## 5.6 Theory of Plastic Analysis

The basic plastic theory has been shown to be a major change in the distribution of stresses after the stresses at certain points in a structure reach the yield stress. The theory is that those parts of the structure that have been stressed to the yield stress cannot resist additional stresses. They instead will yield the amount required to permit the extra load or stresses to be transferred to other parts of the structure where the stresses are below the yield stress, and thus in the elastic range and able to resist increased stress. Plasticity can be said to serve the purpose of equalizing stresses in cases of overload.

For this discussion, the stress–strain diagram is assumed to have the idealized shape shown in Fig. 4-6-1. The yield stress and the proportional limit are assumed to occur at the same point for this steel, and the stress–strain diagram is assumed to be a perfectly straight line in the plastic range. Beyond the plastic range there is a range of strain hardening. This latter range could theoretically permit steel members to withstand additional stress, but from a practical standpoint the strains which arise are so large that they cannot be considered. Furthermore, inelastic buckling will limit the ability of a section to develop a moment greater than $M_p$, even if strain hardening is significant.

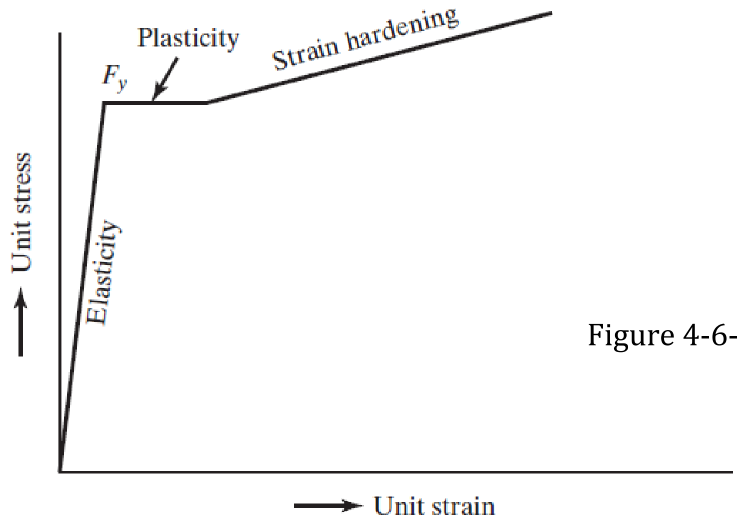



Figure 4-6-1

## 5.7 The Collapse Mechanism

- ✓ A statically determinate beam will fail if one plastic hinge develops.To illustrate this fact, the simple beam of constant cross section loaded with a concentrated load at midspan, shown in Fig. 4-7-1 (a), is considered. Should the load be increased until a plastic hinge is developed at the point of maximum moment (underneath the load in this case), an unstable structure will have been created, as shown in part (b) of the figure. Any further increase in load will cause collapse. $P_n$ represents the nominal, or theoretical, maximum load that the beam can support.

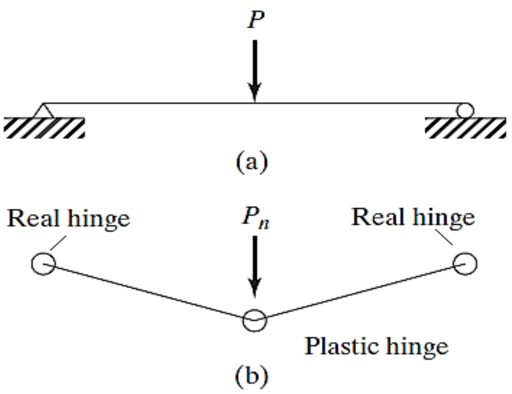



Figure 4-7-1

- ✓ For a statically indeterminate structure to fail, it is necessary for more than one plastic hinge to form. The number of plastic hinges required for failure of statically indeterminate structures will be shown to vary from structure to structure, but may never be less than two. The fixed-end beam of Fig. 4-7-2, part (a), cannot fail unless the three plastic hinges shown in part (b) of the figure are developed.

