جامعة الانبار

كلية العلوم

قسم الرياضيات

نظرية البيانات

Trees

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Lecture (11)

Trees

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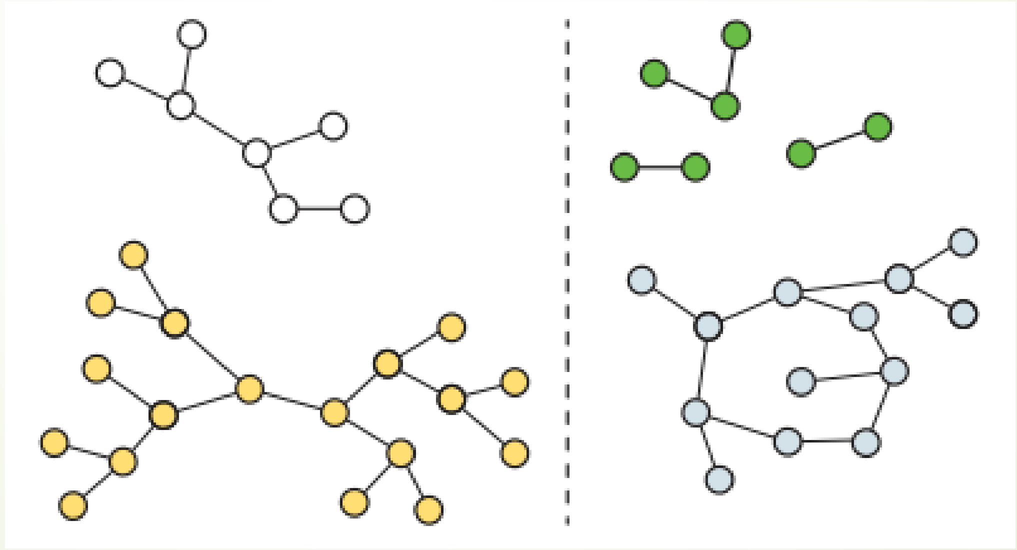
Basic definitions:

Definition: A graph $G(V, E)$ is acyclic if it doesn't include any cycles.
Definition: A tree is a connected, acyclic graph.
Definition: A forest is a graph whose connected components are trees.

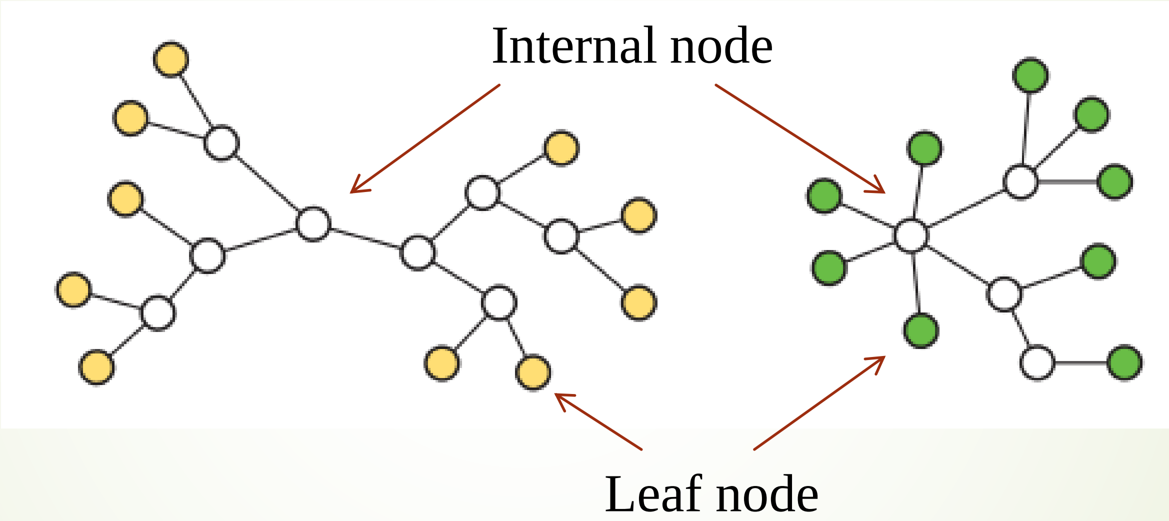


Tree

Forest

Not Tree

Definition: A vertex $v \in V$ in a tree $T(V, E)$ is called a leaf or leaf node if $d(v) = 1$ and it is called an internal node if $d(v) > 1$.



