# University of Anbar 

## College of Science

# Department of Applied Geology 

Advanced Structural Geology<br>Title of the lecture Mohr diagram

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## Mohr diagram for stress

The equations we derived for $\sigma_{\mathrm{n}}$ and $\sigma_{\mathrm{s}}$ do not offer an obvious sense of their values as a function of orientation of a plane in our block of clay. Of course, a programmable calculator or simple computer program will do the job, but a convenient graphical method, known as the Mohr diagram (Figure 1), was introduced over a century ago to solve Equations. A Mohr diagram is an "XY"-type (Cartesian) plot of $\sigma_{s}$ versus $\sigma_{n}$ that graphically solves the equations for normal stress and shear stress acting on a plane within a stressed body. In our experiences, many people find the Mohr construction difficult to comprehend. So, we'll first examine the proof and underlying principles of this approach to try to take the magic out of the method.

$$
\begin{aligned}
& {[\sigma n-1 / 2(\sigma 1+\sigma 3)]^{2}=[1 / 2(\sigma 1-\sigma 3)]^{2} \cos ^{2} 2 \theta} \\
& \Sigma \mathrm{~s}^{2}=[1 / 2(\sigma 1-\sigma 3)]^{2} \sin ^{2} \theta
\end{aligned}
$$



Figure 1 The Mohr diagram for stress. Point Prepresents the plane in our clay experiment.
Adding the last equations
$\left[\sigma_{\mathrm{n}}-1 / 2(\sigma 1+\sigma 3)\right]^{2}+\sigma s^{2}=\left[1 / 2\left(\sigma_{1}-\sigma_{3}\right)\right]^{2} \cdot\left(\cos ^{2} 2 \theta+\sin ^{2} 2 \theta\right)$
Using the trigonometric relationship $\left(\cos ^{2} 2 \theta+\sin ^{2} 2 \theta\right)=1$ gives
$\left[\sigma n-1 / 2\left(\sigma_{1}+\sigma_{3}\right)\right]^{2}+\sigma s^{2}=\left[1 / 2\left(\sigma_{1}-\sigma^{3}\right)\right]^{2}$
Note the equation has the form $(x-a)^{2}+y^{2}=r^{2}$, which is the general equation for a circle with radius $r$ and centered on the x -axis at distance a from the origin. Thus, the Mohr circle has a radius $1 / 2\left(\sigma_{1}-\sigma_{3}\right)$ that is centered on the $\sigma$ axis at a distance $1 / 2\left(\sigma_{1}+\sigma_{3}\right)$ from the origin. The
construction is shown in Figure 1. You also see from this figure that the Mohr circle's radius, 1 $\AA(\sigma 1-\sigma 3)$, is the maximum shear stress, $\sigma s$, max. The stress difference $\left(\sigma_{1}-\sigma_{3}\right)$, called the differential stress, is indicated by the symbol $\sigma_{d}$.