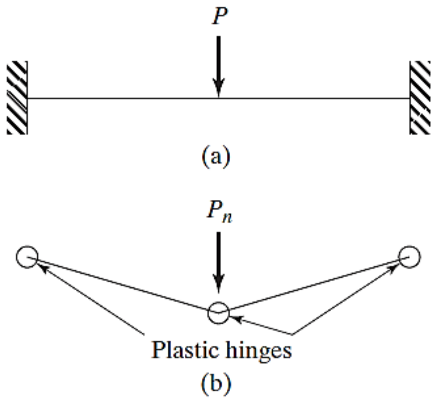


Figure 4-7-2

- ✓ Although a plastic hinge may have formed in a statically indeterminate structure, the load can still be increased without causing failure if the geometry of the structure permits.
- ✓ The plastic hinge will act like a real hinge insofar as increased loading is concerned.
- ✓ As the load is increased, there is a redistribution of moment, because the plastic hinge can resist no more moment.
- ✓ As more plastic hinges are formed in the structure, there will eventually be a sufficient number of them to cause collapse.
- ✓ Actually، some additional load can be carried after this time, before collapse occurs, as the stresses go into the strain hardening range, but the deflections that would occur are too large to be permissible.

The propped beam of Fig. 4-7-3, part (a), is an example of a structure that will fail after two plastic hinges develop. Three hinges are required for collapse, but there is a real hinge on the right end. In this beam, the largest elastic moment caused by the design concentrated load is at the fixed end. As the magnitude of the load is increased, a plastic hinge will form at that point.

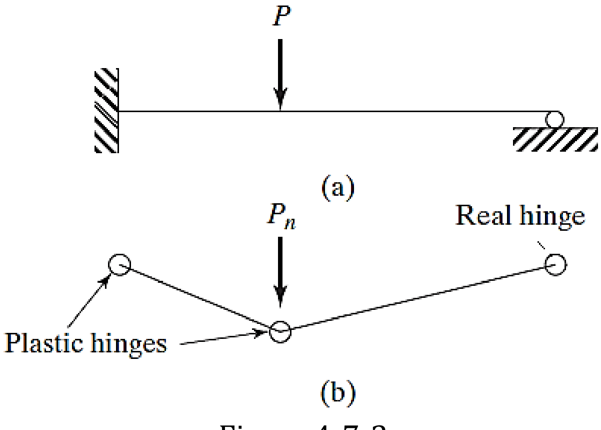


Figure 4-7-3

- ✓ The load may be further increased until the moment at some other point (here it will be at the concentrated load) reaches the plastic moment.
- ✓ Additional load will cause the beam to collapse. The arrangement of plastic hinges and perhaps real hinges that permit collapse in a structure is called the ***mechanism***. Parts (b) of Figs. 4-7-1, 4-7-2, and 4-7-3 show mechanisms for various beams.

## 5.8 The Virtual-Work Method

One very satisfactory method used for the plastic analysis of structures is the ***virtualwork method***.

- ✓ The structure in question is assumed to be loaded to its nominal capacity $M_n$, and is then assumed to deflect through a small additional displacement after the ultimate load is reached.
- ✓ The work performed by the external loads during this displacement is equated to the internal work absorbed by the hinges. For this discussion, the ***small-angle theory*** is used.
- ✓ By this theory, the sine of a small angle equals the tangent of that angle and also equals the same angle expressed in radians. In the pages to follow, the author uses these values interchangeably because the small displacements considered here produce extremely small rotations or angles.

**The uniformly loaded fixed-ended beam Fig. 4-8-1.**

This beam and its collapse mechanism are shown. Owing to symmetry, the rotations at the end plastic hinges are equal, and they are represented by in the figure؛ thus, the rotation at the middle plastic hinge will be $2\theta$.