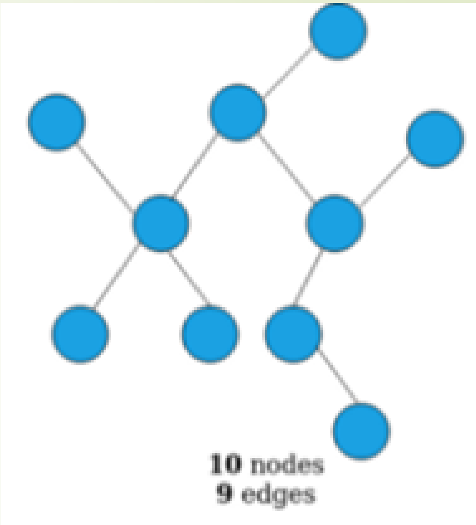
Definition: A graph is called geodetic if any two vertices are joined by a unique shortest path.

Properties of Trees:

Theorem: A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Theorem: All trees are geodetic graphs.

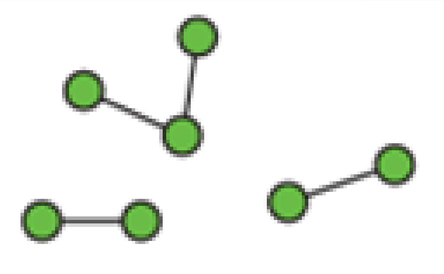
A graph is called geodetic if any two vertices are joined by a unique shortest path.



Theorem : A tree with n vertices has $n-1$ edges.

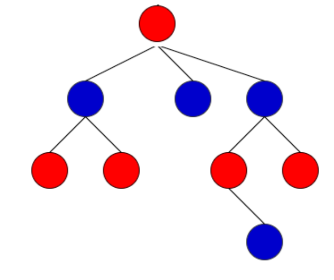
Theorem: A forest of k trees which have a total of n vertices has $n-k$ edges.

$k = 3$ trees
$n = 7$ vertices
$n - k = 4$ edges



Theorem: Any connected graph with n vertices and $n-1$ edges is a tree.

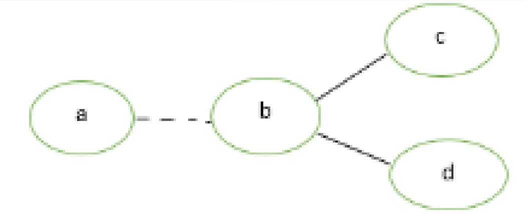
Theorem: Every tree T is bipartite.

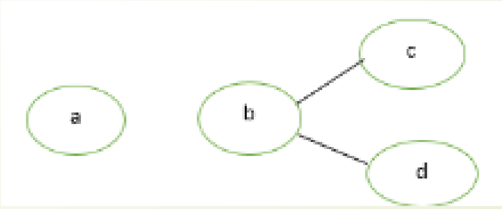


Definition: A graph is said to be minimally connected if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

The following theorem is another characterization of trees.

Theorem: A graph is a tree if and only if it is minimally connected.





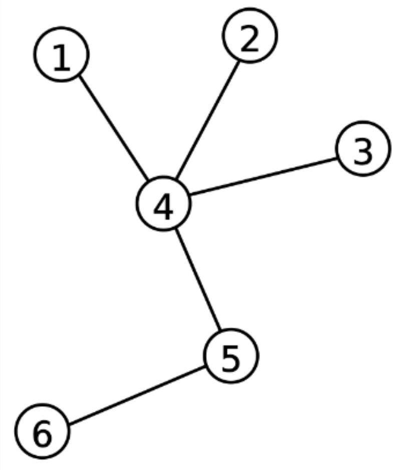
Minimally Connected Graph

Theorem: A graph G with n vertices, $n-1$ edges and no cycles is connected.

Theorem: Any tree with at least two vertices has at least two pendant vertices.

