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## Introduction to Graphs (1)

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# Lecture (1) <br> Graph Theory Introduction to Graphs (1) 

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## The Königsberg Bridge Problem

The city of Königsberg was located on the Pregel river in Prussia. The river divided the city into four separate landmasses, including the island of Kneiphopf. These four regions were linked by seven bridges as shown in the diagram. Residents of the city wondered if it were possible to leave home, cross each of the seven bridges exactly once, and return home. The Swiss mathematician Leonhard Euler (1707-1783) thought about this problem and the method he used to solve it is considered by many to be the birth of graph theory.


See if you can find a round trip through the city crossing each bridge exactly once, or try to explain why such a trip is not possible.

## Outlines

$\checkmark$ Basic Definitions.
$\checkmark$ Degree of Vertex.
$\checkmark$ Minimum and Maximum Degree of a Graph.

## Basic Definitions:

Graph Theory is a well-known area of discrete mathematics which deals with the study of graphs. A graph may be considered as a mathematical structure that is used for modelling the pairwise relations between objects.

Graph Theory has many theoretical developments and applications not only to different branches of mathematics, but also to various other fields of basic sciences, technology, social sciences, computer science etc.





Definition : A graph $G$ can be considered as an ordered triple $(V, E, \varphi)$, where (i) $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ is called the vertex set of $G$ and the elements of $V$ are called the vertices (or points or nodes);
(ii) $E=\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ is the called the edge set of $G$ and the elements of $E$ are called edges (or lines or arcs); and
(iii) $\varphi$ is called the adjacency relation, defined by $\varphi: E \rightarrow V \times V$, which defines the association between each edge with the vertex pairs of $G$.

Usually, the graph is denoted as $\mathrm{G}=(V, E)$. The vertex set and edge set of a graph $G$ are also written as $V(G)$ and $E(G)$ respectively.


Fig. 1: An example of a graph
If two vertices $u$ and $v$ are the (two) end points of an edge $e$, then we represent this edge by $u v$ or $v u$. If $\mathrm{e}=u v$ is an edge of a graph $G$, then we say that $u$ and $v$ are adjacent vertices in $G$ and that $e$ joins $u$ and $v$. In such cases, we also say that $u$ and $v$ are adjacent to each other.

Given an edge $e=u v$, the vertex $u$ and the edge $e$ are said to be incident with each other and so are $v$ and $e$. Two edges $e_{i}$ and $e_{j}$ are said to be adjacent edges if they are incident with a common vertex.

Definition: (Order and Size of a Graph) The order of a graph $G$, denoted by $V(G)$, is the number of its vertices and the size of $G$, denoted by $E(G)$, is the number of its edges.

A graph with $p$ - vertices and $q$ - edges is called a $(p, q)$ - graph. The $(1,0)$ - graph is called a trivial graph. That is, a trivial graph is a graph with a single vertex. A graph without edges is called an empty graph or a null graph.

Definition: (Finite and Infinite Graphs) A graph with a finite number of vertices as well as a finite number of edges is called a finite graph. Otherwise, it is an infinite graph.

Definition: (Self-loop) An edge of a graph that joins a vertex to itself is called loop or a self-loop. That is, a loop is an edge $u v$, where $u=v$.

Definition: (Parallel Edges) The edges connecting the same pair of vertices are called multiple edges or parallel edges.
In Fig. 1., the edges $e_{6}$ and $e_{7}$ are loops and the edges $e_{4}$ and $e_{5}$ are parallel edges.


Definition: (Simple Graphs and Multigraphs) A graph $G$ which does not have loops or parallel edges is called a simple graph. A graph which is not simple is generally called a multigraph.



nonsimple graph with loops

## 13 Degrees \& Minimum and Maximum Degree

Definition: (Degree of a vertex) The number of edges incident on a vertex $v$, with self-loops counted twice, is called the degree of the vertex $v$ and is denoted by $\operatorname{deg}_{G}(v)$ or simply $d(v)$.

Definition: (Isolated vertex) A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are those with zero degree.

Definition: (Pendant vertex) A vertex of degree 1, is called a pendent vertex or an end vertex.

Definition: (Internal vertex) A vertex, which is neither a pendent vertex nor an isolated vertex, is called an internal vertex.

Definition: (Minimum and Maximum Degree of a Graph) The maximum degree of a graph $G$, denoted by $\Delta(G)$, is defined to be $\Delta(G)=\max \{d(v): v \in V(G)\}$. Similarly, the minimum degree of a graph $G$, denoted by $\delta(G)$, is defined to be $\delta(G)=\min \{d(v): v \in V(G)\}$. Note that for any vertex $v$ in $G$, we have $\delta(G) \leq d(v) \leq \Delta(G)$.

The following theorem (first theorem on graph theory) is a relation between the sum of degrees of vertices in a graph $G$ and the size of $G$.

Theorem: In a graph $G$, the sum of the degrees of the vertices is equal to twice the number of edges. That is, $\sum_{v \in V(G)} d(v)=2 E$.
The following two theorems are immediate consequences of the above theorem.
Theorem: For any graph $G, \delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$.
Theorem: For any graph $G$, the number of odd degree vertices is always even.

Theorem: $\sum_{v \in V(G)} d(v)=2 E$.
$\sum_{v \in V(G)} d(v)=18,2 E=18$.
Theorem: $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$.
$|V|=8,|E|=9$,
$2|E| /|V|=18 / 8=2.25$
$1 \leq 2.25 \leq 3$


$$
\begin{aligned}
& d(a)=2, d(b)=3, d(c)=2, d(d)=3 \\
& d(e)=2, d(f)=2, d(g)=3, d(h)=1 \\
& \Delta(G)=3, \delta(G)=1
\end{aligned}
$$

Theorem: For any graph $G$, the number of odd degree vertices is always even.

$$
\{b, d, g, h\}
$$

Thank You

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