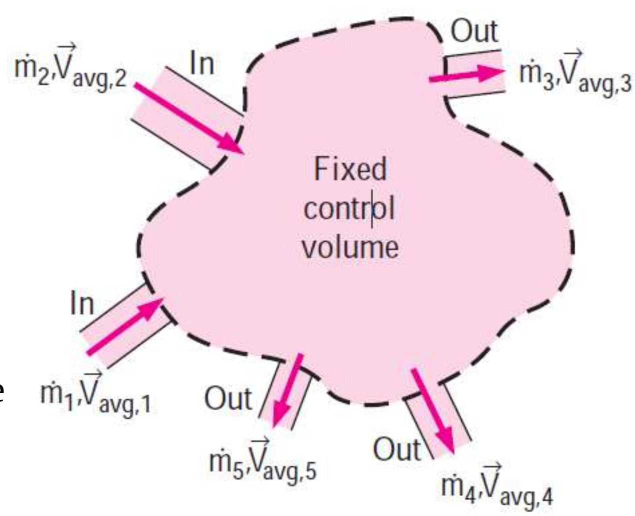


Figure 3.16: In a typical engineering problem, the control volume may contain many inlets and outlets; at each inlet or outlet we define the mass flow rate $\dot{m}$ and the average velocity V_{avg}.

Mass flow rate across an inlet or outlet:

$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n})\, dA_c = \rho V_{avg} A_c$$

Then we could write the rate of inflow or outflow of momentum through the inlet or outlet in simple algebraic form, Momentum flow rate across a uniform inlet or outlet:

$$\int_{A_c} \rho\vec{V}(\vec{V} \cdot \vec{n})\, dA_c = \rho V_{avg} A_c \vec{V}_{avg} = \dot{m}\vec{V}_{avg}$$

The uniform flow approximation is reasonable at some inlets and outlets, e.g., the well-rounded entrance to a pipe, the flow at the entrance to a wind tunnel test section, and a slice through a water jet moving at nearly uniform speed through air (Figure 3-17).

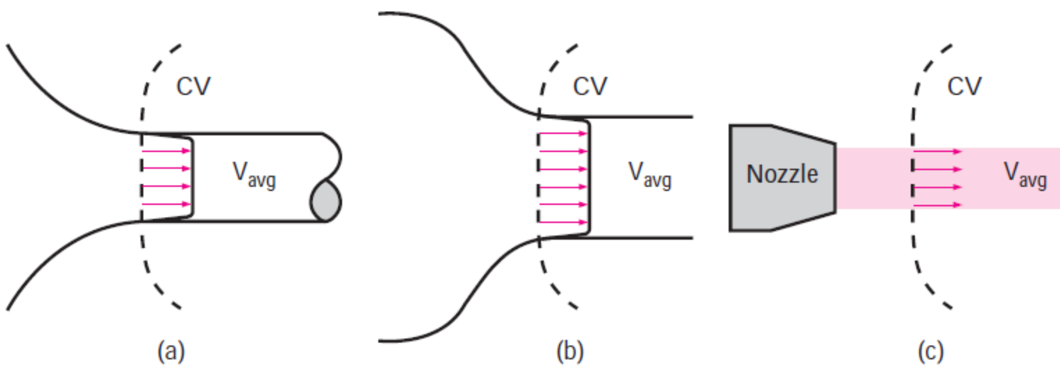


Figure 3.17: Examples of inlets or outlets in which the uniform flow approximation is reasonable: (a) the well-rounded entrance to a pipe, (b) the entrance to a wind tunnel test section, and (c) a slice through a free water jet in air.

1.7 Momentum-Flux Correction Factor, *β*

Unfortunately, the velocity across most inlets and outlets of practical engineering interest is not uniform. Nevertheless, it turns out that we can still convert the control surface integral of Equation,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}\cdot \vec{n})\, dA$$

into algebraic form, but a dimensionless correction factor b, called the momentum-flux correction factor, is required, as first shown by the French scientist *Joseph Boussinesq* (1842–1929). The algebraic form of the above equation for a fixed control volume is then written as,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

where a unique value of momentum-flux correction factor is applied to each inlet and outlet in the control surface. Note that β= 1 for the case of uniform flow over an inlet or outlet, as in Figure 3-17.

Momentum-flux correction factor:

$$\beta = \frac{1}{A_c}\int_{A_c} \left(\frac{V}{V_{avg}}\right)^2 dA_c$$

It turns out that for any velocity profile you can imagine, β is always greater than or equal to unity.

Example 9:

Consider laminar flow through a very long straight section of round pipe. The velocity profile through a cross-sectional area of the pipe is parabolic (Figure 3-18), with the axial velocity component given by

$$V = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

where *R* is the radius of the inner wall of the pipe and V_{avg} is the average velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Figure 3-18.

