LARGE DEFLECTION THEORY (THE ELASTICA)

Although it is not likely to be encountered in the construction of buildings and bridges, a very slender compression member may exhibit a nonlinear elastic large deformation so that a simplifying assumption of the small displacement theory may not be valid. Consider the simply supported wiry column shown in Fig. 9.

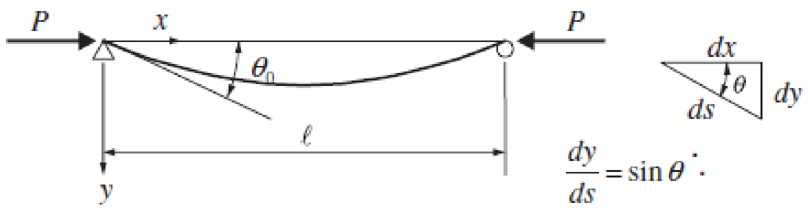

Figure 9: Large deflection model

Aside from the assumption of small deflections, all the other idealizations made for the Euler column are assumed valid. The member is assumed perfectly straight initially and loaded along its centroidal axis, and the material is assumed to obey Hooke's law. From an isolated free body of the deformed configuration of the column, it can be readily observed that the external moment, Py, at any section is equal to the internal moment, $-EI/\rho$. Thus $Py = -EI/\rho$.

where $1/\rho$ is the curvature. Since the curvature is defined by the rate of change of the unit tangent vector of the curve with respect to the arc length of the curve, the curvature and slope relationship is established.

$1/\rho = d\theta/ds$ then $EI d\theta/ds + Py = 0$ but $k^2 = P/EI$ …… $d\theta/ds + k^2 y = 0$

Differentiating Eq. respect to s and replacing dy/ds by $\sin\theta$ yields:

$$\frac{d^2\theta}{ds^2} + k^2 \sin\theta = 0$$

Multiplying each term of Eq. by $2d\theta$ and integrating gives:

$$\int \frac{d^2\theta}{ds^2} 2 \frac{d\theta}{ds} ds + \int 2k^2 \sin\theta \, d\theta = 0$$

Recalling the following mathematical identities

$$\frac{d}{ds}\left(\frac{d\theta}{ds}\right)^2 = 2\left(\frac{d\theta}{ds}\right)\left(\frac{d^2\theta}{ds^2}\right) \quad \text{and} \quad \sin\theta \, d\theta = -d(\cos\theta),$$

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$$\int d\left(\frac{d\theta}{ds}\right)^2 - 2k^2 \int d(\cos\theta) = 0$$

Carrying out the integration gives

$$\left(\frac{d\theta}{ds}\right)^2 - 2k^2 \cos\theta = C$$

$$\frac{d\theta}{ds} = 0 \quad \text{at } x = 0,$$

$$\left(\text{moment} = 0 \Rightarrow \frac{1}{\rho} = 0 \text{ or } \rho = \infty, \text{straight line}\right) \text{and } \theta = \theta_0$$

Hence,

$$C = -2k^2 \cos\theta_0$$

$$\left(\frac{d\theta}{ds}\right)^2 - 2k^2(\cos\theta - \cos\theta_0) = 0 \quad ds = -\frac{d\theta}{\sqrt{2}k\sqrt{\cos\theta - \cos\theta_0}}$$

$$\int_0^{\ell/2} ds = -\frac{1}{\sqrt{2}k}\int_{\theta_0}^{0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \text{ or } \frac{\ell}{2} = \frac{1}{\sqrt{2}k}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\ell = \frac{2}{k}\int_0^{\theta_0} \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}$$

Notice the negative sign is eliminated by reversing the limits of integration. Making use of mathematical identities:

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \quad \text{and} \quad \cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2}$$

$$\ell = \frac{1}{k}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

In order to simplify this Eq. further, let $\sin\frac{\theta_0}{2} = \alpha$

and introduce a new variable ϕ such that

$$\sin\frac{\theta}{2} = \alpha\sin\phi$$

Then $\theta = 0 \Rightarrow \phi = 0$ and $\theta = \theta_0 \Rightarrow \sin\phi = 1 \Rightarrow \phi = \pi/2$.

$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \alpha\cos\phi\,d\phi$$

which can be rearranged to show

$$d\theta = \frac{2\alpha\cos\phi\,d\phi}{\sqrt{1-\sin^2\frac{\theta}{2}}} = \frac{2\alpha\cos\phi\,d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$\ell = \frac{1}{k}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} = \frac{1}{k}\int_0^{\pi/2} \frac{1}{\sqrt{\alpha^2-\alpha^2\sin^2\phi}} \frac{2\alpha\cos\phi\,d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$= \frac{2}{k}\int_0^{\pi/2} \frac{1}{\alpha\cos\phi} \frac{\alpha\cos\phi\,d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$\ell = \frac{2}{k}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2\sin^2\phi}} = \frac{2K}{k}$$

where:

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$\ell = \frac{2K}{k} = \frac{2K}{\sqrt{P/EI}} \quad \text{as} \quad k^2 = \frac{P}{EI}$$

or

$$\frac{P}{P_{cr}} = \frac{4K^2}{\pi^2}$$

as

$$P = \frac{4K^2}{\ell^2/EI} = \frac{4EIK}{\ell^2} \quad \text{and} \quad P_{cr} = \frac{\pi^2 EI}{\ell^2}$$

