First – order ODEs

Consider the differential equation

$Y^{\sim}(t)=f(t, y(t))$, with initial condition

$Y(t_o)=y_{o,\text{ where}}$ the function f is defined on a rectangular of the form

$R=\{(t, y)€RXR^n:|t-t_0|\leq a,|y-y_o|\leq b\}$

Peano's existence theorem states that if f is continuous then the differential equation has at least one solution inarneigh bourhood of the initial condition

"Uniqueness of a solution"

Assume that the mapping f satisfies the caratheodopy

Conditions on R and there is a lebesgue – integreble funetion

$K:[t_o-a,t_o+a]\rightarrow[0,\infty]$,such that

$|f(t,y_1)-f(t,y_2)|\leq K(t)\ |y_1-y_2|$,

$\forall(t,y_1)\in R,(t,y_2)\in R$.

Then,there exists a uniue

Solution

$Y(t)=y(t,t_0,y_2)$ to the initial value problem

$(t,y(t)),y(t_0)=y_0$
$\bar{y}(t)=F$

ملاحظه:. من الجدير بالذكر ان وجود الحل لا يعني امكانية الحصول عليه في صورة مغلقه "closed from"
المضبوطة في جميع الاحوال بل قد يمكن الحصول على الحل بإحدى الطرق التقريبية او العددية

Existence and Uniqueness Theorems for First-Order ODE'S

The general first-order ODE is For a real number x and a positive value 6, the set of numbers x satisfying 2-6 << 20 + 6 is called $y' = F(x,y)\ ,\ y(x_0) = y_0$ (*) open internal centered at x0

We are interested in the following questions

(i) Under what conditions can we be sure that a solution to (*) exist?

(ii) Under what conditions can we be sure that there is a unique solution to (*)?

Here are the answers

. Theorem I (Existence). Suppose that $F(x, y)$is a continues function defined in some region R $\{(x, y): x_0 - \delta < x < x_0 + \delta, y_0 - \epsilon < y < y_0 + \epsilon\}$ containing the point (x0,y0.). Then there exiats a munber (possibly smaller than d) so that a solution y=f(a) to () is defined for -$_1$ << +6$_2$ Theorem (Uniqueness). Suppose that both Fr.) and (y) are continuous functions defined on a re- gion as in Theorem 1. Then there ezists a number 62 (possibly smaller than ay so that the solution y=f(=) to (*), whose existence was guaranteed by Theorem 2, is the unique solution to (") for zo-b$_2$ <=< 20 +0$_2$

. Example 3. Consider the ODE y(1) = 2 In this case, both the function F(x, y)--+1 and its partial derivative (2x)/2 * (x, y) = - 1 are defined and contin uous at all points). The theorem guarantees that a solution to the ODE exists in some open interval cett tered at 1, and that this solution is unique in some (pas sibly smaller) interval centered at 1 In fact, an explicit solution to this equation is (Check this for yourself) This solution exists (and is the unique solution to the equation) for all real numbers In other words, in this example we may choose the numbers & sid 6 large as we

Example 4. Consider the ODE

.y(0)=0

Again, both F(x,y)=1+y^{2} and (y)-2y are des

lined and continuous at all points (2), so by the theorem we can conclude that a solution exists in some open

interval centered at 0, and is unique in some (possibly

.smaller) interval centered at 0

By separating variables and integrating, we derive a

solution to this equation of the form

As an abstract function of z, this is defined for 1=\varphi

However, in order for .....2/3 ,2/-2$_1$/-,2/3-

this function to be considered as a solution to this ODE

we must restrict the domain. (Remember that a solution (to a differential equation must be a continuous function

.Specifically, the function

.y=tan(r)

x<+/2>2/1-

.is a solution to the above ODE

In this example we must choose 6_1 6_2 = 7/2, al

though the initial value & may be chosen a large as we

By separating variables and integrating, we derive so

lutions to this equation of the form

{2}^y(x)=Cx

for any constant C. Notice that all of these solutions

pass through the point (0.0), and that none of them

pass through any point (0.28) with you. So the initial

value problem

.V-2y/z. (0) -0

has infinitely many solutions, but the initial value prob

Jem

/y2 =

,Mo. 10 (0)

has no solutions

For ench (0) with ro0, there is a unique

parabola - C whose graph passes through (za-0)

