

## معادلة برنولي *Bernoulli Equation*

has the form

$$\boxed{\frac{dy}{dx} + p_{(x)}y = Q_{(x)}y^n}$$

Which is Non-Linear first order ODE , such equation can be reduce to linear ODE .

1. Divide Bernoulli equation by  $y^n$ .

$$y^{-n} \frac{dy}{dx} + p_{(x)}y^{-n+1} = Q_{(x)}$$

2. Let  $v = y^{-n+1} \Rightarrow \frac{dv}{dx} = (-n+1)y^{-n} \frac{dy}{dx} \Rightarrow y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx}$

3. Substitute in (1)

$$\frac{1}{1-n} \frac{dv}{dx} + p_{(x)}v = Q_{(x)}$$

وبضرب المعادلة أعلاه بـ  $(1-n)$  ، نحصل على :

$$\frac{dv}{dx} + (1-n)p_{(x)}v = (1-n)Q_{(x)}$$

Which is reduce to linear .

4.  $v = y^{-n+1}$

**Example 1:**  $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$  .

**Sol:-**

$$\frac{dy}{dx} + y = (\cos x - \sin x)y^2 \quad (n=2) * y^{-2}$$

$$y^{-2} \frac{dy}{dx} + y^{-1} = (\cos x - \sin x)$$

$$\text{Let } v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2} \frac{dy}{dx} \Rightarrow y^{-2} \frac{dy}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx} + v = \cos x - \sin x \Rightarrow \frac{dv}{dx} - v = \sin x - \cos x \quad \text{Linear Eq..}$$

$$p = e^{\int p dx} = e^{\int -dx} = e^{-x}$$

$$pv = \int p Q_{(x)} dx = \int e^{-x} (\sin x - \cos x) dx$$

$$= \int \underbrace{e^{-x} \sin x}_{\text{I}} dx - \int \underbrace{e^{-x} \cos x}_{\text{II}} dx$$

**I.**  $u = e^{-x} \Rightarrow du = -e^{-x} dx$

$$dv = \sin x dx \Rightarrow v = -\cos x$$

$$\int u dv = u.v - \int v du$$

$$\int e^{-x} \sin x dx = -e^{-x} \cos x - \int e^{-x} \cos x dx \dots (1)$$

$$\text{Let } u = e^{-x} \Rightarrow du = -e^{-x} dx$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$\int e^{-x} \cos x dx = e^{-x} \sin x + \int e^{-x} \sin x dx \quad \text{نعرض في (1)}$$

$$\int e^{-x} \sin x dx = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx$$

$$2 \int e^{-x} \sin x dx = -e^{-x} \cos x - e^{-x} \sin x .$$

$$\int e^{-x} \sin x dx = \frac{1}{2} [-e^{-x} \cos x - e^{-x} \sin x]$$

**[II].** Let  $u = e^{-x} \Rightarrow du = -e^{-x} dx$

$$du = \cos x dx \Rightarrow v = \sin x$$

$$\int u dv = v \cdot u - \int v du$$

$$\int e^{-x} \cos x dx = e^{-x} \sin x + \int e^{-x} \sin x dx \dots (2)$$

. (2) في (1) نعرض

$$\int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x dx$$

$$2 \int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x$$

$$\int e^{-x} \cos x dx = \frac{1}{2} [e^{-x} \sin x - e^{-x} \cos x].$$

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**Note :** Bernoulli of second kind .

$$\boxed{\frac{dx}{dy} + P_{(y)}x = Q_{(y)}x^n} \dots (3)$$

1. Divided by  $x^n$  .

$$x^{-n} \frac{dx}{dy} + P_{(y)}x^{-n+1} = Q_{(y)} \dots (4)$$

$$2. v = x^{-n+1} \Rightarrow \frac{dv}{dy} = (1-n)x^{-n} \frac{dx}{dy} \Rightarrow x^{-n} \frac{dx}{dy} = \frac{1}{1-n} \frac{dv}{dy}$$