Theorem: A vertex v in a tree is a cut-vertex of T if and only if $d(v) \geq 2$.

vertices 4,5 are
cut - vertex



Theorem: Every edge of a tree is a cut-edge of T.

Distances in Trees:

Definition: A metric on a set A is a function $d : A \times A \rightarrow [0, \infty)$, where $[0, \infty)$ is the set of non-negative real numbers and for all $x, y, z \in A$, the following conditions are satisfied:

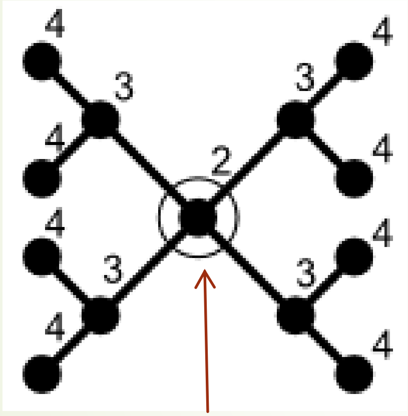
1. $d(x, y) \geq 0$ (non-negativity or separation axiom);
2. $d(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernible);
3. $d(x, y) = d(y, x)$ (symmetry);
4. $d(x, z) \leq d(x, y) + d(y, z)$ (sub- additivity or triangle inequality).

Conditions 1 and 2, are together called a positive-definite function.

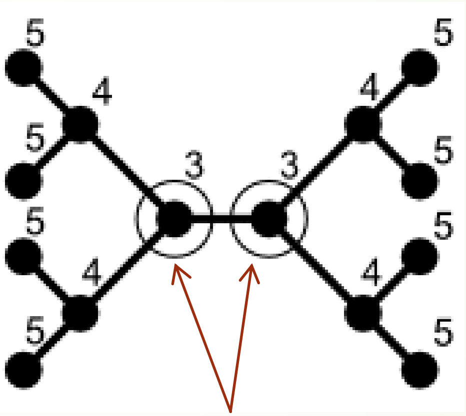
Theorem: The distance between vertices of a connected graph is a metric.

Definition: A vertex in a graph G with minimum eccentricity is called the center of G.

Theorem: Every tree has either one or two centers.



Center



Centers

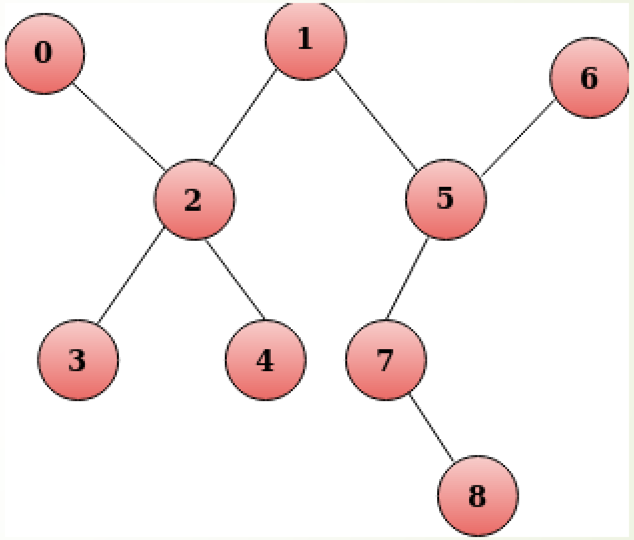
Algorithm:

- Remove all the vertices of degree 1, together with their incident edges.
- Repeat the process until we obtain either a single vertex (the center) or two vertices joined by an edge (the bicenter).

Note: A tree with a center is called a central tree, and a tree with a bicenter is called a bicentral tree. Note that every tree is either central or bicentral, but not both.

Degree Sequences in Trees:

Theorem: The sequence $\langle d_i \rangle; 1 \leq i \leq n$ of positive integers is a degree sequence of a tree if and only if
$i)\ d_i \geq 1\ for\ all\ i, 1 \leq i \leq n\ and$
$ii) \sum d_i = 2n - 2$.

