5. Probability and Counting Rules:

5.1 introduction

The concept of probability originated from the analysis of the games of chance. Even today, a large number of problems exist which are based on the games of chance, such as tossing of a coin, throwing of dice, and playing of cards. Probability is a concept which measures the degree of uncertainty and that of certainty as a corollary. The word probability or ‘chance’ is used commonly in day-to-day life**. It can be defined as the chance of an event occurring.**

5.1. Sample Spaces and Probability

The theory of probability grew out of the study of various games of chance using coins, dice, and cards. Since these devices lend themselves well to the application of concepts of probability, they will be used in this lecture as examples. This section begins by explaining some basic concepts of probability. Then the types of probability and probability rules are discussed.

- **Basic Concepts**
- **Probability experiment** is a chance process that leads to well-defined results called outcomes. such as flipping a coin, rolling a die, or drawing a card from a deck
- **Outcome** is the result of a single trial of a probability experiment. For example, rolling a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6.
- S**ample space** is the set of all possible outcomes of a probability experiment. Show the table below

Experiment	**Sample space**
Toss one coin	Head, tail
Roll a die	1, 2, 3, 4, 5, 6
Answer a true/false question	True, false
Toss two coins	Head-head, tail-tail, head-tail, tail-head

Example 1: Find the sample space for rolling two dice.

Solution

Since each die can land in six different ways, and two dice are rolled, the sample space can be presented by a rectangular array, as shown figure below: -

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Example 2: Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

Solution

There are two genders, male and female, and each child could be either gender. Hence, there are eight possibilities, as shown here.

BBB BBG BGB GBB GGG GGB GBG BGG

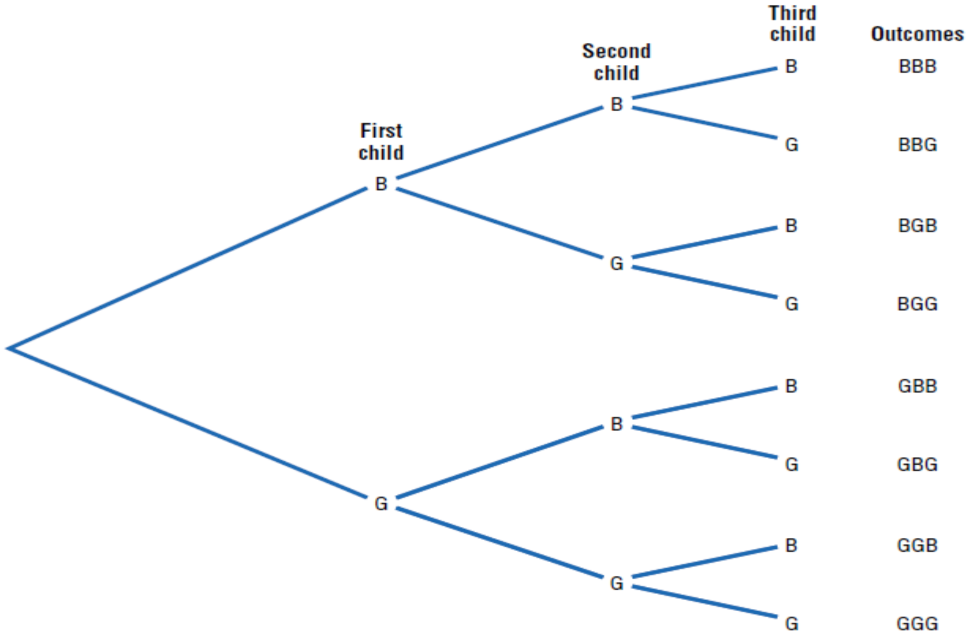
- Tree diagram

It is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.

Example 3: Use a tree diagram to find the sample space for the gender of three children in a family, as in Example 2.

<u>Solution:</u>

Since there are two possibilities (boy or girl) for the first child, draw two branches from a starting point and label one B and the other G. Then if the first child is a boy, there are two possibilities for the second child (boy or girl), so draw two branches from B and label one B and the other G. Do the same if the first child is a girl. Follow the same procedure for the third child. The completed tree diagram is shown in the figure below. To find the outcomes for the sample space, trace through all the possible branches, beginning at the starting point for each one.