3. Substitute in (4)

$$\frac{1}{1-n} \frac{dv}{dy} + P_{(y)}v = Q_{(y)}$$

$$\frac{dv}{dy} + (1-n)P_{(y)}v = Q_{(y)}$$

4. Which is reduced to linear of second kind .

$$5. v = x^{-n+1}.$$

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في الحقيقة أن معادلات بوللي هي معادلات ليست خطية بالرغم من كونها تشبه المعادلات الخطية وتكون قابلة للتحويل الى معادلات خطية بضرب طرفي المعادلة (1) بـ  $y^{-n}$ .

$$\frac{dy}{dx} + yP_{(x)} = Q_{(x)}y^n \dots (1)$$

فحصل على المعادلة (2)

$$y^{-n} \frac{dy}{dx} = y^{1-n}P_{(x)} = Q_{(x)} \dots (2)$$

$$\text{Let } w = y^{1-n} \Rightarrow \frac{dw}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

بضرب المعادلة (2) بـ  $(1-n)$  ثم التعويض عن  $w$  ومشتقاتها نحصل على معادلة خطية :

$$(1-n)y^{-n} \frac{dy}{dx} + y^{1-n}(1-n)P_{(x)} = (1-n)Q_{(n)}$$

$$\Rightarrow \frac{dw}{dx} + w(1-n)P_{(x)} = (1-n)Q_{(x)}.$$

**Example [1]** :- Find the general solution of the following ODE .

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

بضرب المعادلة بـ  $y^{-3}$  نحصل على :

$$y^{-3} \frac{dy}{dx} - xy^{-2} = -e^{-x^2} \dots (*)$$

$$\text{Let } w = y^{-2} \Rightarrow \frac{dw}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{2} \frac{dw}{dx} = y^{-3} \frac{dy}{dx}$$

ثم نعرض في المعادلة (\*)

$$-\frac{1}{2} \frac{dw}{dx} - wx = -e^{-x^2} \quad * (-2)$$

$\frac{dw}{dx} + 2wx = 2e^{-x^2}$ , which is reduced to linear Eq...

$$I.F. e^{\int 2x dx} = e^{x^2} \Rightarrow w.e^{x^2} = \int 2e^{-x^2} \cdot e^{x^2} dx + c$$

$$\Rightarrow w.e^{x^2} = 2x + c \Rightarrow \frac{e^{x^2}}{y^2} = 2x + c \Rightarrow e^{x^2} = y^2(2x + c)$$

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**Example [2]** :- find the general solution of the following ODE .

$$\frac{dx}{dy} + \frac{x}{y} = x^{-1} \cos y .$$

بضرب المعادلة بـ  $x^1$  نحصل على :

$$x \frac{dx}{dy} + \frac{x^2}{y} = \cos y \dots (*)$$

$$\text{Let } w = x^2 \Rightarrow \frac{dw}{dy} = 2x \frac{dx}{dy} \Rightarrow \frac{1}{2} \frac{dw}{dy} = x \frac{dx}{dy}$$

نعرض في المعادلة (\*) عن  $w$  ومشتقاتها :

$$\frac{1}{2} \frac{dw}{dy} + \frac{w}{y} = \cos y \quad * 2$$

$$\frac{dw}{dy} + \left(\frac{2}{y}\right)w = 2 \cos y$$

$$I.F. = e^{\int \frac{2}{y} dy} = e^{2 \ln y} = e^{Lny^2} = y^2 \Rightarrow w.y^2 = \int 2y^2 \cos y dy + c$$

$$w.y^2 = y^2 \sin y + 2y \cos y - 2 \sin y + c$$

$$\Rightarrow x^2 y^2 = y^2 \sin y + 2y \cos y - 2 \sin y + c$$

يمكن في النهاية أن نلخص خطوات حل معادلات بernoulli والتي تأخذ الصورة التالية :

$$\frac{dy}{dx} + P_{(x)}y = Q_{(x)}y^n$$

حيث  $n \neq 0, 1$

والتي يجب تحويلها الى صورة المعادلة الخطية ومن ثم حلها حسب الطرق السابقة .