## Constructing the Mohr Diagram

To construct a Mohr diagram, we draw two mutually perpendicular axes; $\sigma_{\mathrm{n}}$ is the abscissa (xaxis) and $\sigma s$ is the ordinate ( $y$-axis). In our clay deformation experiment, the maximum principal stress $\left(\sigma_{1}\right)$ and the minimum principal stress $\left(\sigma_{3}\right)$ act on plane P that makes an angle $\theta$ with the $\sigma_{3}$ direction (Figure 4); in the Mohr construction we then plot $\sigma_{1}$ and $\sigma_{3}$ on the $\sigma$-axis (Figure 1). These principal stress values are plotted on the on axes because they are normal stresses, but with the special condition that the planes on which they act, the principal planes, have zero shear stress $(\sigma s=0)$. We then construct a circle through points $\sigma_{1}$ and $\sigma_{3}$, with $O$, the midpoint, at 1 $\Omega\left(\sigma_{1}+\sigma_{3}\right)$ as center, and a radius of $1 / 2\left(\sigma_{1}-\sigma_{3}\right)$. Next, we draw a line OP such that angle PO $\sigma 1$ is equal to $2 \theta$. This step often gives rise to confusion and errors. First, remember that we plot twice the angle $\theta$, which is the angle between the plane and $\sigma_{3}$, because of the equations we are solving. Second, remember that we measure $2 \theta$ from the $\sigma_{1}$-side on the $\sigma n$-axis. When this is done, the Mohr diagram is complete and we can read off the value of $\sigma_{n . p}$ along the $\sigma n-a x i s$, and the value of $\sigma_{\text {s.p }}$ along the $\sigma_{s}$-axis for our plane P , as shown in Figure 5. We see that
$\sigma_{\mathrm{n} . \mathrm{p}}=1 / 2\left(\sigma_{1}+\sigma_{3}\right)+1 / 2\left(\sigma_{1}-\sigma_{3}\right) \cos 2 \theta$ and
$\sigma_{\mathrm{s} . \mathrm{p}}=1 / 2\left(\sigma_{1}-\sigma_{3}\right) \sin 2 \theta$
A couple of additional observations can be made from the Mohr diagram (Figure 2). There are two planes, oriented at angle $\theta$ and its complement $(90-\theta)$, with equal shear stresses but different normal stresses. Also, there are two planes with equal normal stress, but with shear stresses of opposite sign (that is, they act in different directions on these planes). In general, for each orientation of a plane, defined by its angle $\theta$, there is a corresponding point on the circle. The coordinates of that point represent the normal and shear stresses that act on that plane. For example, when $\theta=0^{\circ}$ (that is, for a plane parallel to $\sigma_{3}$ ), P coincides with $\sigma_{1}$, which gives $\sigma \mathrm{n}=$ $\sigma_{1}$ and $\sigma s=0$. In other words, for any value of $\sigma_{1}$ and $\sigma_{3}\left(\sigma_{3}=\sigma_{2}\right.$ in our compression experiment), we can determine $\sigma$ n and $\sigma$ s graphically for planes that lie at an angle $\theta$ with $\sigma_{3}$. If we decide to change our earlier experiment by gluing the planks to the clay block and then moving the planks apart (a tension experiment), we must use a negative sign for the least principal stress (in this case, $\sigma_{1}=\sigma_{2}$ and $\sigma_{3}$ is negative). So, the center $O$ of the Mohr circle can lie on either side of the origin, but is always on the $\sigma$-axis.


Figure 2 For each value of the shear stress and the normal stress there are two corresponding planes, as shown in the Mohr diagram (a). The corresponding planes in $\sigma_{1}-\sigma_{3}$ space are shown in (b).

The Mohr diagram also nicely illustrates the attitude of planes along which the shear stress is greatest for a given state of stress. The point on the circle for which $\sigma_{s}$ is maximum corresponds to a value of $2 \theta=90^{\circ}$. For the same point, the magnitude of $\sigma_{s}$ is equal to the radius of the circle, that is, $1 / 2\left(\sigma_{1}-\sigma_{3}\right)$. Thus the $\left(\sigma_{1}-\sigma_{3}\right)$, the differential stress, is twice the magnitude of the maximum shear stress:
$\sigma_{d}=2 \sigma s, \max$

When there are changes in the principal stress magnitudes without a change in the differential stress, the Mohr circle moves along the $\sigma_{\mathrm{n}}$-axis without changing the magnitude of $\sigma_{\mathrm{s}}$. In our experiment, this would be achieved by increasing the air pressure in the classroom or carrying out the experiment under water; this "surrounding" pressure is called the confining pressure $\left(\mathrm{P}_{\mathrm{c}}\right)$ of the experiment. Figure 3a shows six planes in a stressed body at different angles with $\sigma_{3}$. Using the graph in Figure 3b, draw the Mohr circle and estimate the normal and shear stresses for these six planes. You can check your estimates by using Equations.


Figure 3 Adventures with the Mohr circle. To estimate the normal and shear stresses on the six planes in (a) apply the Mohr construction in (b). The principal stresses and angles $\theta$ are given in (a).

## Some Common Stress States

Now that you are familiar with the Mohr construction, let's look at its representation of the various stress states that were mentioned earlier. The three-dimensional Mohr diagrams in Figure 4 may at first appear a lot more complex than those in our earlier examples, because they represent three-dimensional stress states rather than two-dimensional conditions. Threedimensional Mohr constructions simply combine three two-dimensional Mohr circles for ( $\sigma_{1}-$ $\left.\sigma_{2}\right),\left(\sigma_{1}-\sigma_{3}\right)$, and $\left(\sigma_{2}-\sigma_{3}\right)$, and each of these three Mohr circles adheres to the procedures outlined earlier.Figure 4 a shows the case for general triaxial stress, in which all three principal stresses have nonzero values ( $\sigma_{1}>\sigma_{2}>\sigma_{3} \neq 0$ ). Biaxial (plane) stress, in which one of the principal stresses is zero (e.g., $\sigma_{3}=0$ ) is shown in Figure 4b. Uniaxial compression ( $\sigma_{2}=\sigma_{3}=0$; $\sigma_{1}>0$ ) is shown in Figure 4c, whereas uniaxial tension ( $\sigma_{1}=\sigma_{2}=0$; $\sigma_{3}<0$ ) would place the Mohr circle on the other side of the on-axis. Finally, isotropic stress, often called hydrostatic pressure, is represented by a single point on the on-axis of the Mohr diagram (positive for compression, negative for tension), because all three principal stresses are equal in magnitude ( $\sigma_{1}$ $=\sigma_{2}=\sigma_{3}$; Figure 4d).