- **An event** consists of a set of outcomes of a probability experiment. Event can be one or more outcomes, for example a face from one trial dice is called **simple event**, or odd number from a single trial is called **compound event**.

  Single Trial: 1 2 3 4 5 6

There are three basic interpretations of probability:

1. Classical probability

2. Empirical or relative frequency probability

3. Subjective probability

1- Classical Probability

- Classical probability assumes that all outcomes in the sample space are equally likely to occur. For example, when a single die is rolled, each outcome has the same probability of occurring which is (1/6) and for coin (1/2) and so on.

- The probability of any event E can be defined as:

$$\frac{\textbf{Number of outcomes in } \boldsymbol{E}}{\textbf{Total number of outcomes in the sample space}}$$

Or

$$\boldsymbol{P(E) = \frac{n(E)}{n(s)}}$$

Or

$$\boldsymbol{P(E) = \frac{n}{N}}$$

Probabilities can be expressed as fractions, decimals, or where appropriate percentages. If you ask, “What is the probability of getting a head when a coin is tossed?”. typical responses can be any of the following three.

“One-half.”

“Point five.”

“Fifty percent.”

Example 4: If a family has three children, find the probability that two of the three children are girls.

Solution:

The sample space = N = n(S) = 8: (BBB BBG BGB GBB GGG GGB GBG BGG)

The outcomes space = E = n(E) = n(2G) = n = 3: **(GGB GBG BGG)**

$$\boldsymbol{P(E) = \frac{n(E)}{n(s)} = \frac{n(2G)}{n(s)} = \frac{3}{8}}$$

Example 5: When a single die is rolled, find the probability of getting a 9**.**

Solution :

The sample space = N = n(S) = 6: **(1, 2, 3, 4, 5, 6)**

The outcomes space = E = n(E) = n(9) = n = 0: [ ]

$$\boldsymbol{P(E) = \frac{n(E)}{n(s)} = \frac{n(9)}{n(s)} = \frac{0}{6} = 0}$$

Example 6: When a single die is rolled, find the probability of getting an odd number.

Solution:

The sample space = N = n(S) = 6: **(1, 2, 3, 4, 5, 6)**

The outcomes space = E = n(Odd) = n = 3 = **(1, 3, 5)**

$$P(E) = \frac{n(E)}{n(s)} = \frac{n(odd)}{n(s)} = \frac{3}{6} = 0.5$$

❖ **Basic Probability Rules**

- **Probability Rule 1:**

The probability of any event *E* is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by: $\mathbf{0 \le P(E) \le 1}$

- **Probability Rule 2**

If an event E cannot occur (i.e., the event contains no members in the sample space), **its probability is 0**.

- **Probability Rule 3**

If an event E is certain, then the probability of **E is 1**.

- **Probability Rule 4**

The sum of the probabilities of all the outcomes in **the sample space is 1**.

Example 7: A single die is rolled, what is the probability of getting a number less than 7?

Solution

Since all outcomes (1, 2, 3, 4, 5, 6) are less than 7,

the probability is: P(X < 7) = 1 $P(X < 7) = \frac{n}{N} = \frac{6}{6} = 1$

The event of getting a number less than 7 is certain.

➢ **Complement of an event**

If *E* is the set of outcomes in the sample space that are not included in the outcomes of event *E*. The complement of *E* is denoted by $\bar{E}$ (read "*E* bar").

$$P(E) + P(\bar{E}) = 1 \;\Rightarrow\; P(E) = 1 - P(\bar{E})$$

Example 8: If the probability that a person lives in an industrialized country of the world is, find the probability that a person does not live in an industrialized country.

Solution

P(not living in an industrialized country) =

1 - P(living in an industrialized country)=

$$1 - \frac{1}{5} = \frac{4}{5} \text{ OR } P(E) = 1 - P(\bar{E}) = 1 - \frac{1}{5} = \frac{4}{5}$$

2. Empirical Probability

Given a frequency distribution, the probability of an event being in a given class is

$$\boldsymbol{P(E)} = \frac{\textbf{\textit{Frequency for the class}}}{\textbf{\textit{Total frequency in the distribution table}}} = \frac{\boldsymbol{f_i}}{\boldsymbol{\sum f_i}}$$

This probability is called empirical probability and is based on observation.

Example 9: In the travel survey, as shown in Table below, find the probability that a person will travel by airplane over the holiday.

