STRENGTH OF COMPRESSION MEMBERS IN PRACTICE

The highly idealized straight form assumed for the struts considered so far cannot be achieved in practice. Members are never perfectly straight and they can never be loaded exactly at the centroid of the cross section. Deviations from the ideal elastic plastic behavior defined by Fig. 5 are encountered due to strain hardening at high strains and the absence of clearly defined yield point in some steel. Moreover, residual stresses locked-in during the process of rolling also provide an added complexity. Thus the three components, which contribute to a reduction in the actual strength of columns (compared with the predictions from the "ideal" column curve) are

(i) Initial imperfection or initial bow.

(ii) Eccentricity of application of loads.

(iii) Residual stresses locked into the cross section.

ECCENTRICALLY LOADED COLUMNS-SECANT FORMULA

In the derivation of the Euler model, a both-end pinned column, it is assumed that the member is perfectly straight and homogeneous, and that the loading is assumed to be concentric at every cross section so that the structure and loading are symmetric. These idealizations are made to simplify the problem. In real

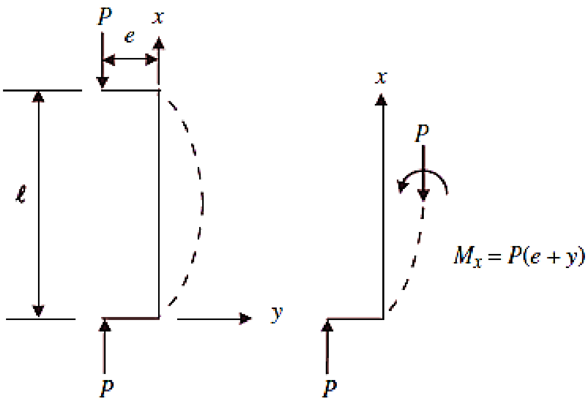

Figure 5: Eccentrically loaded column

life, however, a perfect column that satisfies all three conditions does not exist. It is, therefore, interesting to study the behavior of an imperfect column and compare it with the behavior predicted by the Euler theory. The imperfection of a monolithic slender column is predominantly affected by the geometry and eccentricity of loading. As an imperfect column begins to bend as soon as the initial amount of the incremental load is applied, the behavior of an imperfect column can be investigated successfully by considering either an initial imperfection or an eccentricity of loading. Consider the eccentrically loaded slender column shown in Fig. 5.From equilibrium of the isolated free body of the deformed configuration,

$$EIy'' + P(e+y) = 0$$

$$y'' + k^2 y = -k^2 e \quad \text{with } k^2 = P/EI$$

It should be noted in Eq. above that the system (both-end pinned prismatic column of length (with constant EI) . Eigenvalue remains unchanged from the Euler critical load as it is evaluated from the homogeneous differential equation. The general solution of this Eq. is:

$$y = y_h + y_p = A\sin kx + B\cos kx - e$$

The integral constants are evaluated from the boundary conditions. (The notion of solving an nth order ordinary differential equation implies that a direct or an indirect integral process is applied n times and hence there should be *n* integral constants in the solution of an *nth* order equation.)

$$y = 0 \quad \text{at } x = 0 \qquad\qquad y = 0 \quad \text{at } x = \ell$$

Thus the condition

$$B = e \qquad\qquad A = e\frac{1-\cos k\ell}{\sin k\ell}$$

$$y = e\left(\cos kx + \frac{1 - \cos k\ell}{\sin k\ell}\sin kx - 1\right)$$

$$y\Big|_{x=\ell/2} = \delta = e\left(\cos\frac{k\ell}{2} + \frac{1 - \cos k\ell}{\sin k\ell}\sin\frac{k\ell}{2} - 1\right)$$

$$= e\left(\cos\frac{k\ell}{2} + \frac{1 - 1 + 2\sin^2\frac{k\ell}{2}}{2\sin\frac{k\ell}{2}\cos\frac{k\ell}{2}}\sin\frac{k\ell}{2} - 1\right)$$

$$\delta = e\left(\sec\frac{k\ell}{2} - 1\right) = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_E}}\right) - 1\right] \text{ with } P_E = \frac{\pi^2 EI}{\ell^2}$$

The same deflection curve can be obtained using a fourth-order differential equation,

$$y = A\cos kx + B\sin kx + Cx + D$$

$$y = 0, \quad EIy'' = -Pe \quad \text{at } x = 0 \quad \text{and}$$

$$y = 0, \quad EIy'' = -Pe \quad \text{at } x = \ell.$$

Fig. 6 shows the variation of the mid-height deflection for two values of eccentricity, *e*. The behavior of an eccentrically loaded column is essentially the same as that of an initially bent column except there will be the nonzero initial deflection at the no-load condition in the case of a column initially bent. A slightly imperfect column begins to bend as soon as the load is applied. The bending remains small until the load approaches the critical load, after which the bending increases very rapidly.