١- نقسم طرفي المعادلة على  $y^n$  فتصبح المعادلة بالصورة التالية :

$$y^{-n} \frac{dy}{dx} + P_{(x)}y^{-n+1} = Q_{(x)} \dots (1)$$

٢- نفرض ان  $z = y^{-n+1}$  ثم نستبدل الطرفين بالنسبة الى  $x$  فنحصل على :

$$(1-n)y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$$

٣- نضرب طرفي المعادلة (1) بـ  $(1-n)$  ون التعويض عن  $y$  بدالة  $z$  .

$$\frac{dz}{dx} + (1-n)P_{(x)}z = (1-n)Q_{(x)}$$

٤- نضع  $(1-n)Q_{(x)} = q_{(x)}$  ،  $(1-n)P_{(x)} = p_{(x)}$

فتصبح المعادلة بالصورة  $\frac{dz}{dx} + p_{(x)}z = q_{(x)}$  وهي معادلة خطية في  $z$  .

٥- حل المعادلة هو  $I_{(x)}z = \int I_{(x)}Q_{(x)}dx + c$

٦- ثم باستبدال  $z = y^{1-n}$  ، فنحصل على الحل المطلوب .

$$I_{(x)}y^{1-n} = \int I_{(x)}Q_{(x)}dx + c$$

**Example [3]** :- Find the general solution of the following ODE .

$$dy + 2xy \, dx = xe^{-x^2} y^3 \, dx$$

**Sol:-** we can put the ODE as following :

$$\frac{dy}{dx} + 2xy = xe^{-x^2} y^3, \quad \text{which is Bernoulli Eq.}$$

$$y^{-3} \frac{dy}{dx} + 2xy^{-2} = xe^{-x^2}$$

$$\text{Let } z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow -\frac{1}{2} \frac{dz}{dx} = y^{-3} \frac{dy}{dx}$$

$$-\frac{1}{2} \frac{dz}{dx} + 2xz = xe^{-x^2} \Rightarrow \frac{dz}{dx} - 4xz = -2xe^{-x^2}$$

$$\frac{dz}{dx} + P(x)z = q(x) \quad \text{وهي معادلة خطية على صورة}$$

$$P(x) = -4x \Rightarrow \int P(x)dx = -2x^2, \quad q(x) = -2xe^{-x^2}$$

$$I(x) = e^{\int P(x)dx} = e^{-2x^2}$$

$$\int I(x)q(x)dx = \int e^{-2x^2}(-2xe^{-x^2})dx = -2 \int xe^{-3x^2}dx = \frac{1}{3}e^{-3x^2}$$

Solution of the linear ODE as following

$$I(x)z = \int I(x)q(x)dx + c$$

$$e^{-2x^2}z = \frac{1}{3}e^{-3x^2} + c$$

Put  $z = y^{-2}$ , then we get :

$$e^{-2x^2}y^{-2} = \frac{1}{3}e^{-3x^2} + c$$

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**Example 4** :- Find the general solution of the following ODE .

$$\frac{dy}{dx} - \left[ 1 + \frac{1}{x} \right] y = -2e^x y^2$$

**Sol** :- the above eq. is Bernoulli Eq. , multiply by  $y^{-2}$ .

$$y^{-2} \frac{dy}{dx} - \left[ 1 + \frac{1}{x} \right] y^{-1} = -2e^x$$

$$\text{Let } z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2} \frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2} \frac{dy}{dx}$$

$$-\frac{dz}{dx} - \left[ 1 + \frac{1}{x} \right] z = -2e^x \quad * (-1)$$

$$\frac{dz}{dx} + \left[ 1 + \frac{1}{x} \right] z = 2e^x \text{ , which is linear Eq. } \frac{dz}{dx} + P(x)z = q(x)$$

$$P(x) = 1 + \frac{1}{x} \Rightarrow \int P(x) dx = x + \ln x \text{ , } q(x) = 2e^x$$

$$I(x) = e^{\int P(x) dx} = e^{x + \ln x} = xe^x$$

$$\int I(x) q(x) dx = \int xe^x (2e^x) dx = 2 \int xe^{2x} dx = xe^{2x} - \frac{1}{2} e^{2x}$$