The work performed by the total external load ($w_nL$) is equal to $w_nL$ times the ***average deflection*** of the mechanism. The average deflection equals one-half the deflection at the center plastic hinge **(1/2 x θ x L/2).**

The external work is equated to the internal work absorbed by the hinges, or to the sum of $M_n$ at each plastic hinge times the angle through which it works. The resulting expression can be solved for $M_n$ and $w_n$ as follows:

$$M_n(\theta + 2\theta + \theta) = w_nL\left(\frac{1}{2} \times \theta \times \frac{L}{2}\right)$$

$$M_n = \frac{w_nL^2}{16}$$

$$w_n = \frac{16M_n}{L^2}.$$

For the 18-ft span these values become

$$M_n = \frac{(w_n)(18)^2}{16} = 20.25\, w_n$$

$$w_n = \frac{M_n}{20.25}$$

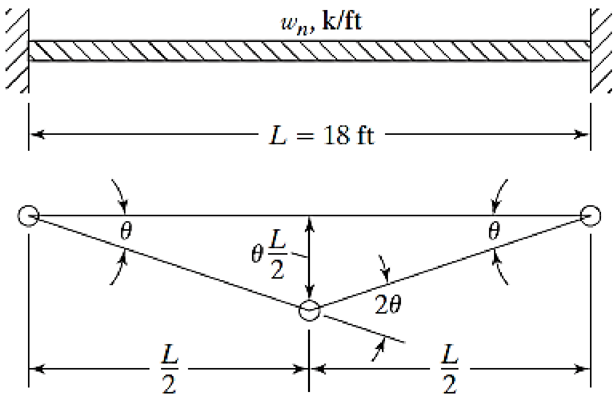



Figure 4-8-1

Plastic analysis can be handled in a similar manner for the propped beam of Fig. 4-8-2. There, the collapse mechanism is shown, and the end rotations (which are equal to each other) are assumed to equal $\theta$.

The work performed by the external load $P_n$ as it moves through the distance **(θ x L/2)** is equated to the internal work performed by the plastic moments

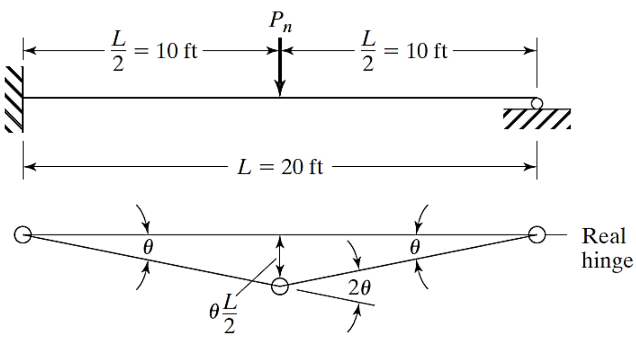



Figure 4-8-2

at the hinges; note that there is no moment at the real hinge on the right end of the beam.

$$M_n(\theta + 2\theta) = P_n\left(\theta \frac{L}{2}\right)$$

$$M_n = \frac{P_n L}{6} \quad (\text{or } 3.33P_n \text{ for the 20-ft beam shown})$$

$$P_n = \frac{6M_n}{L} \quad (\text{or } 0.3M_n \text{ for the 20-ft beam shown})$$

The fixed-end beam of Fig. 4-8-3, together with its collapse mechanism and assumed angle rotations, is considered next. From this figure, the values of $\boldsymbol{M_n}$ and $\boldsymbol{P_n}$ can be determined by virtual work as follows:

$$M_n(2\theta + 3\theta + \theta) = P_n\left(2\theta \times \frac{L}{3}\right)$$

$$M_n = \frac{P_n L}{9} \quad (\text{or } 3.33P_n \text{ for this beam})$$

$$P_n = \frac{9M_n}{L} \quad (\text{or } 0.3M_n \text{ for this beam}).$$

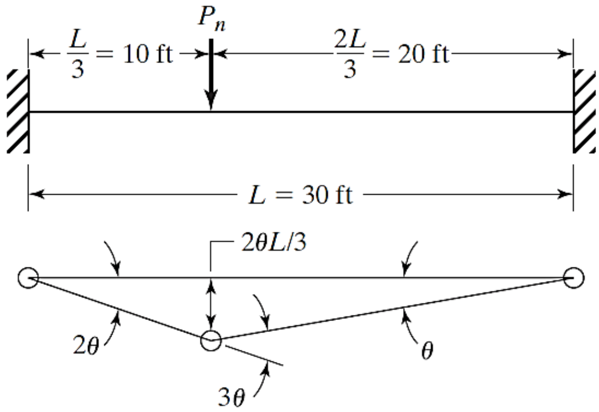


Figure 4-8-3

The plastic analysis of the propped beam of Fig. 4-8-4 is done by the virtual-work method. The beam with its two concentrated loads is shown, together with four possible collapse mechanisms and the necessary calculations. It is true that the mechanisms of parts (b), (d), and (e) of the figure do not control, but such a fact is not obvious to the average student until he or she makes the virtual-work calculations for each case. Actually, the mechanism of part (e) is based on the assumption that the plastic moment is reached at both of the concentrated loads simultaneously (a situation that might
very well occur).

**Note**: The value for which the collapse load $\boldsymbol{P_n}$ is the smallest in terms of $\boldsymbol{M_n}$ is the correct value (or the value where $\boldsymbol{M_n}$ is the greatest in terms of $\boldsymbol{P_n}$ ). For this beam, the second plastic hinge forms at the $\boldsymbol{P_n}$ concentrated load, and $\boldsymbol{P_n}$ equals **0.154** $\boldsymbol{M_n}$**.**