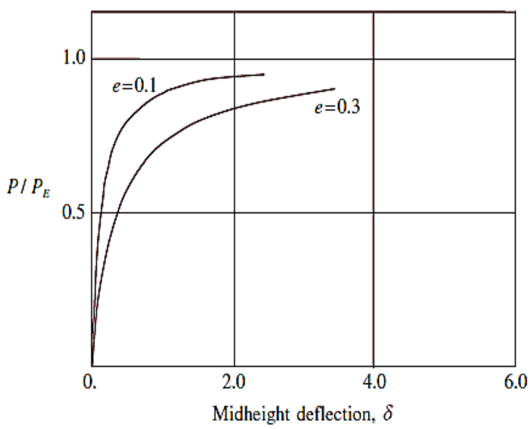

Figure 6: Load vs. deflection, eccentrically loaded column

Hence, the Euler theory provides a reasonable design criterion for real imperfect columns if the imperfections are small. The maximum stress in the extreme fiber is due to the combination of the axial stress and the bending stress. Hence,

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max}c}{I} = \frac{P}{A} + \frac{P(\delta + e)c}{I} = \frac{P}{A} + \frac{ceP\sec\left(\frac{\ell}{2}\sqrt{\frac{P}{EI}}\right)}{I}$$

$$= \frac{P}{A}\left[1 + \frac{ecA}{I}\sec\left(\frac{\ell}{2}\sqrt{\frac{P}{EI}}\right)\right]$$

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\ell}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$

This equation is known as the secant formula.

STRESS AMPLIFICATION IN COLUMNS

The behavior of a compression member under increasing load can be seen most clearly by calculating the bending stresses and lateral deflections that occur as the axial load is gradually applied. Consider a perfectly straight, slender member supporting a nominal axial load P. The ends of the member are assumed free to rotate in this case. If the member were perfectly straight and homogeneous and the load were perfectly centered, the stress in the column at any section would be simply $\sigma_a = P/A$, where A is the cross-sectional area of the column. No actual member ever would be perfectly straight and homogeneous, nor would the load be perfectly centered. Even when great efforts are made to achieve such perfection in laboratory tests, it is not completely attained. Therefore, the actual case is best represented by assuming a slight initial imperfection of loading or an initial crookedness represented by a deflection Y_0 at mid-height of the member as shown in Fig. 7.

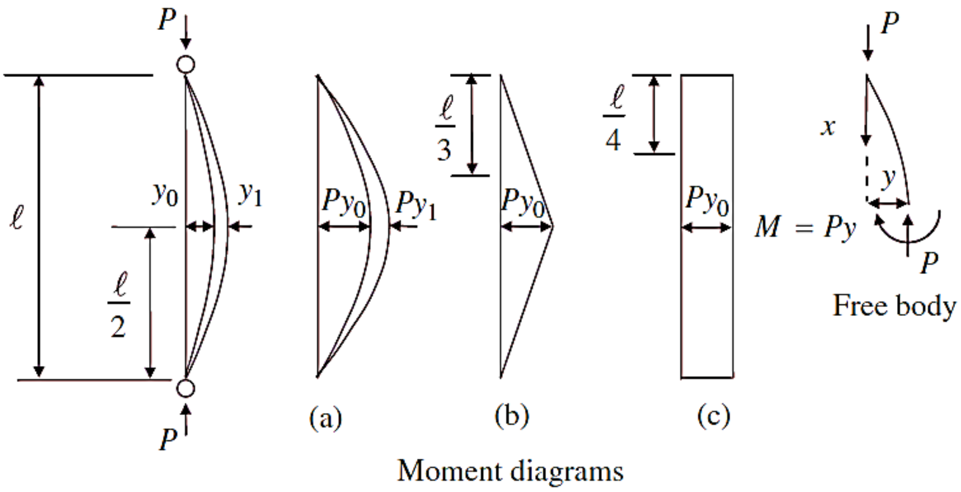

Figure 7: Load and moment diagrams of imperfect column.