Figure 4 Mohr diagrams of some representative stress states: (a) triaxial stress, (b) biaxial (plane) stress, (c) uniaxial compression, and (d) isotropic stress or hydrostatic pressure, P (compression is shown).

## Mean stress and deviatoric stress

Because of a body's response to stress, we subdivide the stress into two components, the mean stress and the deviatoric stress (Figure 5). The mean stress is defined as $\left(\sigma_{1}+\sigma_{2}+\sigma_{3}\right) / 3$, using the symbol $\sigma_{\mathrm{m}}$. The difference between mean stress and total stress is the deviatoric stress ( $\sigma_{\mathrm{dev}}$ ), so
$\sigma=\sigma_{\text {mean }}+\sigma_{\text {dev }}$
The mean stress is often called the hydrostatic component of stress or the hydrostatic pressure, because a fluid is stressed equally in all directions. Because the magnitude of the hydrostatic stress is equal in all directions it is an isotropic stress component. When we consider rocks at depth in the Earth, we generally refer to lithostatic pressure, $\mathrm{P}_{1}$, rather than the hydrostatic pressure. The lithostatic stress component is best explained by a simple but powerful calculation. Consider a rock at a depth of 3 km in the middle of a continent. The lithostatic pressure at this
point is a function of the weight of the overlying rock column because other (tectonic) stresses are unimportant. The local pressure is a function of rock density, depth, and gravity:
$\mathrm{P}_{\mathrm{l}}=\rho \cdot \mathrm{g} \cdot \mathrm{h}$
If $\rho$ (density) equals a representative crustal value of $2700 \mathrm{~kg} / \mathrm{m} 3, \mathrm{~g}$ (gravity) is $9.8 \mathrm{~m} / \mathrm{s} 2$, and h (depth) is 3000 m , we get

$$
\mathrm{P}_{1}=2700 \cdot 9.8 \cdot 3000=79.4 \cdot 106 \mathrm{~Pa} \approx 80 \mathrm{MPa} \text { (or } 800 \text { bars) }
$$



Figure 5 The mean (hydrostatic) and deviatoric components of the stress. (a) Mean stress causes volume change and (b) deviatoric stress causes shape change.

In other words, for every kilometer in the Earth's crust the lithostatic pressure increases by approximately 27 MPa . With depth the density of rocks increases, so you cannot continue to use the value of $2700 \mathrm{~kg} / \mathrm{m}^{3}$. For crustal depths greater than approximately 15 km the average density of the crust is $2900 \mathrm{~kg} / \mathrm{m}^{3}$. Deeper into Earth the density increases further, reaching as much as $13,000 \mathrm{~kg} / \mathrm{m}^{3}$ in the solid inner core.

Because the lithostatic pressure is of equal magnitude in all directions, it follows that $\sigma_{1}=\sigma_{2}=$ $\sigma_{3}$. The actual state of stress on a body at depth in the Earth is often more complex than only that from the overlying rock column. Anisotropic stresses that arise from tectonic processes, such as the collision of continental plates or the drag of the plate on the underlying material, contribute to the stress state at depth. The differential stresses of these anisotropic stress components, however, are many orders of magnitude less than the lithostatic stress. In the crust, differential stresses may reach a few hundred megapascals, but in the mantle, where lithostatic pressure is
high, they are only on the order of tens of megapascals or less. Yet, such low differential stresses are responsible for the slow motion of "solid" mantle that is a critical element of our planet's plate dynamics. Let's return to Figure 9 and the preceding comments. Why divide a body's stress state into an isotropic (lithostatic/hydrostatic) and an anisotropic (deviatoric) component? For our explanation we return to look at the deformation of a stressed body. Because isotropic stress acts equally in all directions, it results in a volume change of the body (Figure 5a). Isotropic stress is responsible for the consequences of increasing water pressure at depth on a human body. Place an air-filled balloon under water and you will see that isotropic stress maintains the spherical shape of the balloon, but reduces the volume. Deviatoric stress, on the other hand, changes the shape of a body (Figure 5b). Distortion of a body can often be measured in structural geology, but volume change is considerably more difficult to determine. As in determining distortions, knowledge about the original volume of a body is the obvious way to determine any volume change. Reliable volume markers, however, are rare in rocks and we resort to indirect approaches such as chemical contrasts between deformed and undeformed samples. The division between the isotropic and anisotropic components of stress provides the connection between the volumetric and distortional components of deformation, respectively.