Method	Frequency
Drive	41
Fly	6
Train or bus	3
	50

Solution :

$\boldsymbol{P(E)} = \frac{\boldsymbol{f_i}}{\boldsymbol{\sum f_i}} = \frac{\mathbf{6}}{\mathbf{50}} = \frac{\mathbf{3}}{\mathbf{25}}$ is the probability of the person traveling by fly**.**

Example 10: In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities. **a.** A person has type O blood.

b. A person has type A or type B blood. **c.** A person has neither type A nor type O blood. **d.** A person does not have type AB blood.

Type	A	B	AB	O	Total
Frequency	22	5	2	21	50

Solution

a. $P(O) = \frac{f_i}{\sum f_i} = \frac{21}{50}$

b. $P(A\ or\ B) = \frac{f_i}{\sum f_i} = P(A) + P(B) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

c. $P(neither\ A\ nor\ O) = P(B\ and\ AB) = \left(P(AB) + P(B)\right)$

$$= \frac{2}{50} + \frac{5}{50} = \frac{7}{50}$$

d. $P(not\ AB) = 1 - P(AB) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$

Example 10: Hospital records indicated that knee replacement patients stayed in the hospital for the number of days shown in the distribution Table. Find these probabilities: -

a. A patient stayed exactly 5 days.
b. A patient stayed less than 6 days.
c. A patient stayed at most 4 days.
d. A patient stayed at least 5 days.

Number of days stayed	**3**	**4**	**5**	**6**	**7**	**Total**
Frequency	**15**	**32**	**56**	**19**	**5**	**127**

Solution :-

a. $P(5) = \frac{f_i}{\sum f_i} = \frac{56}{127}$

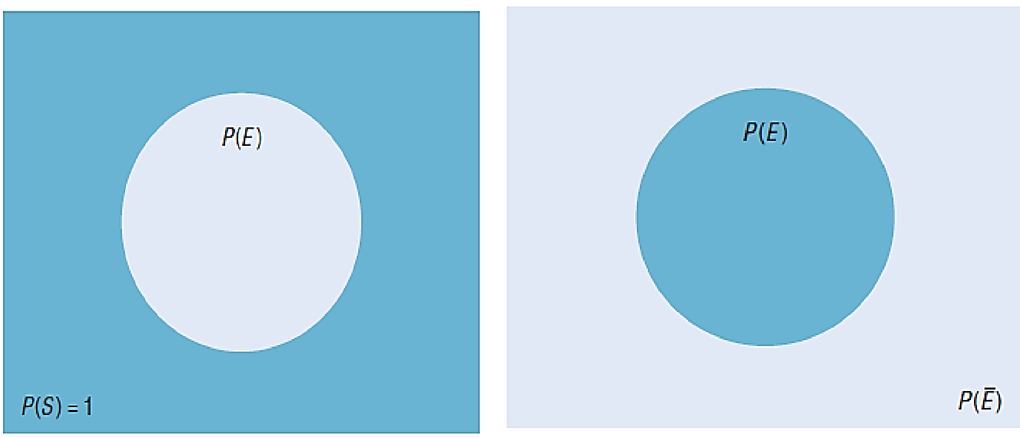
b. $P(fewer\ than\ 6\ days) = P(5) + P(4) + P(3) = \frac{56}{127} + \frac{32}{127} + \frac{15}{127} = \frac{103}{127}$

c. $P(at\ most\ 4\ days) = P(4) + P(3) = \frac{32}{127} + \frac{15}{127} = \frac{47}{127}$

d. $P(at\ least\ 5\ days) = P(5) + P(6) + P(7) = \frac{56}{127} + \frac{19}{127} + \frac{5}{127} = \frac{80}{127}$

- Venn Diagram

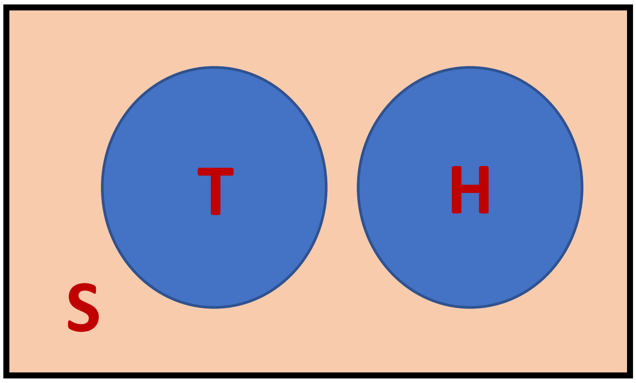
- It is an illustration that uses circles to show the relationships among things or finite groups of things.
- It is often useful to use a Venn diagram to visualize the probabilities of multiple events.



