ELASTIC BUCKLING OF COLUMNS

EULER LOAD

It is informative to begin the formulation of the column equation with a much idealized model, the Euler column. The axially loaded member shown in Fig. 1 is assumed to be prismatic (constant cross-sectional area) and to be made of homogeneous material. In addition, the following further assumptions are made:

1. The member's ends are pinned. The lower end is attached to an immovable hinge, and the upper end is supported in such a way that it can rotate freely and move vertically, but not horizontally.

2. The member is perfectly straight, and the load P, considered positive when it causes compression, is concentric.

3. The material obeys Hooke's law.

4. The deformations of the member are small so that the term $(y')^2$ is negligible compared to unity in the expression for the curvature:

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}} \approx y''$$

$$-\frac{M}{EI} = \frac{y''}{[1 + (y')^2]^{3/2}} \simeq y''$$

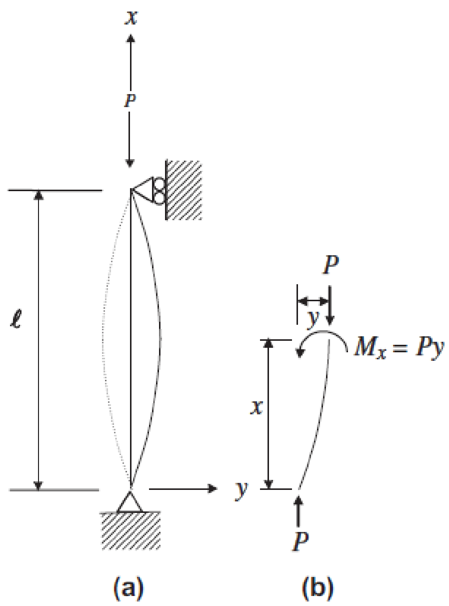

Figure1: Euler column

From the free body, part (b) in Fig.1, the following becomes immediately obvious:

$$EIy'' = -M(x) = -Py \quad \text{or} \quad EIy'' + Py = 0 \quad \text{........(1)}$$

This Equation is a second-order linear differential equation with constant coefficients. Its boundary conditions are: $y = 0$ at $x = 0$ and $x = \ell$

Let $k^2 = P/EI$, then $y'' + k^2 y = 0$. Assume the solution to be of a form $y = \alpha e^{mx}$ for which $y' = \alpha m e^{mx}$ and $y'' = \alpha m^2 e^{mx}$. Substituting these into Eq. above yields $(m^2 + k^2)\ \alpha e^{mx} = 0$. Since αe^{mx} cannot be equal to zero for a nontrivial solution, $m^2 + k^2 = 0$. Substituting gives

$$y = C_1 \alpha e^{kix} + C_2 \alpha e^{-kix} = A \cos kx + B \sin kx$$

A and B are integral constants, and they can be determined by boundary conditions.

$$y = 0 \quad at\ x = 0 \Rightarrow A = 0$$

$$y = 0 \quad \text{at } x = \ell \Rightarrow B \sin k\ell = 0$$

As $B \neq 0$ (if $B = 0$, then it is called a trivial solution; $0 = 0$), $\sin k\ell = 0 \Rightarrow k\ell = n\pi$
where $n = 1, 2, 3, \ldots$ but $n \neq 0$. Hence, $k^2 = P/EI = n^2\pi^2/\ell^2$, from which it follows immediately

$$P_{cr} = \frac{n^2\pi^2 EI}{\ell^2} \quad (n = 1, 2, 3, ..) \quad \text{..........(2)}$$

The eigenvalues P_{cr}, called critical loads, denote the values of load P for which a nonzero deflection of the perfect column is possible. The deflection shapes at critical loads, representing the eigen modes or eigenvectors, are given by: $y = B \sin \dfrac{n\pi x}{\ell}$

Note that B is undetermined, including its sign; that is, the column may buckle in any direction. Hence, the magnitude of the buckling mode shape cannot be determined, which is said to be immaterial. The smallest buckling load for a pinned prismatic column corresponding to n=1 is:

$$P_E = \frac{\pi^2 EI}{\ell^2}$$

If a pinned prismatic column of length ‘ is going to buckle, it will buckle at n = 1 unless external bracings are provided in between the two ends. A curve of the applied load versus the deflection at a point in a structure such as that shown in part (a) of Fig. 1-3 is called the equilibrium path.