**5.9**

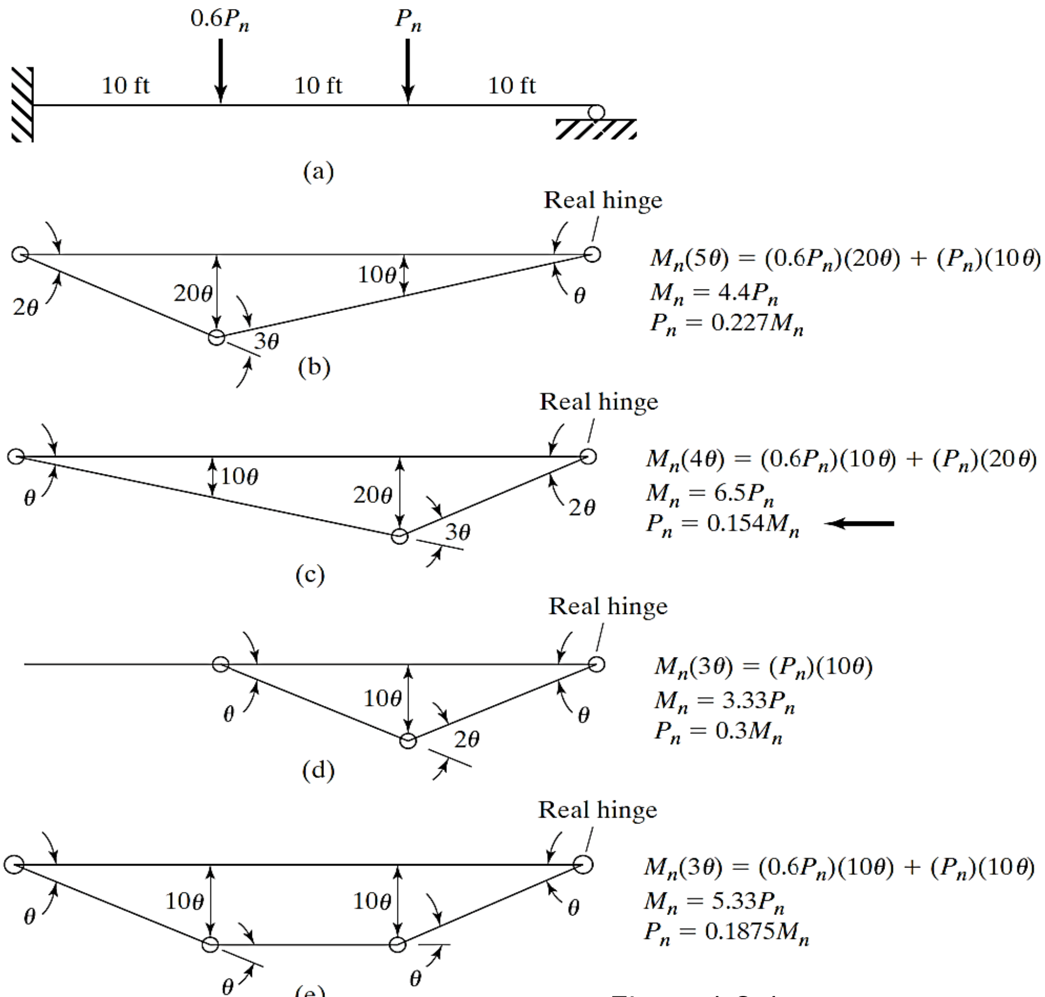



Figure 4-8-4

## 5.10 Location of Plastic Hinge for Uniform Loadings

There was no difficulty in locating the plastic hinge for the uniformly loaded fixed-end beam, but for other beams with uniform loads, such as propped or continuous beams, the problem may be rather difficult.

The elastic moment diagram for this beam is shown as the solid line in part (b) of the figure. As the uniform load is increased in magnitude, a plastic hinge will first form at the fixed end. At this time, the beam will, in effect, be a "simple" beam (so far as increased loads are concerned) with a plastic hinge on one end and a real hinge on the other. Subsequent increases in the load will cause the moment to change, as represented by the dashed line in part (b) of the figure. This process will continue until the moment at some other point (a distance *x* from the right support in the figure) reaches $\boldsymbol{M_n}$ and creates another plastic hinge.

The virtual-work expression for the collapse mechanism of the beam shown in part (c) of Fig. 4-9-1 is written as follows:

$$M_n\left(\theta+\theta+\frac{L-x}{x}\theta\right)=\left(w_nL\right)\left(\theta\right)\left(\mathrm{L}-\mathrm{x}\right)\left(\frac{1}{2}\right)$$

Solving this equation for $\boldsymbol{M_n}$, taking $\boldsymbol{dM_n/dx = 0}$, the value of $\boldsymbol{x}$ can be calculated to equal **0.414*L***.This value is also applicable to uniformly loaded end spans of continuous beams with simple end supports.

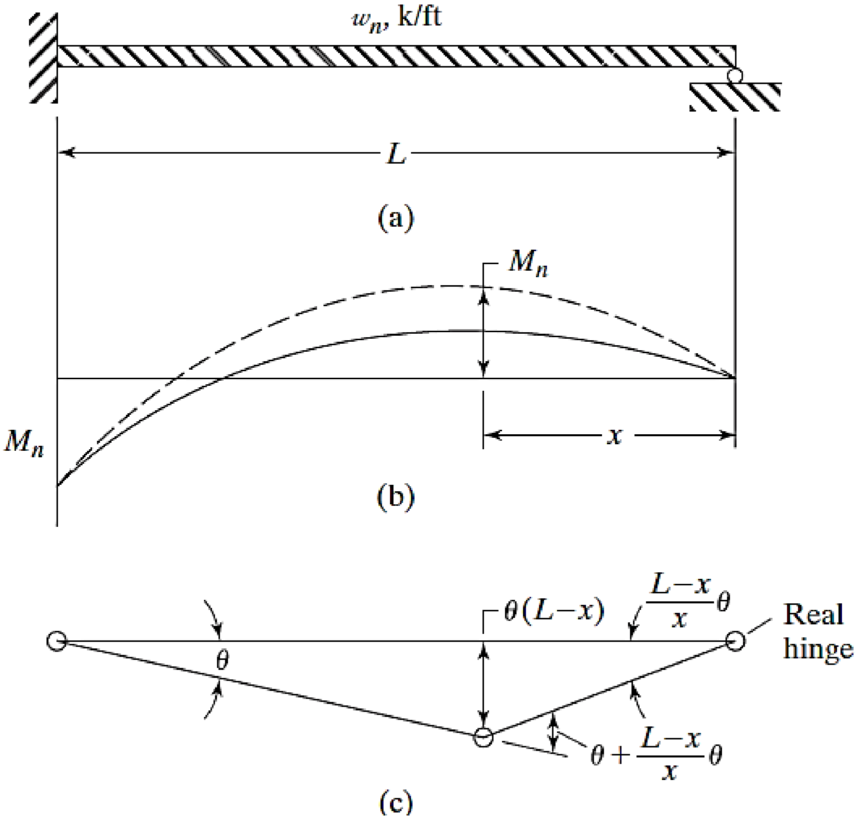



Figure 4-9-1

The beam and its collapse mechanism are redrawn in Fig. 4-9-2, and the following expression for the plastic moment and uniform load are written by the virtual-work procedure:

$$M_n(\theta + 2.414\theta) = (w_nL)(0.586\theta L)\left(\frac{1}{2}\right)$$

$$M_n = 0.0858 w_n L^2$$

$$w_n = 11.65\frac{M_n}{L^2}$$

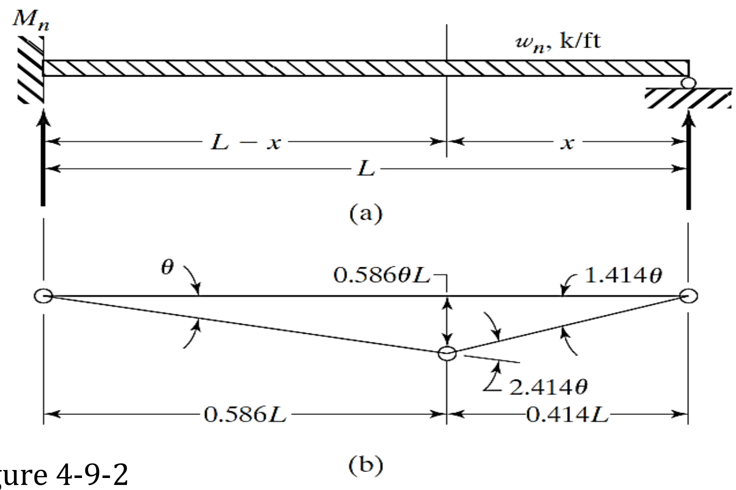



Figure 4-9-2

**Example 4-2**

A **W18 x 55** has been selected for the beam shown in Fig. 4-2.Using **50 ksi** steel and assuming full lateral support, determine the value of $w_n$.

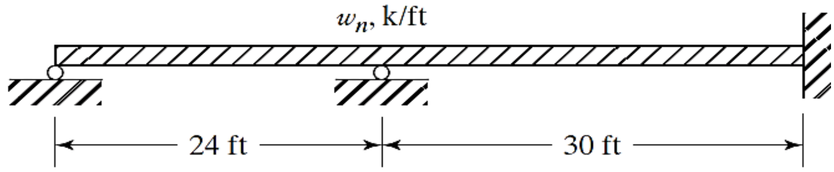


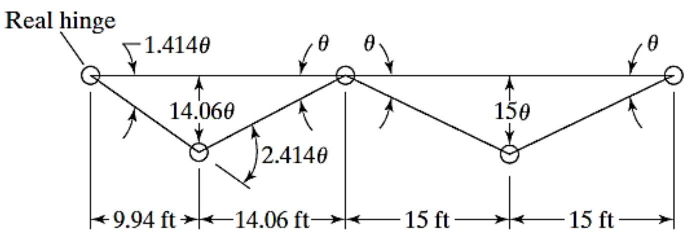

Figure 4-2

***Solution***

From the **Table 3-2** of the AISC Manual. $Z_x$ **= 112 in³**

$$M_n = F_y Z = \frac{(50 \text{ ksi})(112 \text{ in}^3)}{12 \text{ in/ft}} = 466.7 \text{ ft-k}$$

Drawing the (collapse) mechanisms for the two spans:




Left-hand span:

$$(M_n)(3.414\theta) = (24w_n)\left(\frac{1}{2}\right)(14.06\theta)$$

$$w_n = 0.0202\ M_n = (0.0202)(466.7) = 9.43 \text{ k/ft}$$

Right-hand span:

$$(M_n)(4\theta) = (30w_n)\left(\frac{1}{2}\right)(15\theta)$$

$$w_n = 0.0178\ M_n = (0.0178)(466.7) = 8.31 \text{ k/ft} \leftarrow$$

**Example 4-3**

A **W12 x 72** is used for the beam and columns of the frame shown in Fig. 4-3. If **Fy = 50 ksi,** determine the value of $P_n$.