(a) Simple probability (b) $P(\bar{E}) = 1 - P(E)$

5.2. The Addition Rules for Probability:

Mutually exclusive events are two or more events which cannot occur at the same time (i.e., they have no outcomes in common). For example coin experiment (H or T), on trial dice (1, or 2 or………6)



5.2.1. Addition Rule 1

- When two events A and B are mutually exclusive, the probability that A or B will occur is**:**

$$P(A \textbf{ or } B) = P(A) + P(B)$$

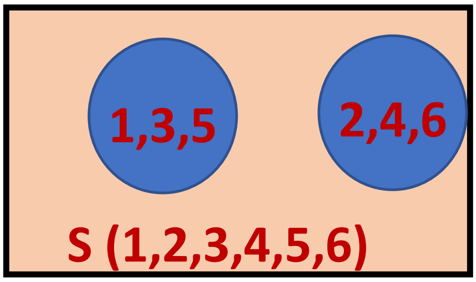
- More than two events:

$$P(A \textbf{ or } B \textbf{ or } C) = P(A) + P(B) + P(C)$$

Example 11: Determine which events are mutually exclusive and which are not, when a single die is rolled: (a) Getting an odd number and getting an even number; (b) Getting a 3 and getting an odd number; (c) Getting an odd number and getting a number less than 4; (d) Getting a number greater than 4 and getting a number less than 4.

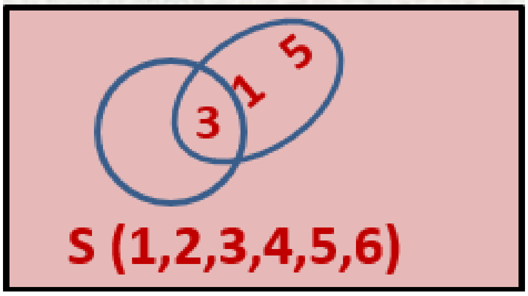
Solution

(a) The events are mutually exclusive, since the first event can be 1, 3, or 5 and the second event can be 2, 4, or 6.

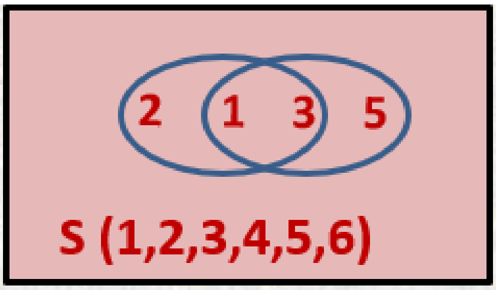


(a)

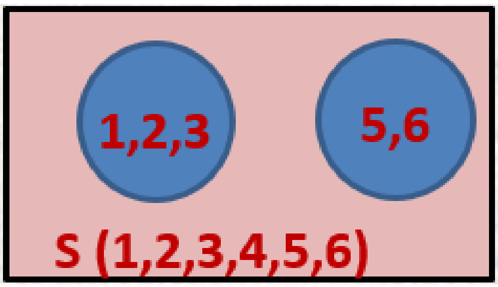
(b) The events are not mutually exclusive, since the first event is a 3 and the second can be 1, 3, or 5. Hence, 3 is contained in both events.



(c) The events are not mutually exclusive, since the first event can be 1, 3, or 5 and the second can be 1, 2, or 3. Hence, 1 and 3 are contained in both events.



(d) The events are mutually exclusive, since the first event can be 5 or 6 and the second event can be 1, 2, or 3.



Example 12: A city has 9 Steel factories: 3 high strength, 2 high carbon, and 4 recycled steel. If a contract selects one factory at random to buy tons of steel, find the probability that it is either high strength or recycled steel.

Solution

Since there are 3 high strength, and 4 recycled steel, and a total of 9 factories.