Solution:

We substitute the given velocity profile for *V* in the above equation and integrate, noting that $dAc = 2\pi r dr$,

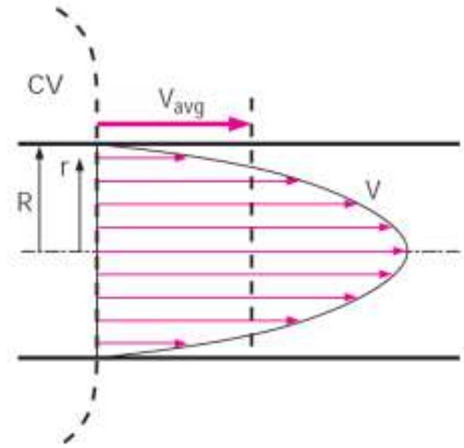


Figure 3.18: Velocity profile over a cross section of a pipe in which the flow is fully-developed and laminar.

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R\left(1-\frac{r^2}{R^2}\right)^2 2\pi r\, dr$$

Defining a new integration variable $y = 1 - r^2/R^2$ and thus $dy = -2r\,dr/R^2$ (also, $y = 1$ at $r = 0$, and $y = 0$ at $r = R$) and performing the integration, the momentum-flux correction factor for fully developed laminar flow becomes

Laminar flow: $$\beta = -4\int_1^0 y^2\,dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3}$$

Notice: For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected. It is wise to include β in all momentum control volume problems.

1.8 Steady Flow

If the flow is also steady, the time derivative term in Equation:

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

vanishes and we are left with,

Steady linear momentum equation: $\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$

where we dropped the subscript "*avg*" from average velocity. Above Equation states that the net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows. This statement is illustrated in Figure 3.19. It can also be expressed for any direction, since above equation is a vector equation.

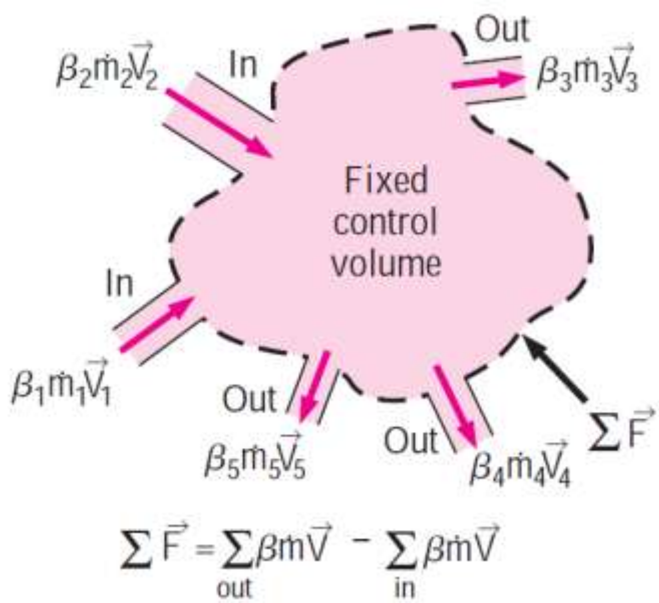


Figure 3.19: Velocity profile over a cross section of a pipe in which the flow is fully-developed and laminar.

Steady Flow with One Inlet and One Outlet: Many practical problems involve just one inlet and one outlet (Figure 3.20). The mass flow rate for such single-stream systems remains constant, and above equation reduces to,

One inlet and one outlet: $\sum \vec{F} = \dot{m}\,(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1)$

Example 10:

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it as shown in figure 3.20. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm2 at the inlet and 7 cm2 at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (**a**) the gage pressure at the center of the inlet of the elbow and (**b**) the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be β= 1.03.

Solution:

(**a**) We take the elbow as the control volume and designate the inlet by ① and the outlet by ②. We also take the *x*- and *z*-coordinates as shown.

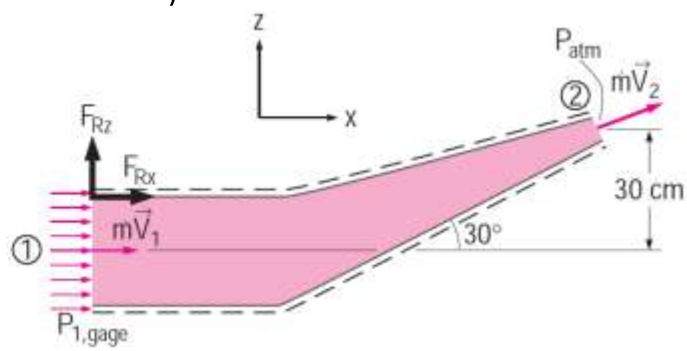


Figure 3.20: Schematic for Example 10.

