مُعادلات من المرتبة الأولى والدرجة الثانية

لحل هذا الصنف من المعادلات نقوم بتخفيض درجتها الى الدرجة الأولى لحلها بالطرق السابقة وذلك بفرض

$$\frac{dy}{dx} = P \Rightarrow \left(\frac{dy}{dx}\right)^2 = P^2$$

ثم نقوم بتحليل المعادلة وتجزئتها الى قوسين (مقدارين) من الدرجة الأولى لنحلهما على انفراد :

Example (1) : solve the ODE .

$$xy\left(\frac{dy}{dx}\right)^2 + (x+y)\frac{dy}{dx} + 1 = 0$$

Solution : let $\frac{dy}{dx} = P \Rightarrow \left(\frac{dy}{dx}\right)^2 = P^2$

$$\Rightarrow xyP^2 + (x+y)P + 1 = 0 \quad \Rightarrow \quad (xp+1)(yp+1) = 0$$

$$\Rightarrow \quad xp + 1 = 0 \quad \cdots\cdots (1) \qquad and \qquad yp + 1 = 0 \cdots\cdots (2)$$

$$(1) \Rightarrow x\frac{dy}{dx} + 1 = 0 \Rightarrow dy = -\frac{dy}{dx} \Rightarrow y = -\ln x + \ln c_1$$

$$\Rightarrow y = \ln\left(\frac{c_1}{x}\right) \quad \cdots\cdots (3)$$

$$(2) \Rightarrow \quad y\frac{dy}{dx} + 1 = 0 \quad \Rightarrow \quad y\,dy + dx = 0 \quad \Rightarrow \quad \frac{y^2}{2} + x = c_2$$

$$\Rightarrow y = \pm\sqrt{2(c_2 - x)} \quad \cdots\cdots (4)$$

كل من العلاقتين (3) و (4) تمثلان حلاً للمعادلة .

Example (2) : solve the following ODE . $y^2P^2 - a^2 + y^2 = 0$

Solution : let $\frac{dy}{dx} = P \Rightarrow \left(\frac{dy}{dx}\right)^2 = P^2$

$$y^2P^2 = a^2 - y^2 \Rightarrow yp = \pm\sqrt{a^2 - y^2} \quad \Rightarrow y\frac{dy}{dx} = \pm\sqrt{a^2 - y^2}$$

$$dx = \frac{y\,dy}{\sqrt{a^2 - y^2}} \quad \cdots\cdots (1) \qquad and \qquad dx = \frac{-y\,dy}{\sqrt{a^2 - y^2}} \quad \cdots\cdots (2)$$

$$x = c_1 - \sqrt{a^2 - y^2} \quad \cdots\cdots (3) \qquad and$$

$$x = c_1 + \sqrt{a^2 - y^2} \quad \cdots\cdots (4)$$

كل من العلاقتين (٣) و (٤) تمثلان حلاً للمعادلة .

H.W (1) :

$1 - x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0$

$2 - xp^2 + (1 - x^2y)p - xy = 0$

$3 - p^2 - x^2y^2 = 0$

$4 - x^2p^2 - 5xyp + 6y^2 = 0$

معادلات من المرتبة الثانية والدرجة الأولى

لحل هذا الصنف من المعادلات نقوم بتخفيض رتبتها الى الرتبة الأولى لحلها بالطرق السابقة وذلك بفرض

$$\frac{dy}{dx} = y' = P \implies \frac{d^2y}{dx^2} = y'' = P' = \frac{dp}{dx}$$

هذا في حالة المعادلات التي لا يظهر فيها المتغير غير المستقل (y) [المتغير المعتمد]

Example (3) : solve the ODE . $xy'' + y' = 4x \quad \cdots\cdots (1)$

Solution : let $\frac{dy}{dx} = y' = P \implies \frac{d^2y}{dx^2} = y'' = P' = \frac{dp}{dx}$

then the ODE (1) become , as following , $x\frac{dp}{dx} + P = 4x \implies \frac{dp}{dx} + \frac{p}{x} = 4$

which is linear Equation

$$I \cdot F \cdot = e^{\int \frac{dx}{x}} = e^{\ln x} = x \implies px = \int 4x\,dx + c$$