***P* (high strength(HS) or 4 recycled steel(RS)) = *P*(*HS*) + *P*(RS)**$=\frac{3}{9}+\frac{4}{9}=\frac{7}{9}$

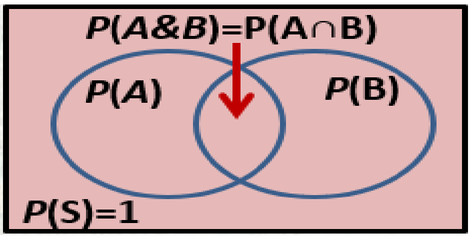
The events are mutually exclusive.

5.2.2. Addition Rule 2:

This rule can also be used when the events are mutually exclusive, since P (A and B) will always equal 0. However, it is important to make a distinction between the two situations.

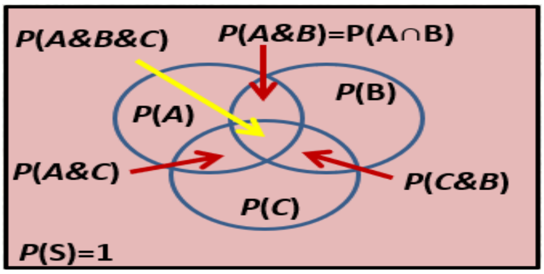
- If *A* and *B* are *not* mutually exclusive, then:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



- For three events that are *not* mutually exclusive,

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \,\&\, C) - P(B \,\&\, C) + P(A \,\&\, B \,\&\, C)$$

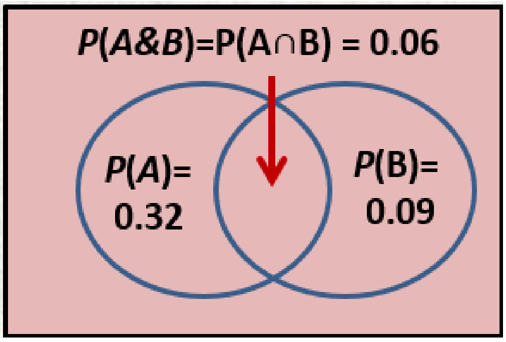


Example 13: The probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

Solution

$$P(\text{intoxicated or accident}) = P(\text{intoxicated}) + P(\text{accident}) - P(\text{intoxicated and accident})$$

$$P(\text{intoxicated or accident}) = 0.32 + 0.09 - 0.06 = 0.35$$



5.3. The Multiplication Rules and Conditional Probability:

- **The Multiplication Rules**

  - The *multiplication rules* can be used to find the probability of two or more events that occur in sequence (dependent and independent events).
  - Two events A and B are independent events if the fact that A occurs does not affect the probability of B occurring.
  - For example: Rolling a die and getting a 6, and then rolling a second die and getting a 3.

Multiplication Rule 1:

✓ When two events are independent, the probability of both occurring is:

$$\boldsymbol{P(A \text{ and } B) = P(A)\, .\, P(B)}$$

Example 14: A coin is flipped, and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

$P(\text{head and } 4) = P(\text{head})\ .\ P(4) = \frac{1}{2}.\frac{1}{6} = \frac{1}{12}$

Sample space: coin (T,H) and Die (1,2,3,4,5,6) (both are independent): ***[T1, T2, T3, T4,T5, T6, H1, H2, H3, H4, H5, H6] = 12***

Example 15: A box contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color noted. Then it is replaced. A second ball is selected, and its color noted. Find the probability of each of these.

a. Selecting 2 blue balls

b. Selecting 1 blue ball and then 1 white ball

c. Selecting 1 red ball and then 1 blue ball

Solution

$P(blue) = \frac{n}{N} = \frac{2}{10}$, $P(red) = \frac{3}{10}$ $P(white) = \frac{5}{10}$

a. *P(blue and blue) = P(blue).P(blue)*

$$= \frac{2}{10}\ .\ \frac{2}{10} = \frac{4}{100}$$ (B1B1, B1B2, B2B1, B2B2)

b. *P(blue and white) = P(blue).P(white)* $= \frac{2}{10}\ .\ \frac{5}{10} = \frac{1}{10}$

c. *P(red and blue) = P(red).P(blue)*

$$= \frac{3}{10}\ .\ \frac{2}{10} = \frac{6}{100}$$

Sample space: B1B1, B1B2, B2B1, B1R1, B1R2, B1R3, B1W1, B1W2, B1W3, B1W4, B1W5, B2B1, B2R1, B2B2, B2R3, B2W1, ……… R1B1, R1B2, R1R1, ……..W1B1, W1B2, W1W1, …..