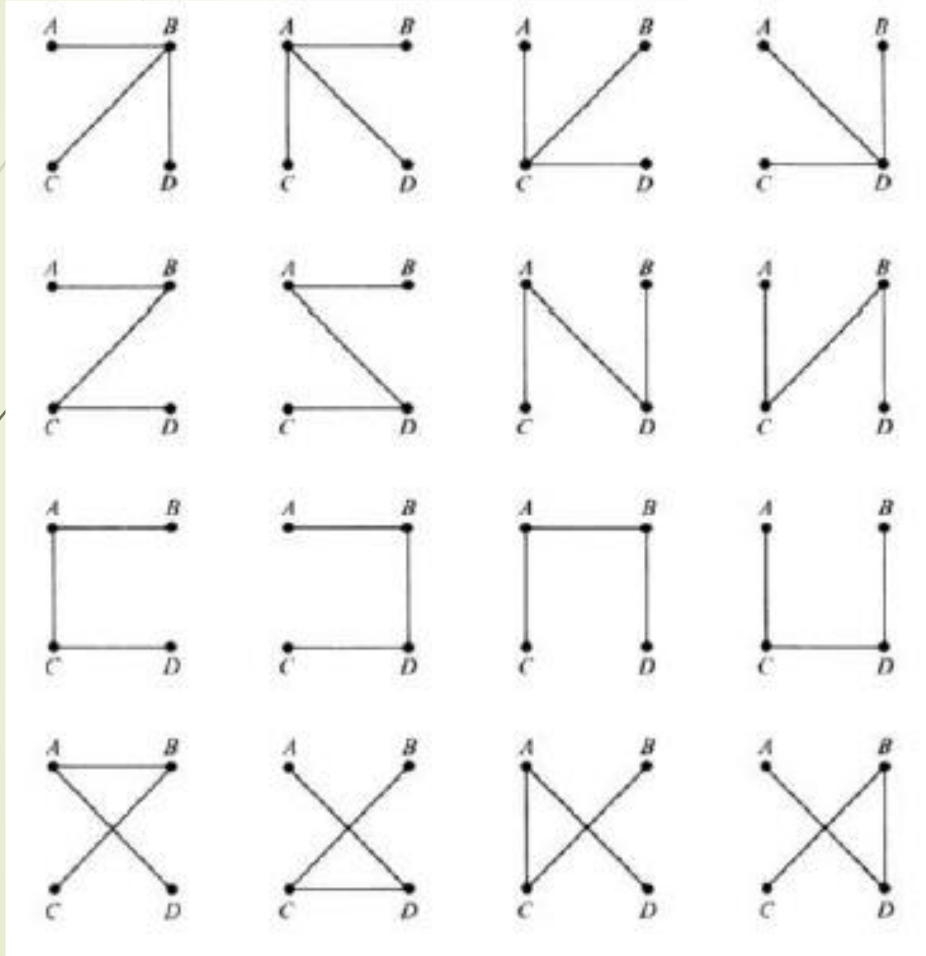


$\langle 4,3,2,2,1,1,1,1,1 \rangle$

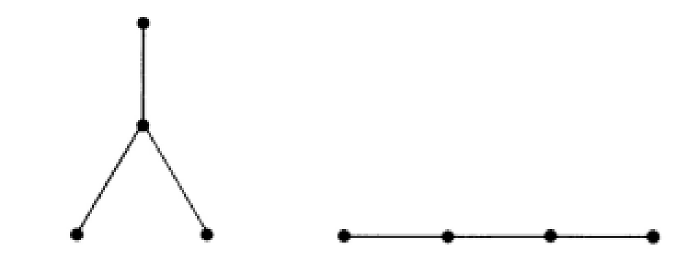
Theorem: Let T be a tree with k edges. If G is a graph whose minimum degree satisfies $\delta(G) \geq k$, then G contains T as a subgraph. In other words, G contains every tree of order at most $\delta(G) + 1$ as a subgraph.

Definition: A labelled graph is a graph, each of whose vertices (or edges) is assigned a unique name.