$$\implies px = 2x^2 + c \implies p = 2x + \frac{c}{x}$$

$$\implies \frac{dy}{dx} = 2x + \frac{c}{x} \implies y = x^2 + c\ln x + c_1$$

Example (4) : solve the following ODE . $x^2y'' - (y')^2 - 2xy' = 0$

Solution : let $\frac{dy}{dx} = y' = P \implies \frac{d^2y}{dx^2} = y'' = P' = \frac{dp}{dx}$

then the a bore ODE become as following

$$x^2\frac{dp}{dx} - P^2 - 2xP = 0 \implies \frac{dp}{dx} - \frac{2P}{x} = \frac{P^2}{x^2}$$

$$\Rightarrow \quad -P^{-2}\frac{dp}{dx} + \frac{2P^{-1}}{x} = \frac{-1}{x^2}$$

$$let\ W = P^{-1} \qquad \Rightarrow \qquad \frac{dw}{dx} = -P^{-2}\frac{dp}{dx}$$

$$\frac{dw}{dx} + \frac{2w}{x} = \frac{-1}{x^2} \qquad \Rightarrow \qquad I \cdot F \cdot = e^{\int \frac{2dx}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$wx^2 = \int \frac{-1}{x^2} \cdot x^2\, dx + c \qquad \Rightarrow \qquad wx^2 = -x + c$$

$$\frac{x^2}{P} = c - x \qquad \Rightarrow \quad P = \frac{x^2}{c-x}$$

$$\frac{dy}{dx} = \frac{x^2}{c-x} \qquad \Rightarrow \qquad dy = \left(\frac{x^2}{c-x}\right) dx$$

$$\Rightarrow \quad dy = \left(-x - c + \frac{c^2}{c-x}\right) dx$$

$$\Rightarrow \quad y = \left(-\frac{x^2}{2} - cx + c^2 \ln(c-x) + c_1\right)$$

* وفي حالة المعادلات التي لا يظهر فيها المتغير المستقل (x) سيكون الفرض كالاتي :

$$\frac{dy}{dx} = y' = P \qquad \Rightarrow \qquad \frac{d^2y}{dx^2} = y'' = P' \quad \Rightarrow \quad \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = P\frac{dp}{dy}$$

Example (5) : solve the following ODE . $\quad yy'' + (y')^2 + 1 = 0$

Solution : let $\quad y' = P \Rightarrow \frac{d^2y}{dx^2} = y'' = P' \Rightarrow \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = P\frac{dp}{dy}$

then the a bore ODE become as following

$$yP\frac{dp}{dy} + P^2 + 1 = 0 \qquad \Rightarrow \qquad \frac{dp}{dy} + \frac{P}{y} = -P^{-1}y^{-1}$$

$$2P\frac{dp}{dy} + \frac{2P^2}{y} = -2y^{-1}$$

$$\Rightarrow \ let \quad w = P^2 \qquad \Rightarrow \qquad \frac{dw}{dy} = 2P\frac{dp}{dy}$$

$$\frac{dw}{dy} + \frac{2w}{y} = \frac{-2}{y} \quad \Rightarrow \quad I \cdot F \cdot = e^{\int \frac{2dy}{y}} = e^{2\ln y} = e^{\ln y^2} = y^2$$

$$wy^2 = \int -2\,y\,dy + c \quad \Rightarrow \quad wy^2 = y^2 + c$$

$$P^2y^2 = c - y^2 \quad \Rightarrow \quad P^2 = \frac{c - y^2}{y^2}$$

$$P = \pm\frac{\sqrt{c - y^2}}{y} \quad \Rightarrow \frac{dy}{dx} = \pm\frac{\sqrt{c - y^2}}{y} \Rightarrow dx = \frac{ydy}{\sqrt{c - y^2}}$$

$$x_1 = c_1 - \sqrt{c - y^2} \quad or \quad x_2 = c_1 + \sqrt{c - y^2}$$

H.W (2) :

1 – $xy'' - (y')^3 - y' = 0$

2 – $yy'' + (y')^3 = 0$

3 – $y'' = e^x(y')^2$

4 – $3yy'y'' = (y')^3 - 1$

5 – $x^2y'' + (y')^2 - = 0$

6 – $y'' - a^2y = 0$

7 – $yy'' + 2y' - 2(y')^2 = 0$

8 – $x^3y'' - x^2y' = 3 - x^2$

9 – $y'' = x(y')^2$

10 – $yy'' + (y')^3 - (y')^2 = 0$

11 – $y'' + y = 0$

12 – $xy'' = y' - (2 - 3xy')$

13 – $x(x+1)y'' + xy' - 1 = 0$