- ✓ For three or more independent events by using the formula:

P(A and B and C and . . . and K) = P(A) . P(B) . P(C) . . . P(K)

Example 16: At a signalized intersection, three cars come one by one, at the end, they have to turn left or write, determine the probability of? a) RRR, b) LRL, c) 2L1R.?

Solution

Each car will turn left or write (**independent events) ????** .

P(R) = P(L) = $\frac{1}{2}$

Sample space = RRR, RRL, RLR, LRR, LLL, LLR, LRL, RLL = 8

a) P(RRR) = P(R). P(R) . P(R) = $\frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$

   (in traditional probability = $\frac{n}{N} = \frac{1}{8}$)

b) P(LRL) = P(L).P(R).P(L) = $\frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$

   (in traditional probability = $\frac{n}{N} = \frac{1}{8}$)

c) P(2L1R) = P(LLR)+P(LRL)+P(RLL)

$$= \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{3}{8}$$

(in traditional probability = $\boldsymbol{\frac{n}{N} = \frac{3}{8}}$)

Multiplication Rule 2

- When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be dependent events. For example, when one ball is drawn without replacement by one.
- When two events are dependent, the probability of both occurring is

$$P(A \text{ and } B) = P(A)\,.\,P(B|A)$$

Where ($P(B|A)$: the probability that event *B* occurs when event *A* has already occurred.

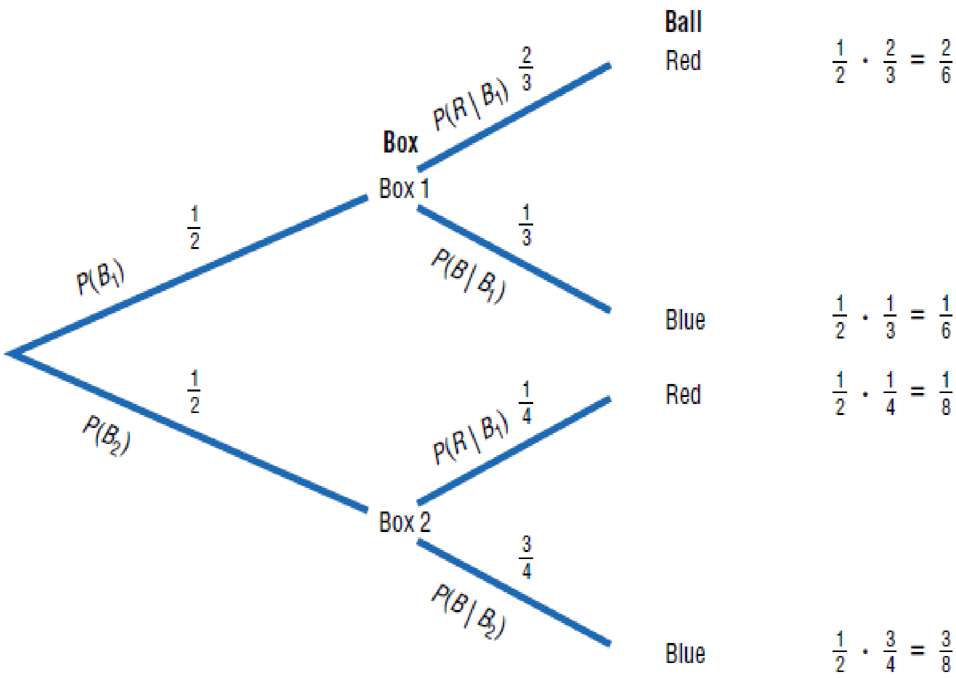
Example 16: At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004.

Solution

In this case, the events are dependent since the researcher wishes to investigate two distinct cases. Hence the first case is selected and not replaced.

$$P(C_1 \& C_2) = P(C_1).P(C_2|C_1) = \frac{16}{53}.\frac{15}{52} = \frac{60}{689}$$

Example 17: Box 1 contains 2 red balls and 1 blue ball. Box 2 contains 3 blue balls and 1 red ball. A coin is tossed. If it falls heads up, box 1 is selected and a ball is drawn. If it falls tails up, box 2 is selected and a ball is drawn. Find the tree probability.