When a load P is acting on the column, the stresses in the extreme fibers at the mid-height section are: $\sigma = \frac{P}{A} \pm \frac{Mc}{I}$

where c is the distance measured from the centroidal axis. At any section of the column, the bending moment is the load times the eccentricity, and the bending moment diagram has the same shape as the curve of the deflected member (see Fig. 2). This bending moment produces a further deflection at the mid-height y_o.

$$y_o = \sum_{n=1}^{\infty} a_n sin\frac{n\pi x}{l} \qquad M = -EI\frac{d^2(y-y_o)}{d^2x} \qquad EI\frac{d^2y}{d^2x} + Py = EI\frac{d^2y_o}{d^2x}$$

$$EI\frac{d^2y}{d^2x} + Py = -EI\sum_{n=1}^{\infty} a_n(\frac{n\pi}{L})^2 sin\frac{n\pi x}{l}$$

$$y = Asinkx + Bcoskx + y_p \quad .... \quad y_P = \sum_{n=1}^{\infty} F_n sin\frac{n\pi x}{l}$$

$$F_n = \frac{a_n}{1 - \frac{P}{EI}\cdot\frac{L^2}{n^2L^2}} = \frac{a_n}{1 - \frac{P}{P_{cr}}\cdot\frac{1}{n^2}} \; .... \quad P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$y = Asinkx + Bcoskx + \sum_{n=1}^{\infty}\frac{a_n}{1-\frac{P}{P_{cr}}\cdot\frac{1}{n^2}} sin\frac{n\pi x}{l}$$

B.C.: y=0 @ x=0 .... B=0 & y=0 @ x=L....... A=0

$$y = \sum_{n=1}^{\infty}\frac{a_n}{1 - \frac{P}{P_{cr}}\cdot\frac{1}{n^2}}$$

$$@\ x=L/2 \quad \& \quad n=1 .... \quad y_{max} = \frac{a_1}{1-\frac{P}{P_{cr}}}$$

In this case we have a buckling state as P approach P_{cr} as for initially streak column. However the initially deformed column does not represent

$(P < P_{cr})$ eign value problem. Since the deflected shape can defound for each values of P.

$$\sigma = \frac{P}{A} \pm \frac{Py_0c}{I} \frac{1}{1 - \frac{P}{P_E}}$$

Let $\sigma_a = \frac{P}{A}$ and $\sigma_{cr} = \frac{P_E}{A} = \frac{\pi^2 E A r^2}{A \ell^2} = \frac{\pi^2 E}{\left(\frac{\ell}{r}\right)^2}$

Further, recall that

$$\frac{1}{1 - \frac{P}{P_E}} = \frac{1}{1 - \frac{\sigma_a}{\sigma_{cr}}}$$

and

$$\frac{Py_0c}{I} = \frac{Py_0c}{Ar^2} = \sigma_a \frac{y_0c}{r^2} = \sigma_a \left(\frac{c}{r}\right)^2 \frac{y_0}{c}$$

The total stress is then

$$\sigma = \sigma_a \pm \sigma_a \left(\frac{c}{r}\right)^2 \frac{y_0}{c} \frac{1}{1 - \frac{\sigma_a}{\sigma_{cr}}}$$

Thus, the magnitude of the bending stress, the second term in Eq. above depends on P represented in sa; the shape of the cross section (c/r); and the initial curvature (y_0/c). As the critical stress, $\sigma_{cr}=\pi^2 E/(l/r)^2$, is a function of the stiffness of the material of the column and the slenderness ratio, it is convenient to make the expression for stress dimensionless by dividing the stress Eq. by σ_{cr}. Thus

$$\frac{\sigma}{\sigma_{cr}} = \frac{\sigma_a}{\sigma_{cr}} \pm \frac{\sigma_a}{\sigma_{cr}} \left(\frac{c}{r}\right)^2 \frac{y_0}{c} \frac{1}{1 - \frac{\sigma_a}{\sigma_{cr}}}$$