Choose C=y_{0}/x_{1}^{2}. So the initial value problem)

y/2, V(0) Mo00, has a unique solution2

defined on some interval centered at the point ro. In

fact, in this case, there exists a solution which is de

fined for all values of ar (6 may be chosen as large as

we plasse), but that there is a unique solution only on the interval and <<+dy, where dy 120l

This examples shows that the values and d may be

.different

Example 5. Consider the ODE

y/22 = ²/1

.In this example, F(r.) 2y/2 (y) - 2/a

Both of these functions are defined for all a 0, so

Theorem 2 tells us that for each re 0 there exists a

unique solution defined in an open interval around o

Summary. The initial vaine problemu y(20) = 3/0, kame

amique solution is an open interval containing to

Do solution if y= 0 and in Z •

.(0,0) = infinitely many solutions if (ru)

المعادلات التفاضليه الاعتياديه من المرتبه الاولئ والدرجه الاولئ
لمعادلات التي يمكن ايجاد حلها بصوره مباشره هي :.

1-طريقه فصل المتغيرات
2- معادلات تفاضليه متجانسه
3- معادلات تفاضليه تامه
4- طريقه تعيين عامل التكامل(y.x)l
5- المعادلات التفاضليه الخطيه
6- معادلات تفاضليه توؤل الئ خطيه (برنولي معادله ريكارتي التفاضليه علئ صوره
(x)Q=(y)f(x)P+xd/yd'(y)f
معادلات تفاضليه من مرتبه الاولئ قابله لفصل متغيرين
snoitauqE edrao tsrif ItalaPes
في حالات كثيره يمكن وضع المعادلات
(y ,x)=y'
علئ الشكل
0=(x)h +xd/yd (y)g

او مايكافئ ذالك

0=xd(x)h+yd(y)g

ويقال عن هذه المعادله انها معادله تفاضليه قابله لفصل متغيرات او للسهوله قابله لفصل snoitauqEltalaPes وذالك امكن فصل متغير x عن متغير y تماما وبمعنى اخر يتم فصل المتغيرين اذا كان معامل تفاضل xداله من x فقط ومعامل تفاضل y داله من y فقط وبمكامله الطرفين نحصل على

A=xd(x)h∫+yd (y)g∫

حيث A ثابت اختياري واستخدمنا ثابت واحد لان المعادله من الرتبه الاولى وباجراء التكاملين ينتج

$$G(y)+H(x)=A$$

وتكون قد حصلنا على الحل العام للمعادله التفاضليه

ملاحظه : قد نجد صور اخرى للمعادله التفاضليه القابله للفصل

EX/ $g_1(y)f_2(x)dy+g_2(y)f_2(x)dx=0 \qquad \ldots\ldots\ldots..(1)$

$$\frac{dy}{dx}+f(x)h(y)=0 \qquad \ldots\ldots\ldots\ldots(2)$$

حيث يمكن فصل المتغيرات في (1) بالضرب في عامل التكميل (التكامل)

$$\frac{1}{f_2(x)g_2(y)}$$

لنحصل على $\int \frac{g_1(y)}{g_2(y)}dy+\int \frac{f_1(x)}{f_2(x)}dx=0$

بينما يمكن فصل المتغيرات في المعادله (2) بالضرب في العامل التكميلي $\frac{1}{h(x)}$ لنحصل على

$$\frac{1}{h(y)}dy+f(x)dx=0$$

ومنها

$$\int \frac{1}{h(y)}dy+\int f(x)dx=0$$

Ex/

$$\frac{\partial y}{\partial x} - xy = 0$$

الحل المعادله المعطاه على شكل معادله (2) السابقه بالقسمه على

y يمكن فصل المتغيرين $\frac{dy}{y} - xdx = 0$ by

integrate

$$ln\, y - \frac{x^2}{2} = ln\, A$$

هنا وضعنا الثابت الاختياري على صوره $l_n A$ لكونها اكثر ملائمه

$$ln \frac{y}{A} = \frac{x^2}{2} \qquad\qquad y = A_e^{x^2/2}$$

و هذا هو الحل العام للمعادله التفاضليه المعطاه و هو عباره عن طائفه منحنيات الاسيه