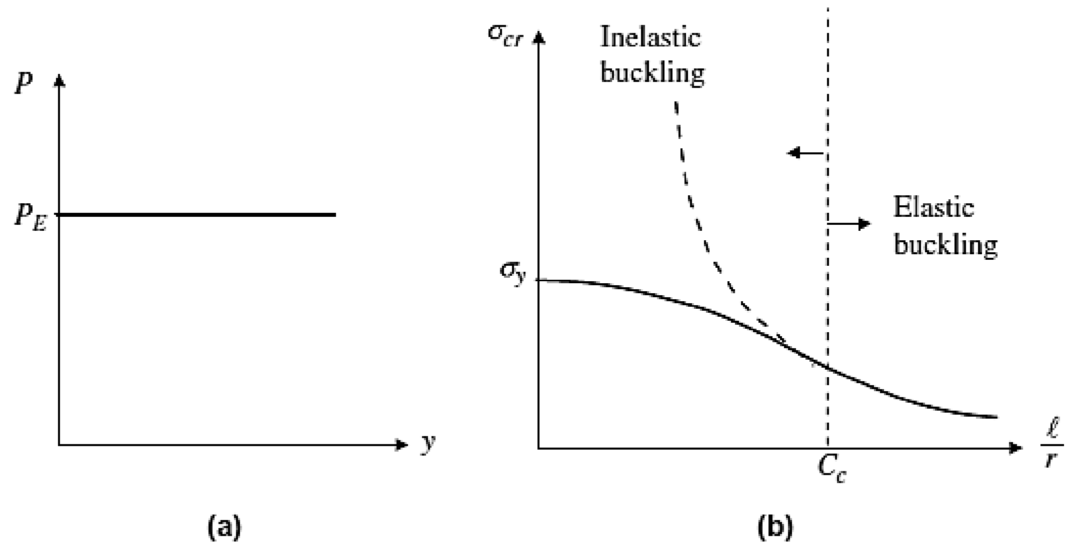

Figure 1-3 Euler load and critical stresses

It will be recalled that the equilibrium condition is determined based on the deformed geometry of the structure in part (b) of Fig. 1-2. The theory that takes into account the effect of deflection on the equilibrium conditions is called the second-order theory. The governing equation, Eq. (1), is an ordinary linear differential equation. It describes neither linear nor nonlinear responses of a structure. It describes an eigenvalue problem. Any nonzero loading term on the right-hand side of Eq. (1) will induce a second-order (nonlinear) response of the structure. Dividing Eq. (2) by the cross-sectional area A gives the critical stress

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{\ell^2 A} = \frac{\pi^2 EAr^2}{\ell^2 A} = \frac{\pi^2 E}{(\ell/r)^2}$$

where *l/r* is called the slenderness ratio and $r = (I/A)^{1/2}$ is the radius of gyration of the cross section. Note that the critical load and hence, the critical buckling stress is independent of the yield stress of the material. They are only the function of modulus of elasticity and the column geometry. In Fig. 2 (b), Cc is the threshold value of the slenderness ratio from which elastic buckling commences.

$$\text{eigen pair} \begin{cases} \text{eigenvalue} = P_{cr} = \dfrac{n^2\pi^2 EI}{\ell^2} \\ \text{eigenvector} = y = B\sin\dfrac{n\pi x}{\ell} \end{cases}$$

Example 1 Consider a both-ends-fixed column shown in Fig. 3.

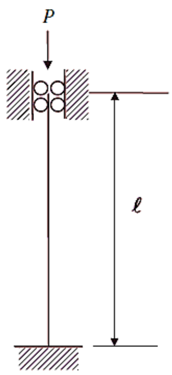

Figure 3: Both-ends-fixed column

$$y' = Ak\cos kx - Bk\sin kx + C$$

$$y'' = -Ak^2\sin kx - Bk^2 kx$$

$$y = 0 \quad \text{at } x = 0 \Rightarrow B + D = 0$$

$$y' = 0 \quad \text{at } x = 0 \Rightarrow Ak + C = 0$$

$$y = 0 \quad \text{at } x = \ell \Rightarrow A\sin k\ell + B\cos k\ell + C\ell + D = 0$$

$$y' = 0 \quad \text{at } x = \ell \Rightarrow Ak\cos k\ell - Bk\sin k\ell + C = 0$$

For a nontrivial solution for *A*, *B*, *C*, and *D* (or the stability condition equation), the determinant of coefficients must vanish. Hence,

$$Det = \begin{vmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin k\ell & \cos k\ell & \ell & 1 \\ k\cos k\ell & -k\sin k\ell & 1 & 0 \end{vmatrix} = 0$$