The value of shape factor (c/r) ranges from 1.0 for a section in which all (most) of the area is assumed concentrated in the flanges to √3 for rectangular section and 2.0 for a solid circular section. Rolled shapes generally used for columns have $(c/r)^2$ in the vicinity of 1.4 about the strong axis and 3.8 about the weak axis. S shapes (wide flange shapes with sloped flanges) run the values of 5.0 and over about the weak axis. Reasonable values of (y_0/c) are more difficult to estimate since the initial crookedness may be the result of either lack of straightness of the member itself or imperfection of the alignment of loading through the connections. Pending better establishment of the values, the combined constant $[(c/r)^2 (y_0/c)]$ has been assumed to range from 0.01 to 1.0. Since it is usually more convenient to express the initial crookedness y_0 in terms of the length of the member, $[(c/r)^2 (y_0/c)]$ may be written as:

$$[(c/r)^2 (y_0/c)] = (y_0/l) (l/r) (c/r)$$

Where: y_0/l = lack of straightness, l/r = slenderness ratio, and c/r = shape factor. The acceptable tolerances for straightness of rolled shapes are listed in some specifications (AISC 2005).

BUCKLING OF COLUMN WITH CHANGE IN CROSS-SECTION

The critical load on columns of stepped (variable) cross section as used in telescopic power cylinders can be computed applying differential equations considering continuity at the junctures. In order to limit the computational complexity, only two-stepped columns shown in the sketch are considered. Multiple-stepped columns.

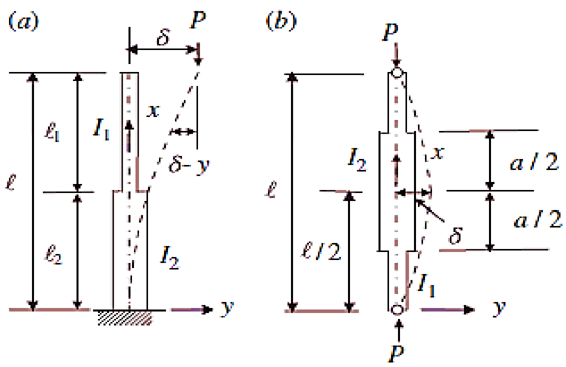

Figure 8: Stepped columns

are best analyzed by a means of computerized structural analysis methods. Consider the stepped cantilever column shown in Fig. 8(a). The bending moment of the column at any section along the member *x*-axis can be written for each segment as

$$EI_1 y_1'' = P(\delta - y_1) \quad \text{and} \quad EI_2 y_2'' = P(\delta - y_2)$$

Let $k_1^2 = \dfrac{P}{EI_1}$ and $k_2^2 = \dfrac{P}{EI_2}$, then the equations becomes

$$y_1'' + k_1^2 y_1 = k_1^2 \delta \qquad y_1 = \delta + C\cos k_1 x + D \sin k_1 x$$

$$y_2'' + k_2^2 y_2 = k_2^2 \delta \qquad y_2 = \delta + A\cos k_2 x + B \sin k_2 x$$

In order to determine the integral constants A and B for segment 2, consider the following boundary conditions:

$$y_2 = 0 \quad \text{at } x = 0 \Rightarrow A = -\delta$$

$$y_2' = 0 \quad \text{at } x = 0 \Rightarrow B = 0 \Rightarrow y_2 = \delta(1 - \cos k_2 x)$$

At the top of the column for y_1, it requires that

$$\delta + C\cos k_1\ell + D\sin k_1\ell = \delta \Rightarrow C\cos k_1\ell + D\sin k_1\ell = 0 \Rightarrow$$

$$C = -D\tan k_1\ell$$

The continuity at the juncture requires that

$$\delta + C\cos k_1\ell_2 + D\sin k_1\ell_2 = \delta(1 - \cos k_2\ell_2) = \delta - \delta\cos k_2\ell_2$$

$$-\tan k_1\ell \cos k_1\ell_2 D + D\sin k_1\ell_2 = -\left(\frac{\sin k_1\ell}{\cos k_1\ell}\cos k_1\ell_2 - \sin k_1\ell_2\right)D$$
$$= -\delta\cos k_2\ell_2$$

$$D = \frac{\delta\cos k_2\ell_2\cos k_1\ell}{\sin k_1\ell\cos k_1\ell_2 - \sin k_1\ell_2\cos k_1\ell}$$
$$= \frac{\delta\cos k_2\ell_2\cos k_1\ell}{\sin k_1(\ell_1+\ell_2)\cos k_1\ell_2 - \sin k_1\ell_2\cos k_1(\ell_1+\ell_2)}$$
$$= \frac{\delta\cos k_2\ell_2\cos k_1\ell}{\sin k_1\ell_1}$$

$$C = -\tan k_1\ell \frac{\delta \cos k_2\ell_2 \cos k_1\ell}{\sin k_1\ell_1} = \frac{\delta \cos k_2\ell_2 \sin k_1\ell}{\sin k_1\ell_1}$$