## A brief summary of stress

Let's summarize where we are in our understanding of stress. You have seen that there are two ways to talk about stress. First, you can refer to stress on a plane (or traction), which can be represented by a vector (a quantity with magnitude and direction) that can be subdivided into a component normal to the plane ( $\sigma_{\mathrm{n}}$, the normal stress) and a component parallel to the plane ( $\sigma_{\mathrm{s}}$, the shear stress). If the shear stress is zero, then the stress vector is perpendicular to the plane, but this is a special case; in general, the stress vector is not perpendicular to the plane on which it acts. It is therefore meaningless to talk about stress without specifying the plane on which it is acting. For example, it is wrong to say "the stress at 1 km depth in the Earth is $00^{\circ} / 070^{\circ}$," but it is reasonable to say "the stress vector acting on a vertical, north-south striking joint surface is oriented $00^{\circ} / 070^{\circ}$." In this example there must be a shear stress acting on the fracture; check this for yourself. If the magnitude of this shear stress exceeds the frictional resistance to sliding along the fracture, then there might be movement. The stress state at a point cannot be described by a single vector. Why? Because a point represents the intersection of an infinite number of planes, and without knowing which plane you are talking about, you cannot define the stress vector. If you want to describe the stress state at a point you must have a tool that will allow you to calculate the stress vector associated with any of the infinite number of planes. We introduced three tools: (1) the stress ellipsoid, (2) the three principal stress axes, and (3) the stress tensor. The stress ellipsoid is the envelope containing the tails or tips (for compression and tension, respectively) of the stress vectors associated with the infinite number of planes passing through
the point, with each of the specified vectors and its opposite associated with one plane. On all but three of these planes the vectors have shear stress components. As a rule, there will be three mutually perpendicular planes on which the shear component is zero; the stress vector acting on each of these planes is perpendicular to the plane. These three planes are called the principal planes of stress, and the associated stress vectors are the principal axes of stress, or principal stresses ( $\sigma_{1} \geq \sigma_{2} \geq \sigma_{3}$ ). Like any ellipsoid, the stress ellipsoid has three axes, and the principal stresses lie parallel to these axes. Given the three principal stresses, you have uniquely defined the stress ellipsoid; given the stress ellipsoid, you can calculate the stress acting on any random plane that passes through the center of the ellipsoid (which is the point for which we defined the stress state). So, the stress ellipsoid and the principal stresses give a complete description of the stress at a point. Structural geologists find these tools convenient to work with because they are easy to visualize. Thus, we often represent the stress state at a point by picturing the stress ellipsoid, or we talk about the values of the principal stresses at a location. For example, we would say that "the orientation of the maximum principal stress at the New York-Pennsylvania border trends about $070^{\circ}$." For calculations, these tools are a bit awkward and a more general description of stress at a point is needed; this tool is the stress tensor. The stress tensor consists of the components of three stress vectors, each associated with a face of an imaginary cube centered in a specified Cartesian frame of reference. Each face of the cube contains two of the Cartesian axes. If it so happens that the stress vectors acting on the faces of the cube have no shear components, then by definition they are the principal stresses, and the axes in your Cartesian reference frame are parallel to the principal stresses. But if you keep the stress state constant and rotate the reference frame, then the three stress vectors will have shear components. The components of the three stress vectors projected onto the axes of your reference frame (giving one normal stress and two shear stresses) are written as components in a $3 \times 3$ matrix (a second-rank tensor). If the axes of the reference frame happen to be parallel to the principal stresses, then the diagonal terms of the matrix are the principal stresses and the off-diagonal terms are zero (that is, the shear stresses are zero). If the axes have any other orientation, then the diagonal terms are not the principal stresses and some, or all, of the off-diagonal terms are not equal to zero. When using the three principal stresses or the stress ellipsoid, you are merely specifying a special case of the stress tensor at a point.

## References

Ben A. van der Pluijm, and Stephen Marshak. (2004) EARTH STRUCTURE AN INTRODUCTION TO STRUCTURAL GEOLOGY AND TECTONICS. Second edition.