Expanding the determinant gives: $2(\cos k\ell - 1) + k\ell \sin k\ell = 0$

Know the following mathematical identities:

$$\begin{cases} \sin k\ell = \sin\left(\frac{k\ell}{2} + \frac{k\ell}{2}\right) = \sin\frac{k\ell}{2}\cos\frac{k\ell}{2} + \cos\frac{k\ell}{2}\sin\frac{k\ell}{2} = 2\sin\frac{k\ell}{2}\cos\frac{k\ell}{2} \\ \cos k\ell = \cos\left(\frac{k\ell}{2} + \frac{k\ell}{2}\right) = \cos\frac{k\ell}{2}\cos\frac{k\ell}{2} - \sin\frac{k\ell}{2}\sin\frac{k\ell}{2} = 1 - 2\sin^2\frac{k\ell}{2} \\ \quad \Rightarrow \cos k\ell - 1 = -2\sin^2\frac{k\ell}{2} \end{cases}$$

Rearranging the determinant given above yields:

$$2\left(-2\sin^2\frac{k\ell}{2}\right) + k\ell\left(2\sin\frac{k\ell}{2}\cos\frac{k\ell}{2}\right) = 0$$

$$\Rightarrow \sin\frac{k\ell}{2}\left(\frac{k\ell}{2}\cos\frac{k\ell}{2} - \sin\frac{k\ell}{2}\right) = 0$$

Let $u = k\ell/2$, then the solution becomes $\sin u = 0$ *or* $\tan u = u$.
For $\sin u = 0 \Rightarrow u = n\pi$ *or* $k\ell = 2n\pi \Rightarrow P_{cr} = 4n^2\pi^2 EI/\ell^2$. Substituting the eigenvalue $k = 2n\pi/\ell$ into the buckling mode shape yields

$$y = c_1 \sin\frac{2n\pi x}{\ell} + c_2\cos\frac{2n\pi x}{\ell} + c_3 x + c_4$$

$y = 0$ at $x = 0 \Rightarrow 0 = c_2 + c_4 \Rightarrow c_4 = -c_2$ Hence, $y = c_1 \sin(2n\pi x/\ell) + c_2\left(\cos(2n\pi x/\ell) - 1\right) + c_3 x$

$$y = 0 \text{ at } x = \ell \Rightarrow 0 = c_1 \sin 2n\pi + c_2(\cos 2n\pi - 1) + c_3\ell \Rightarrow c_3 = 0$$

$$y' = -\frac{2n\pi}{\ell} c_2 \sin\frac{2n\pi x}{\ell} + \frac{2n\pi}{\ell} c_1 \cos\frac{2n\pi x}{\ell}$$

Hence, y = $\Longrightarrow$ hape as shown in Fig. 4.

$$y' = 0 \quad \text{at } x = 0 \Rightarrow y' = 0 + \frac{2n\pi}{\ell} c_1 \Rightarrow c_1 = 0$$

$$\text{If } n = 1, \quad p_{cr} = \frac{\pi^2 EI}{\left(\frac{\ell}{2}\right)^2} = \frac{\pi^2 EI}{(\ell_e)^2}$$

where $l_e = l/2$ is called the effective buckling length of the column.

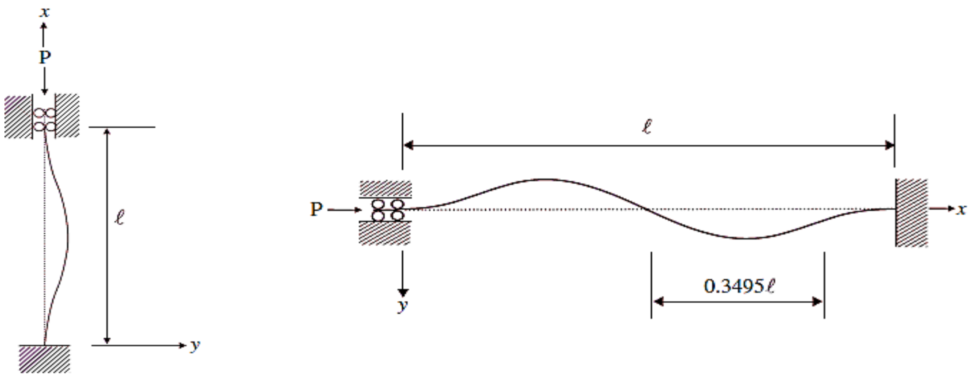

Figure 4: Mode shape, first mode and second mode

EFFECTS OF BOUNDARY CONDITIONS ON THE COLUMN STRENGTH

The critical column buckling load on the same column can be increased in two ways. 1. Change the boundary conditions such that the new boundary condition will make the effective length shorter.

(a) pinned-pinned: $l_e = l$

(b) pinned-fixed: $l_e = 0.7l$

(c) fixed-fixed: $l_e = 0.5l$

(d) flag pole (cantilever): $l_e = 2l$

2. Provide intermediate bracing to make the column buckle in higher modes i.e. achieve shorter effective length.