- **Conditional Probability**

- The probability that the second event *B* occurs given that the first event *A* has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred. The formula is

$$P(B|A) = \frac{P(A\ and\ B)}{P(A)}$$

Example 18: A box has 6 red balls and 4 black balls, if two balls has been drawn one by one without replacement. Determine the probability of the second try is being red if the first is red as well?

Solution

P(R1)=6/10

P(R2)=5/9

P(R2/R1)= P(R1 . R2)/P(R1)=

(6/10×5/9)/6/10=(1/3)/(6/10)=5/9

Example 19: In a residential complex 1000 apartments, 500 residents in the northern sector and 500 others in the southern sector in each sector, 200 of the apartments contain large windows, 100 central heated, and 30% of the apartments with large windows are centrally heated. When choosing an apartment randomly, determine the probability of:

1. In the northern sector
2. In the northern sector, with a large window
3. In the northern sector, with a large window that are centrally heated.

Solution:

E1 = apartment in northern sector

E2 = apartment with a large window

E3 = apartment centrally heated

N(E1)=500

N(E2)=200; P(E2) = 200/1000= 0.2 northern

N(E3)=160; P(E3) = 160/1000 = 0.16

N(E1&E2)=200; P(E1&E2) = 200/1000= 0.2

N(E1&E3)=100; P(E&E3) = 0.1

N(E2&E3)=(400*0.3)=120

N(E1&E2&E3)=200*0.3=60; (P(E1E2E3) = 0.06

1. P(E1)= $\frac{n}{N} = \frac{500}{1000}$ = 0.5
2. P(E2|E1)= $\frac{P(E1).P(E2)}{P(E1)} = \frac{0.2\times0.5}{0.5} = 0.20$
3. P(E1E2|E3)=$\frac{P(E1E2).P(E3)}{P(E1E2)} = \frac{0.2\times0.06}{0.2}$ = 0.06

5.4. Counting Rules:

- **Factorial Notation:**

For any counting *n, factorial formula is:*

0! = 1

$$n! = n(n\ \ 1)(n\ \ 2)\ ......\ 1$$

Example 20: 5! = 5 . 4 . 3 . 2 . 1 = 240

- Permutations

The arrangement of n objects in a specific order using r objects at a time is called a permutation of n objects taking r objects at a time. It is written as $_n\mathbf{P}_r$, and the formula is:

$$nPr = \frac{n!}{(n-r)!}$$

Example 21: The advertising director for a television show has 7 ads to use on the program. If she selects 1 of them for the opening of the show, 1 for the middle of the show, and 1 for the ending of the show, how many possible ways can this be accomplished?

Solution :

Since order is important, the solution is:

$$\mathbf{nPr} = \frac{\mathbf{n}!}{(\mathbf{n-r})!} \Rightarrow 7P3 = \frac{7!}{(7-3)!} = 210$$

means there would be 210 ways to show 3 ads. $P_1P_1P_1$, $P_1P_1P_2$, $P_1P_1P_3$, $P_1P_1P_4$, $P_1P_1P_5$, $P_1P_1P_6$, …………………

- Combinations

The number of combinations of r objects selected from n objects is denoted by $_n\mathbf{C}_r$ and is given by the formula:

$$nCr = \frac{n!}{(n-r)!\,r!}$$

<u>Note</u>: Combinations are used when the order or arrangement is not important, as in the selecting process.

Example 22: Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters. Using permutation and combination.

Solution :-

The permutations are:

AB	**BA**	**CA**	**DA**
AC	**BC**	**CB**	**DB**
AD	**BD**	**CD**	**DC**

$$4P2 = \frac{4!}{(4-2)!} = 12$$

The combination are:

$$nCr = \frac{n!}{(n-r)!r!} = \frac{4!}{2!2!} = 6$$

<u>Homework four</u>

1- The corporate research and development centers for three local companies have the following number of employees:

U.S. Steel	**110**
Alcoa	**750**
Bayer Material Science	**250**

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

2- There are four blood types, A, B, AB, and O. Blood can also be Rh and Rh. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled? (Use tree diagram)

3- A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

4- In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?