$(v_1, v_2, v_3, \cdots);$
$(A, B, C, \cdots);$
$(1, 2, 3, \cdots).$



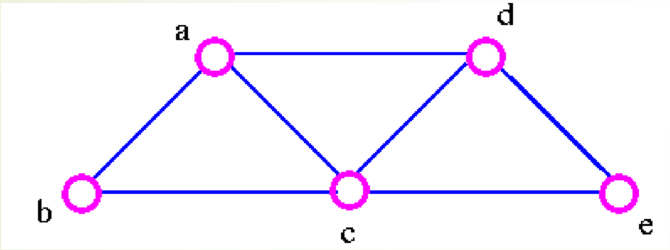
The distinct vertex labelled trees on 4 vertices

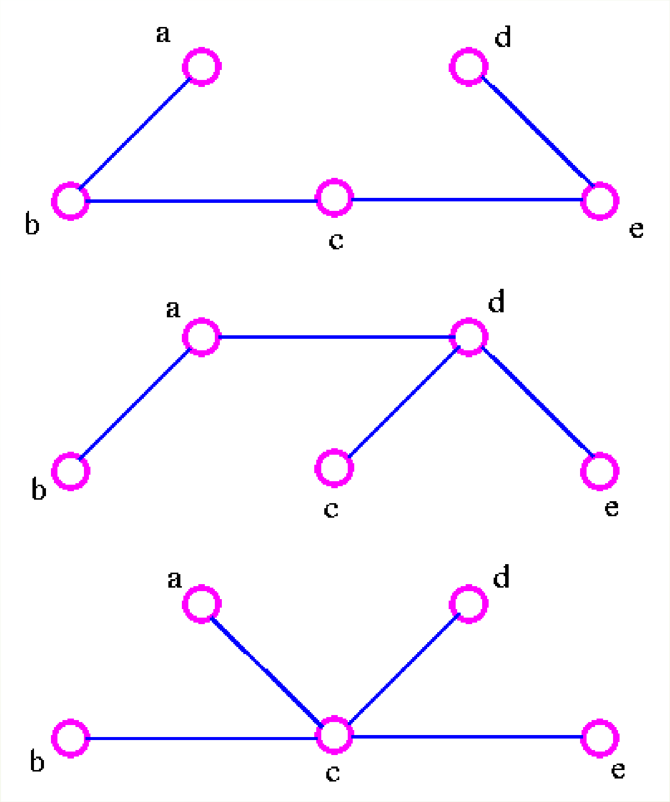


The distinct unlabelled trees on 4 vertices

Spanning Trees:

Definition: A spanning tree of a connected graph G is a tree containing all the vertices of G. A spanning tree of a graph is a maximal tree subgraph of that graph.

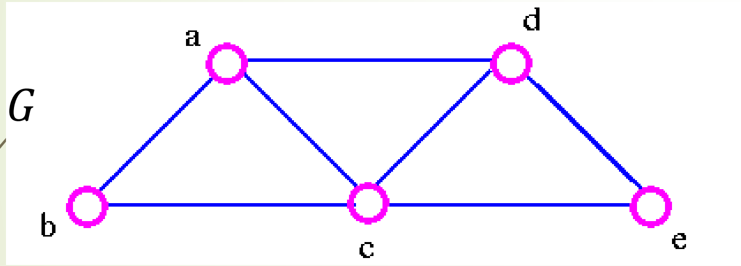


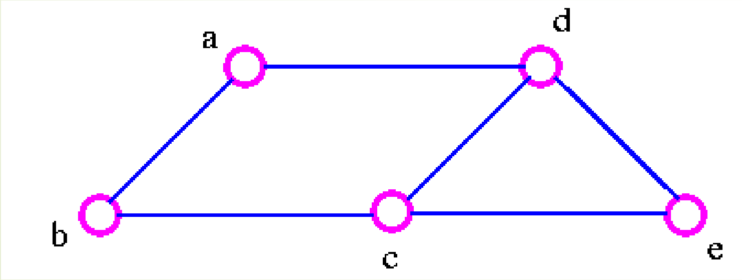


We can find a spanning tree systematically by using either of two methods.