If the lateral deflection of the member is very small (just after the initial bulge), then θ_0 is small and consequently $\alpha_2 \sin^2 \phi$ in the denominator of K becomes negligible. The value of K approaches $\pi/2$ and

$$P = P_{cr} = \pi^2 EI/\ell^2$$

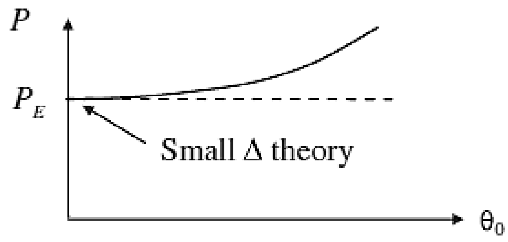

Figure 10: Postbuckling behavior

The midheight deflection, y_m (or δ), can be determined from $dy = ds \sin\theta$.

$$dy = -\frac{\sin\theta \, d\theta}{\sqrt{2}k\sqrt{\cos\theta - \cos\theta_0}}$$

Integrating the above equation gives

$$\int_0^{y_m} dy = -\frac{1}{2k}\int_{\theta_0}^{0} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \quad \text{or} \quad y_m = \frac{1}{2k}\int_0^{\theta_0} \frac{\sin\theta\, d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Recall $\sin(\theta/2) = \alpha\sin\phi$ and $d\theta = 2\alpha\cos\phi\, d\phi/\sqrt{1-\alpha^2\sin^2\phi}$

Hence,

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\sqrt{1-\sin^2\frac{\theta}{2}} = 2\alpha\sin\phi\sqrt{1-\alpha^2\sin^2\phi}$$

$$y_m = \frac{1}{2k}\int_0^{\theta_0} \frac{\sin\theta\, d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

$$= \frac{1}{2k}\int_0^{\pi/2} \frac{2\alpha\sin\phi\sqrt{1-\alpha^2\sin^2\phi}\,2\alpha\cos\phi\, d\phi}{\sqrt{\alpha^2 - \alpha^2\sin^2\phi}\sqrt{1-\alpha^2\sin^2\phi}}$$

$$y_m = \delta = \frac{2\alpha}{k}\int_0^{\pi/2} \sin\phi \, d\phi = \frac{2\alpha}{k} \quad \text{or} \quad \frac{y_m}{\ell} = \frac{2\alpha}{\pi\sqrt{\dfrac{P}{P_E}}}$$

The distance between the two load points (x-coordinates) can be determined from

$$dx = ds\cos\theta$$

$$dx = -\frac{\cos\theta \, d\theta}{\sqrt{2}k\sqrt{\cos\theta - \cos\theta_0}}$$

Integrating (x_m is the x-coordinate at the midheight) the above equation gives

$$\int_0^{x_m} dx = -\frac{1}{\sqrt{2}k}\int_{\theta_0}^{0} \frac{\cos\theta \, d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = -\frac{1}{\sqrt{k}}\int_{\theta_0}^{0} \frac{\cos\theta \, d\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}} \text{ or}$$

$$x_m = \frac{1}{2k}\int_0^{\theta_0} \frac{\cos\theta \, d\theta}{\sqrt{\sin^2\dfrac{\theta_0}{2} - \sin^2\dfrac{\theta}{2}}}$$

Recall $\sin(\theta/2) = \alpha\sin\phi$ and $d\theta = 2\alpha\cos\phi \, d\phi/\sqrt{1-\alpha^2\sin^2\phi}$

and $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2) = 1 - 2\alpha^2\sin^2\phi$

$$x_m = \frac{1}{2k}\int_0^{\theta_0} \frac{\cos\theta \, d\theta}{\sqrt{\sin^2\dfrac{\theta_0}{2} - \sin^2\dfrac{\theta}{2}}}$$

$$= \frac{1}{2k}\int_0^{\pi/2} \frac{(1-2\alpha^2\sin^2\phi)2\alpha\cos\phi \, d\phi}{\sqrt{\alpha^2 - \alpha^2\sin^2\phi}\sqrt{1-\alpha^2\sin^2\phi}}$$

$$= \frac{1}{k}\int_0^{\pi/2} \frac{(1-2\alpha^2\sin^2\phi)d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$x_0 = 2x_m = \frac{2}{k}\int_0^{\pi/2} \frac{[2(1-\alpha^2\sin^2\phi)-1]d\phi}{\sqrt{1-\alpha^2\sin^2\phi}}$$

$$= \frac{4}{k}\int_0^{\pi/2} \sqrt{1-\alpha^2\sin^2\phi} \, d\phi$$

$$-\frac{2}{k}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2\sin^2\phi}} = \frac{4}{k}E(\alpha) - \ell$$

where $E(\alpha)$ is the complete elliptic integral of the second kind

$$\frac{x_0}{\ell} = \frac{4E(\alpha)}{\ell\sqrt{\frac{P}{EI}}} - 1 = \frac{4E(\alpha)}{\pi\sqrt{\frac{P}{P_E}}} - 1$$

The complete elliptic integral of the first kind can be evaluated by an infinite series given by

$$\begin{aligned} K &= \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\alpha^2\sin^2\phi}} \\ &= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2\alpha^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\alpha^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\alpha^6 + \cdots\right] \text{ with } \alpha^2 < 1 \end{aligned}$$

Summing the first four terms of the above infinite series for $\alpha = 0.5$ yields $K = 1.685174$.

Likewise, the complete elliptic integral of the second kind can be evaluated by an infinite series given by

$$\begin{aligned} E &= \int_0^{\pi/2} \sqrt{1-\alpha^2\sin^2\phi}\, d\phi \\ &= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2\alpha^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{\alpha^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{\alpha^6}{5} - \cdots\right] \text{ with } \alpha^2 < 1 \end{aligned}$$

Summing the first four terms of the above infinite series for $\alpha = 0.5$ yields E =1.46746.

Question: How can you explain the contraction between analytical and experimental results.