The solution of Eq. as following

$$I(x)z = \int I(x)q(x) dx + c$$

$$xe^x z = e^{2x} \left( x - \frac{1}{2} \right) + c$$

put  $x = y^{-1}$  , then we get

$$xe^x y^{-1} = e^{2x} \left( x - \frac{1}{2} \right) + c$$

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**Example [5]** :- Find the general solution of the following ODE .

$$\frac{dy}{dx} + \frac{1}{x-2}y = 5(x-2)\sqrt{y}$$

**Sol** :- the ODE is Bernoulli Eq.

$$y^{-\frac{1}{2}} \frac{dy}{dx} + \frac{1}{x-2} y^{\frac{1}{2}} = 5(x-2).$$

$$\text{Let } z = y^{\frac{1}{2}} \Rightarrow \frac{dz}{dx} = \frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx} \Rightarrow 2 \frac{dz}{dx} = y^{-\frac{1}{2}} \frac{dy}{dx}$$

$$2 \frac{dz}{dx} + \frac{1}{x-2}z = 5(x-2)$$

$$\frac{dz}{dx} + \frac{1}{2(x-2)}z = \frac{5(x-2)}{2}, \text{ which is Linear Eq.}$$

$$\frac{dz}{dx} + P(x)z = q(x)$$

$$P(x) = \frac{1}{2(x-2)} \Rightarrow \int P(x) dx = \frac{1}{2} \ln(x-2), q(x) = \frac{5(x-2)}{2}$$

$$I(x) = e^{\int P(x) dx} = e^{\int \frac{1}{2} \ln(x-2)} = (x-2)^{\frac{1}{2}}$$

$$\int I(x) q(x) dx = \int (x-2)^{\frac{1}{2}} \left( \frac{5(x-2)}{2} \right) dx$$

$$= \frac{5}{2} \int (x-2)^{\frac{3}{2}} dx = (x-2)^{\frac{5}{2}}, \text{ The solution of Eq. as following}$$

$$I(x)z = \int I(x) q(x) dx + c \Rightarrow (x-2)^{\frac{1}{2}}z = (x-2)^{\frac{5}{2}} + c$$

$$\text{Put } z = y^{-\frac{1}{2}}, \text{ then we get: } (x-2)^{\frac{1}{2}} \sqrt{y} = (x-2)^{\frac{5}{2}} + c$$

$$y = (x-2)^4 + \frac{c}{(x-2)}$$

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**H.W** :- Find the general solution of the following ODE .

$$1. \ y' + 3x^2y = y^2x e^{x^2}.$$

$$2. \ 3 dy - y dx = 3e^{4x/3} y^3.$$

$$3. \ y' + xy = x y^2.$$

$$4. \ y' + \frac{3}{x}y = x^4 \sqrt[3]{y}.$$

$$5. \ \frac{dy}{dx} + 3x^2y = y^2 x e^{x^3} dx.$$

$$6. \ 2y dx - x dy + x^3 y \cos y dy = 0.$$

$$7. \ (2xy^2 - y)dx + 2x dy = 0.$$

$$8. \ (2 + y^2)dx - (xy + 2y + y^3)dy = 0.$$

$$9. \ y(1 + y^2)dz = 2(1 - 2xy^2)dy.$$

$$10. \ \frac{dy}{dx} - 6y = 10 \sin 2x.$$

$$11. \ \frac{dx}{dy} - \frac{x}{6y} = -\frac{x^4}{3y^2}.$$

$$12. \ \frac{dy}{dx} - y \left( \frac{x}{1-x^2} \right) = \left( \frac{x^2}{1-x^2} \right) y^2.$$

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