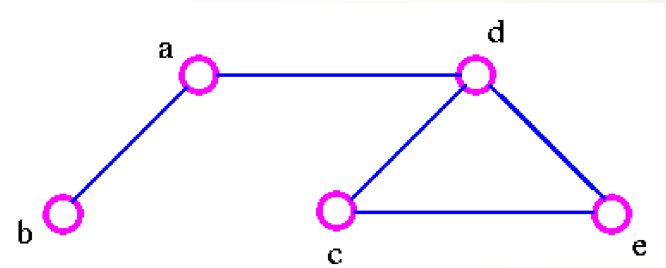
1) Cutting-down Method

- Start choosing any cycle in G.
- Remove one of cycle's edges.
- Repeat this procedure until there are no cycle left.

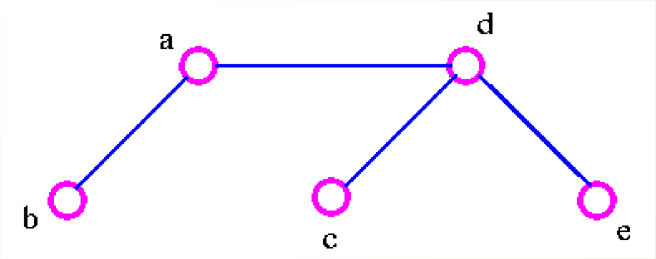




1. Remove the edge ac which destroy the cycle $adca$ in G.



2. Remove the edge cb, which destroy the cycle $adcba$ in G.



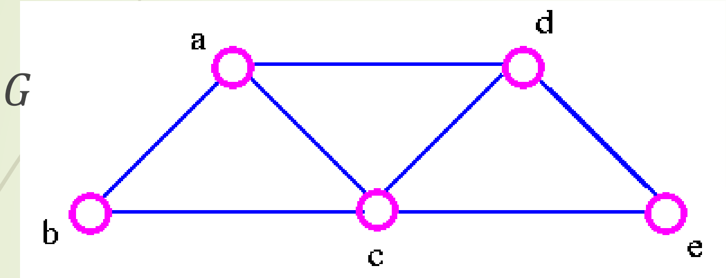
spanning tree

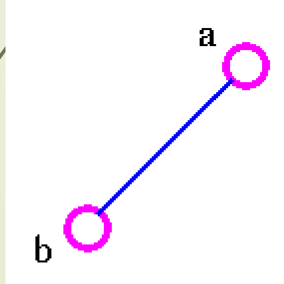
3. Remove the edge ec, which destroy the cycle $decd$ in G and thus obtained the following spanning tree.

2) Building-up Method:

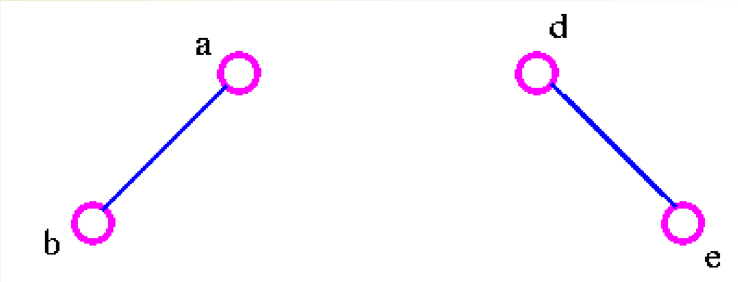
- Select edges of G one at a time. in such a way that no cycles are created.
- Repeat this procedure until all vertices are included.