تتلخص طريقه فصل المتغيرات

1)نعزل الحدود التي تحتوي على x مع dx في طرف والحدود التي تحتوي dy, y في طرف اخر نحصل على معادله بالشكل

$$g(y)dy = f(x)dx \qquad \ldots..(1)$$

2) نكامل طرفي المعادله (1) فيكون c ثابت اختياري $\int g(y)dy = \int f(x)dx$

3) قدر الامكان ان نضع حل المعادله y بدلاله

dy / dx =2x + 5

sol: dy / dx =2x + 5

dy =(2x + 5)dx.

∫dy =∫(2x + 5)dx

y =x² + 5x + c

****

Example: dy / dx =x-1 / y ;

solve the ODE

Sol: $dy/dx = x-1/y$

$ydy= (x-1)dx$

$\int ydy = \int(x-1)dx$

$1/2y^2= 1/2x^2 -x +c$

$y^2 = x^2 -2x +2c$

$y = \pm\sqrt{x^2-2x+c1}$

when $c2=c1$

Example: solve the following ODE

$dy = \sin x \cos^2 y\, dx$;whan $\cos y \neq 0$ and $y \neq (2n + 1)\, \pi/2$

sol: $dy= \sin x \cos^2 y\, dx$

$dy/\cos^2 y = \sin x\, dx$

$\sec^2 y\, dy = \sin x\, dx$

$\int \sec^2 y = \int dy \sin x\, dx$

$\tan y = -\cos x + c$

ملاحظه: المعادلة التفاضلية الاعتيادية من المرتبة الاولى والدرجة الاولى تأخذ إحدى الصيغتين

EX/ solve the ODE

$$\frac{dy}{dx} = 2x + 5$$

$$\frac{dy}{dx} = 2x + 5 \rightarrow dy = (2x + 5\,)dx$$

$$\int dy = \int (2x+5)dx \;\rightarrow y = \; x^2 + 5x + c$$

Solve the OD

EX/ 2 _

$$\frac{dy}{dx} = \frac{x-1}{y}$$

$$\frac{dy}{dx} = \frac{x-1}{y} \rightarrow ydy = (x-1)dx$$

$$\int ydy = \int (x-1)dx \;\rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 - x + c$$

$$y^2 = x^2 - 2x + 2c \;\rightarrow y = \mp\sqrt{x^2} - 2x + c_1$$

$$2c = c_1$$

EX/3_SOLVE THE ODE

$$dy = sinx\, cos^2 y\, dx$$

$$cosy \;\neq o \; and \; y \neq (2n+1)\frac{\pi}{2}$$

$$sol\_\_)(dy = \sin x\, cos^2 y\, dy$$

$$\rightarrow \frac{dy}{cos^2 y} \; = sinx\, dx \rightarrow \; sec^2 y\, dy = sinx\, dx$$

$$\int sec^2 y\, dy = \int sinx\, dx \;\rightarrow tany = \; -cosx + c$$

ملاحظة :- المعادلة التفاضلية الاعتيادية من الرتبة الاولى والدرجة الاولى تأخذ احدى الصيغتين

Ex/ solve the following initial value problem (I V P) where y=0

when x=o or y(0)=0 , $\frac{dy}{dx} = e^{x+y}$

$$\frac{dy}{dx} = e^{2x}e^{y} \qquad \frac{dy}{e^{y}} = e^{2x}dx$$

$$\int e^{-y}\,dy = \int e^{2x}dx \qquad -e^{-y} = \frac{1}{2}e^{2x} + C$$

$$y(0) = 0 \qquad -e^{0} = \frac{1}{2}e^{0} + C$$

$$C = \frac{-3}{2}$$

$$-e^{-y} = \frac{1}{2}e^{2x} + \frac{2}{3}$$