The continuity equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m} = 14$ kg/s. Noting that $\dot{m} = \rho A V$, the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{atm} = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2) \times \left(\frac{(20 \text{ m/s})^2 - (1.24 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0.3 - 0\right)\left(\frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2}\right)$$

$$P_{1,\text{ gage}} = 202.2 \text{ kN/m}^2 = \mathbf{202.2 \text{ kPa}} \quad \text{(gage)}$$

(**b**) The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$$

We let the x- and z-components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the x- and z-axes become

$$F_{Rx} + P_{1,\,gage} A_1 = \beta \dot{m} V_2 \cos\theta - \beta \dot{m} V_1$$
$$F_{Rz} = \beta \dot{m} V_2 \sin\theta$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned} F_{Rx} &= \beta \dot{m}(V_2 \cos\theta - V_1) - P_{1,\,gage} A_1 \\ &= 1.03(14\ \text{kg/s})[(20 \cos 30^\circ - 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) \\ &\quad - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= 232 - 2285 = \mathbf{-2053\ N} \end{aligned}$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta = (1.03)(14\ \text{kg/s})(20 \sin 30^\circ\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{144\ N}$$

Example 11:

A reversing elbow such that the fluid makes a 180° *U-turn* before it is discharged, as shown in Figure 3.21. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be β= 1.03.

V_2= 20 m/s

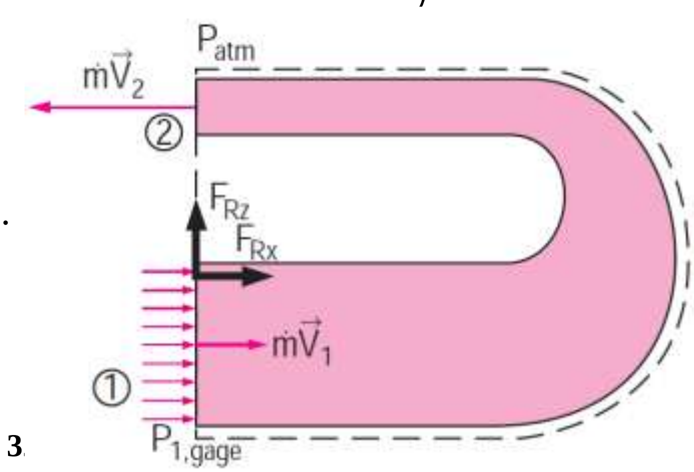


Figure 3.21: Schematic for Example 11.

V_1= 1.204 m/s

$P_{1,\,gage}$= 2022oo Pa

A_1= 0.0113 m^2

Solution:

The vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case (F_{Rz}= 0) since there is no other force or momentum flux in the vertical direction.

$$F_{Rx} + P_{1,\,gage}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for F_{Rx} and substituting the known values,

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\,gage}A_1$$

$$= -(1.03)(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\ N}$$

Noting that the outlet velocity is negative since it is in the negative x-direction. Therefore, the horizontal force on the flange is 2591 N acting in the negative x-direction (the elbow is trying to separate from the pipe).

Example 12:

Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s (Figure 3.22). After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream. Take the momentum-flux correction factor to be β= 1.

Solution:

The momentum equation for steady one-dimensional flow is given as,

$$\sum \vec{F} = \sum_{out} \beta\dot{m}\vec{V} - \sum_{in} \beta\dot{m}\vec{V}$$

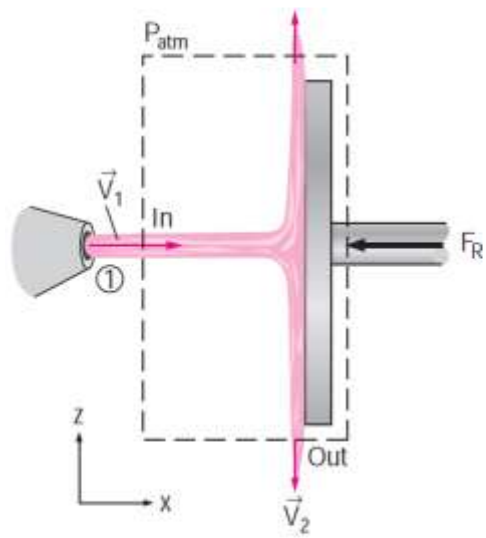


Figure 3.22: Schematic for Example 12.