G

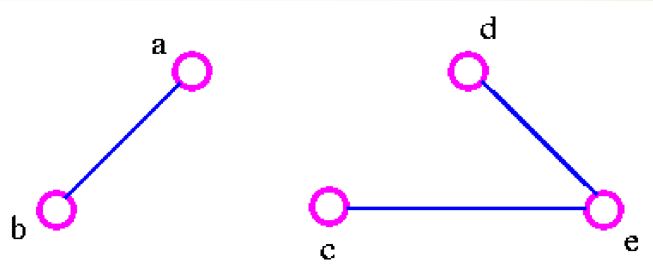




1. Choose the edge ab

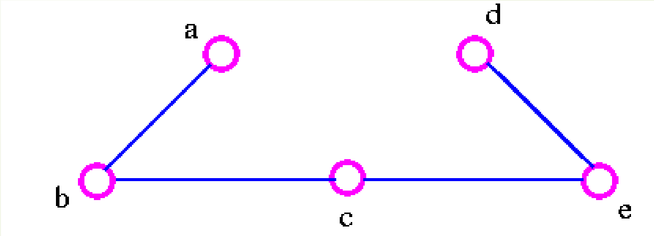


2. Next choose the edge de



3. After that choose the edge ec

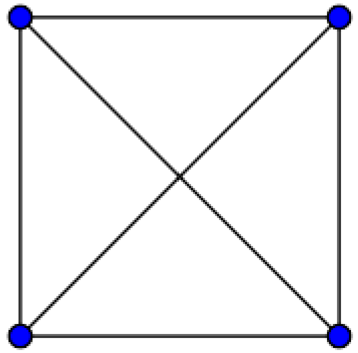
4. Finally, we choose the edge cb and thus obtain the following spanning tree.



Theorem: A graph is connected if and only if it has a spanning tree.

Theorem: (Cayley's Theorem) The number of spanning trees of a complete graph on n vertices is n^{n-2}.

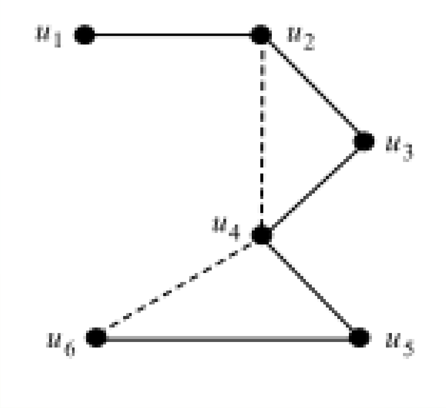
K_4



Theorem: (Cayley's Theorem on Labelled Trees) For $n \geq 2$, the number of labelled trees with n vertices is n^{n-2}.

Theorem: Every connected graph has at least one spanning tree.

Definition: An edge in a spanning tree T is called a branch of T. An edge of G that is not in a given spanning tree T is called a chord. It may be noted that branches and chords are defined only with respect to a given spanning tree. An edge that is a branch of one spanning tree T_1 (in a graph G) may be a chord with respect to another spanning tree T_2. In Figure below, $u_1u_2u_3u_4u_5u_6$ is a spanning tree, u_2u_4 and u_4u_6 are chords.



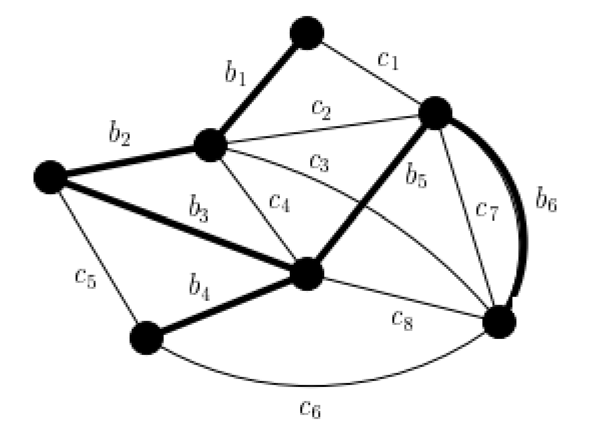
A connected graph G can be considered as a union of two subgraphs T and $\bar{T}$, that is $G = T \cup \bar{T}$, where T is a spanning tree, $\bar{T}$ is the complement of T in G. $\bar{T}$ being the set of chords is called the cotree, or chord set.

The following result provides the number of chords in any graph with a spanning tree.

Theorem : With respect to any of its spanning trees, a connected graph of n vertices and m edges has $n-1$ tree branches and $m-n+1$ chords.

Definition: Let G be a graph with n vertices, m edges and k components. The $rank(G)$ (The number of branches in a spanning tree T of a graph G) and $nullity\,(G)$ (The number of chords of a graph G) of G are defined as $rank(G) = n-k$ and $nullity\,(G) = m-n+k$. Therefore we have, $rank(G) + nullity(G) = |E(G)|$.

Theorem: Let T and T_0 be two distinct spanning trees of a connected graph G and $e \in \{E(T) - E(T_0)\}$. Then, there exists an edge $e_0 \in \{E(T_0) - E(T)\}$ such that $T - e + e_0$ (or $T_0 + e - e_0$) is a spanning tree of G.



Thank You

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