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## Graphs and Their Operations م. د. امين شامان امين

# Lecture (4) <br> Graphs and Their Operations 

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## Outlines

$\checkmark$ Union, Intersection and Ringsum of Graphs.
$\checkmark$ The complement of a graph.
$\checkmark$ The join of two graphs.
$\checkmark$ Edge Deletion \& Vertex Deletion in Graphs.
$\checkmark$ Fusion of Vertices \& Edge Contraction in Graphs.
$\checkmark$ Subdivision of an Edge.
$\checkmark$ Homeomorphic Graphs.
$\checkmark$ Smoothing Vertices in Graphs.

## ${ }^{3}$ Union, Intersection and Ringsum of Graphs

Definition: The union of two graphs $G_{1}$ and $G_{2}$ is a graph $G$, written by $G=G_{1} \cup G_{2}$, with vertex set $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and the edge set $E\left(G_{1}\right) \cup E\left(G_{2}\right)$.

Definition: The intersection of two graphs $G_{1}$ and $G_{2}$ is a graph $G$, written by $G=G_{1} \cap G_{2}$, with vertex set $V\left(G_{1}\right) \cap V\left(G_{2}\right)$ and the edge set $E\left(G_{1}\right) \cap E\left(G_{2}\right)$.

Definition: The ringsum of two graphs $G_{1}$ and $G_{2}$ is another graph $G$, written by $G=G_{1} \oplus G_{2}$, with vertex set $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and the edge set $E\left(G_{1}\right) \oplus E\left(G_{2}\right)=$ $\left(E\left(G_{1}\right) \cup E\left(G_{2}\right)\right)-\left(E\left(G_{1}\right) \cap E\left(G_{2}\right)\right)$.

The following examples explain the union, intersection and ringsum of two given graphs.

(a) $G_{1}$

(c) $G_{1} \cup G_{2}$

(b) $G_{2}$

$$
\begin{aligned}
& G=G_{1} \cup G_{2}=\left(V_{1} \cup V_{2}, E_{1} \cup E_{2}\right) \\
& G=G_{1} \cap G_{2}=\left(V_{1} \cap V_{2}, E_{1} \cap E_{2}\right)
\end{aligned}
$$


(d) $G_{1} \cap G_{2}$


$$
\mathrm{G}=G_{1} \oplus G_{2}=\left(V_{1} \cup V_{2},\left(E_{1} \cup E_{2}\right)-\left(E_{1} \cap E_{2}\right)\right)
$$


$G_{1}$
$\mathrm{G}_{2}$

i) The union, intersection and ringsum operations of graphs are commutative. That is, $G_{1} \cup G_{2}=G_{2} \cup G_{1}, G_{1} \cap G_{2}=G_{2} \cap G_{1}$ and $G_{1} \oplus G_{2}=G_{2} \oplus G_{1}$.
ii) If $G_{1}$ and $G_{2}$ are edge-disjoint, then $G_{1} \cap G_{2}$ is a null graph (i.e. $G(\},\{ \})$ ), and $G_{1} \oplus G_{2}=G_{1} \cup G_{2}$.
iii) If $G_{1}$ and $G_{2}$ are vertex-disjoint, then $G_{1} \oplus G_{2}$ is empty (i.e. $\left.G(V,\{ \})\right)$
iv) For any graph $G, G \cap G=G \cup G$ and $G \bigoplus G$ is a null graph.

Definition: A graph $G$ is said to be decomposed into two subgraphs $G_{1}$ and $G_{2}$, if $G_{1} \cup G_{2}=G$ and $G_{1} \cap G_{2}$ is a empty graph.

For example the graph $K_{5}$ decomposed into the following two subgraphs $G_{1} \& G_{2}$.



Definition: The complement or inverse of a graph $G$, denoted by $\bar{G}$ is a graph with $V(G)=V(\bar{G})$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in $G$.

Note: that for a graph $G$ and its complement $\bar{G}$, we have
i) $G \cup \bar{G}=K_{n}$;
ii) $V(G)=V(\bar{G})$;
iii) $E(G) \cup E(\bar{G})=E\left(K_{n}\right)$;
iv) $|E(G)|+|\mathrm{E}(\bar{G})|=\left|E\left(K_{n}\right)\right|=\binom{n}{2}=\frac{n(n-1)}{2}$.