The continuity condition that the two segments of the deflected curve have the same slope at the juncture ($@x = \ell_2$) gives

$$\delta k_2 \sin k_2\ell_2 = -Ck_1 \sin k_1\ell_1 + Dk_1 \cos k_1\ell_2$$
$$= -\frac{\delta \cos k_2\ell_2 \sin k_1\ell}{\sin k_1\ell_1} k_1 \sin k_1\ell_2 + \frac{\delta \cos k_2\ell_2 \cos k_1\ell}{\sin k_1\ell_1} k_1 \cos k_1\ell_2$$

Rearranging gives

$$k_2 \sin k_2\ell_2 \sin k_1\ell_1 = k_1 \cos k_2\ell_2(\sin k_1\ell_1 \cos k_1\ell_2 + \cos k_1\ell_1 \sin k_1\ell_2) \sin k_1\ell_2 + k_1 \cos k_2\ell_2(\cos k_1\ell_1 \cos k_1\ell_2 - \sin k_1\ell_1 \sin k_1\ell_2) \cos k_1\ell_2$$
$$= k_1(\cos k_1\ell_1 \cos k_2\ell_2)$$

which leads to $\tan k_1\ell_1 \tan k_2\ell_2 = \frac{k_1}{k_2} \Leftarrow$ stability condition equation.

The same stability condition equation can be obtained by setting the coefficient determinant equal to zero. There are a total of four integral constants to be determined. As the governing differential equation is in second order, only one boundary condition at each support is to be used. Hence, the other two conditions are to be extracted from the continuity condition as used above.

$$y_2' = 0 \quad \text{at } x = 0 \Rightarrow B = 0 \Rightarrow y_2 = \delta + A \cos k_2x \qquad \text{(a)}$$

$$y_1 = \delta \; (\text{or } y_1'' = 0) \quad \text{at } x = \ell \Rightarrow C \cos k_1\ell + D \sin k_1\ell = 0 \qquad \text{(b)}$$

$$y_1 = y_2 \quad \text{at } x = \ell_2 \Rightarrow A \cos k_2\ell_2 - C \cos k_1\ell_2 - D \sin k_1\ell_2 = 0 \qquad \text{(c)}$$

$$y_1' = y_2' \quad \text{at } x = \ell_2 \Rightarrow Ak_2 \sin k_2\ell_2 - Ck_1 \sin k_1\ell_2 + Dk_1 \cos k_1\ell_2 = 0 \qquad \text{(d)}$$

Setting the determinant for the coefficients, *A*, *C*, and *D* equal to zero yields the identical stability condition equation. Knowing $I_1=I_2$ and $l_1=l_2$, the solution of the transcendental equation can be found. By substituting a/2 for l_2 and $l/2$ for l, the result obtained can be directly applied to the column shown in sketch (b). Coefficient m for $P_{cr} = mEI_2/l^2$ is given in Table 1. The table should be used for the case shown in sketch (b). For the case of stepped columns shown in sketch (a), values for m should be taken from Table 2. Consider a stepped cantilever column similar to that shown in Fig. 8 (a). The length of each segment is 20 inches. The cross-sectional area of the bottom segment is 4 in^2 and the upper segment is 1 in^2 The modulus of elasticity of the material is assumed to be 29,000 ksi. The stability condition equation now becomes $tan(80k_2)\ tan(20k_2)= 4$
Gives k_2 = 0.0184315, which leads to P_{cr} =13.136 kips.

Table 1: Buckling coefficients for step column

I_1/I_2	a/ℓ			
	0.2	0.4	0.6	0.8
0.01	0.15	0.27	0.60	2.26
0.1	1.47	2.40	4.50	8.59
0.2	2.80	4.22	6.69	9.33
0.4	5.09	6.68	8.51	9.67
0.6	6.98	8.19	9.24	9.78
0.8	8.55	9.18	9.63	9.84

Table 2: Buckling coefficients for step column

I_1/I_2	ℓ_2/ℓ			
	0.2	0.4	0.6	0.8
0.01	0.038	0.068	0.150	0.563
0.1	0.367	0.600	1.124	2.147
0.2	0.699	1.056	1.674	2.332
0.4	1.272	1.669	2.127	2.419
0.6	1.745	2.046	2.311	2.446
0.8	2.138	2.294	2.408	2.459