Writing it for this problem along the *x*-direction (without forgetting the negative sign for forces and velocities in the negative *x*-direction) and noting that $V_{1,x} = V_1$ and $V_{2,x} = 0$ gives,

$-F_R = 0 - \beta \dot{m}\vec{V}_1$ Substituting the given values,

$$F_R = \beta \dot{m}\vec{V}_1 = (1)(10\ \text{kg/s})(20\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{200\ N}$$

Example 13:

A wind generator with a 30-ft-diameter blade span has a cut-in wind speed (minimum speed for power generation) of 7 mph, at which velocity the turbine generates 0.4 kW of electric power (Figure 3–23). Determine (**a**) the efficiency of the wind turbine–generator unit and (**b**) the horizontal force exerted by the wind on the supporting mast of the wind turbine. What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same, and take the density of air to be 0.076 lbm/ft^3. Take the momentum-flux correction factor to be β= 1.

Solution:

The power potential of the wind is proportional to its kinetic energy, which is $V^2/2$ per unit mass, and thus the maximum power is $\dot{m}V^2/2$ for a given mass flow rate:

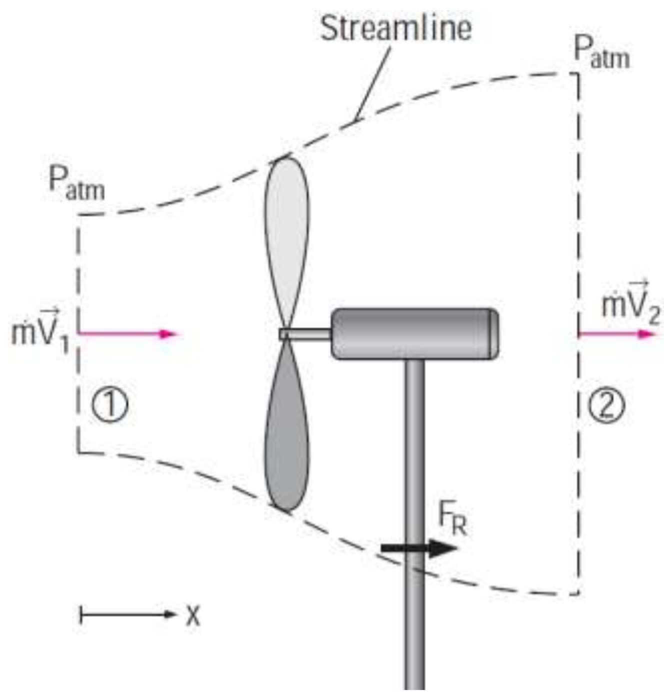


Figure 3.23: Schematic for Example 13.

$$V_1 = (7\text{ mph})\left(\frac{1.4667\text{ ft/s}}{1\text{ mph}}\right) = 10.27\text{ ft/s}$$

$$\dot{m} = \rho_1 V_1 A_1 = \rho_1 V_1 \frac{\pi D^2}{4} = (0.076\text{ lbm/ft}^3)(10.27\text{ ft/s})\frac{\pi(30\text{ ft})^2}{4} = 551.7\text{ lbm/s}$$

$$\dot{W}_{max} = \dot{m}ke_1 = \dot{m}\frac{V_1^2}{2}$$

$$= (551.7\text{ lbm/s})\frac{(10.27\text{ ft/s})^2}{2}\left(\frac{1\text{ lbf}}{32.2\text{ lbm}\cdot\text{ft/s}^2}\right)\left(\frac{1\text{ kW}}{737.56\text{ lbf}\cdot\text{ft/s}}\right)$$

$$= 1.225\text{ kW}$$

Therefore, the available power to the wind turbine is 1.225 kW at the wind velocity of 7 mph. Then the turbine–generator efficiency becomes

$$\eta_{\text{wind turbine}} = \frac{\dot{W}_{act}}{\dot{W}_{max}} = \frac{0.4\text{ kW}}{1.225\text{ kW}} = \mathbf{0.327}$$

Noting that the mass flow rate remains constant, the exit velocity is determined to be

$$\dot{m}ke_2 = \dot{m}ke_1(1 - \eta_{\text{wind turbine}}) \quad \rightarrow \quad \dot{m}\frac{V_2^2}{2} = \dot{m}\frac{V_1^2}{2}(1 - \eta_{\text{wind turbine}})$$

$$V_2 = V_1\sqrt{1 - \eta_{\text{wind turbine}}} = (10.27\text{ ft/s})\sqrt{1 - 0.327} = 8.43\text{ ft/s}$$

The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{out} \beta\dot{m}\vec{V} - \sum_{in} \beta\dot{m}\vec{V} \qquad F_R = \dot{m}V_2 - \dot{m}V_1 = \dot{m}(V_2 - V_1)$$

Substituting the known values gives

$$F_R = \dot{m}(V_2 - V_1) = (551.7\text{ lbm/s})(8.43 - 10.27\text{ ft/s})\left(\frac{1\text{ lbf}}{32.2\text{ lbm}\cdot\text{ft/s}^2}\right)$$

$$= -31.5\text{ lbf}$$

Then the force exerted by the wind on the mast becomes $F_{mast} = -F_R = 31.5$ lbf.