***Solution***

The virtual-work expressions are written for parts (b), (c), and (d) of Fig. 4-3 and shown with the respective parts of the figure. The combined beam and sidesway case is found to be the critical case, and from it, the value of $P_n$ is determined as follows:

$Z_x$ **= 108 in³**

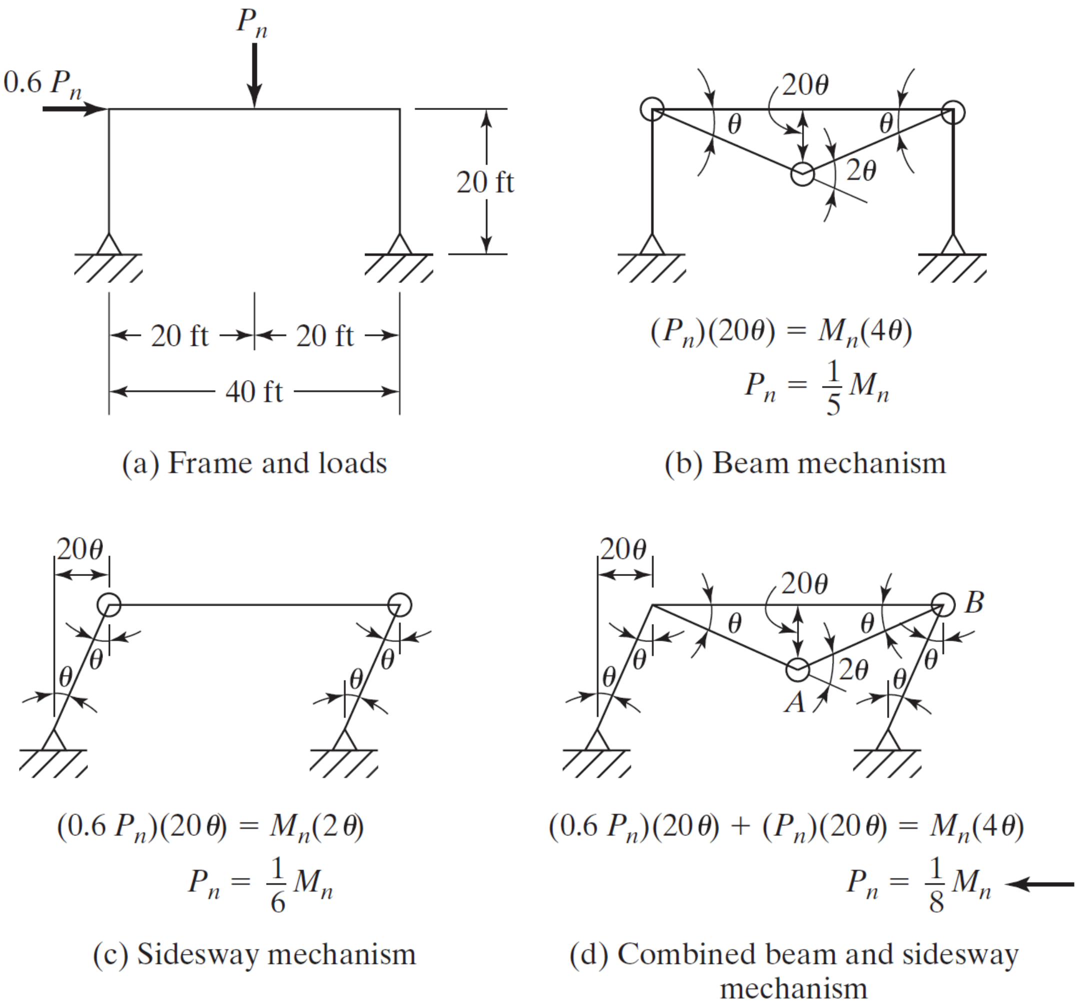



Figure 4-3

$$P_n = \frac{1}{8}M_n = \left(\frac{1}{8}\right)\left(F_y Z\right) = \left(\frac{1}{8}\right)\left(\frac{50x\,108}{12}\right) = 56.25\,\text{k}$$