A graph and its complement are illustrated below.

(a) $G$

(b) $\bar{G}$

Definition: A graph $G$ is said to be self-complementary if $G$ is isomorphic to its complement $\bar{G}$. If $G$ is self- complementary, then

$$
|E(G)|=|E(\bar{G})|=\frac{1}{2}\left|E\left(K_{n}\right)\right|=\frac{1}{2}\binom{n}{2}=\frac{n(n-1)}{4}
$$

The following is example of self-complementary graphs.


G


$$
f: \begin{array}{ccccc}
v_{1} & v_{2} & v_{3} & v_{4} & v_{5} \\
1 & 3 & 5 & 2 & 4
\end{array}
$$

Definition: The join of two graphs $G$ and $H$, denoted by $G+H$ is the graph such that $V(G+H)=V(G) \cup V(H)$ and $E(G+H)=E(G) \cup E(H) \cup\{x y: x \in V(G), y$ $\in V(H)\}$.

In other words, the join of two graphs $G$ and $H$ is defined as the graph in which every vertex of the first graph is adjacent to all vertices of the second graph.


Definition: (Edge Deletion in Graphs) If $e$ is an edge of $G$, then $G-e$ is the graph obtained by removing the edge $e$ of $G$. The subgraph of $G$ thus obtained is called an edge-deleted subgraph of $G$. Clearly, $G-\mathrm{e}$ is a spanning subgraph of $G$.

Definition: (Vertex Deletion in Graphs) If $v$ is a vertex of $G$, then $G-v$ is the graph obtained by removing the vertex $v$ and all edges $G$ that are incident on $v$. The subgraph of $G$ thus obtained is called a vertex-deleted subgraph of $G$. Clearly, $G-v$ will not be a spanning subgraph of $G$.

The following figure illustrates the edge deletion and the vertex deletion of a graph $G$.

(a) $G$

(b) $G-e$

(c) $G-v$

12 Definition: (Fusion of Vertices) A pair of vertices $u$ and $v$ are said to be fused (or merged) together if the two vertices are together replaced by a single vertex $w$ such that every edge incident with either $u$ or $v$ is incident with the new vertex $w$.

Note that the fusion of two vertices does not change the number of edges, but reduces the number of vertices by 1 .


Fusion of two vertices $u, v$.

Definition: (Edge Contraction in Graphs) An edge contraction is operation occurs relative to a particular edge $e$. The edge $e$ is removed and its two incident vertices $u$ and $v$ are merged into a new vertex $w$, where the edges incident to $w$ each correspond to an edge incident to either $u$ or $v$. In other words The contraction of $e$ results in a new graph $G^{\prime}$ $=\left(V^{\prime}, E^{\prime}\right)$, where $\left(V^{\prime}, E^{\prime}\right)=(V \backslash\{u, v\} \cup\{w\}, E \backslash\{e\})$. A graph obtained by contracting an edge $e$ of a graph $G$ is denoted by $G \circ e$. Vertex fusion is a less restrictive form of this operation.


Edge contraction of a graph $G \circ u v$.

Definition: (Subdivision of an Edge) Let $e=u v$ be an arbitrary edge in $G$. The subdivision of the edge $e$ yields a path of length 2 with end vertices $u$ and $v$ with a new internal vertex $w$ (That is, the edge $e=u v$ is replaced by two new edges, $u w$ and $w v$ ).


Subdivision of an edge

Definition: A subdivision of a graph $G$ is a graph resulting from the subdivision of (some or all) edges in $G$. The newly introduced vertices in the subdivisions are represented by white vertices.

Definition: (Homeomorphic Graphs) Two graphs are said to be homeomorphic, if one graph can be obtained from the others by creation of edges or merges of edges in series (Two adjacent edges are said to be in series if their common vertex is of degree two).


Definition: (Smoothing Vertices in Graphs) The reverse operation, smoothing out or smoothing a vertex $w$ of degree 2 with regards to the pair of edges $\left(e_{i}, e_{j}\right)$ incident on $w$, removes $w$ and replaces the pair of edges $\left(e_{i}, e_{j}\right)$ containing $w$ with a new edge e that connects the other endpoints of the pair $\left(e_{i}, e_{j}\right)$.

Smoothing of the vertex w

Thank You

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