BEAM-COLUMN BUCKLING

DIFFERENTIAL EQUATIONS OF BEAM-COLUMNS

Bifurcation-type buckling is essentially flexural behavior. Therefore, the free-body diagram must be based on the deformed configuration as the examination of equilibrium is made in the neighboring equilibrium position. Summing the forces in the horizontal direction in Fig. 1-4(a) gives:

$$\sum F_y = 0 = (V + dV) - V + q dx$$
$$\frac{dV}{dx} = V' = -q(x)$$

Summing the moment at the top of the free body gives

$$\sum M_{\text{top}} = 0 = (M + dM) - M + V dx + P dy - q(dx)\frac{dx}{2}$$

Taking derivatives on both sides of Eq.: $\frac{dM}{dx} + P\frac{dy}{dx} = -V$

Taking derivatives on both sides of Eq. above give:

$$M'' + (Py')' = -V'$$

Equation (3) is the fundamental beam-column governing differential equation. Consider the free-body diagram shown in Fig. 1-4(d). Summing forces in the y direction gives

$$\sum F_y = 0 = -(V + dV) + V + q dx \Rightarrow \frac{dV}{dx} = V' = q(x)$$

Si
pc
a
fo

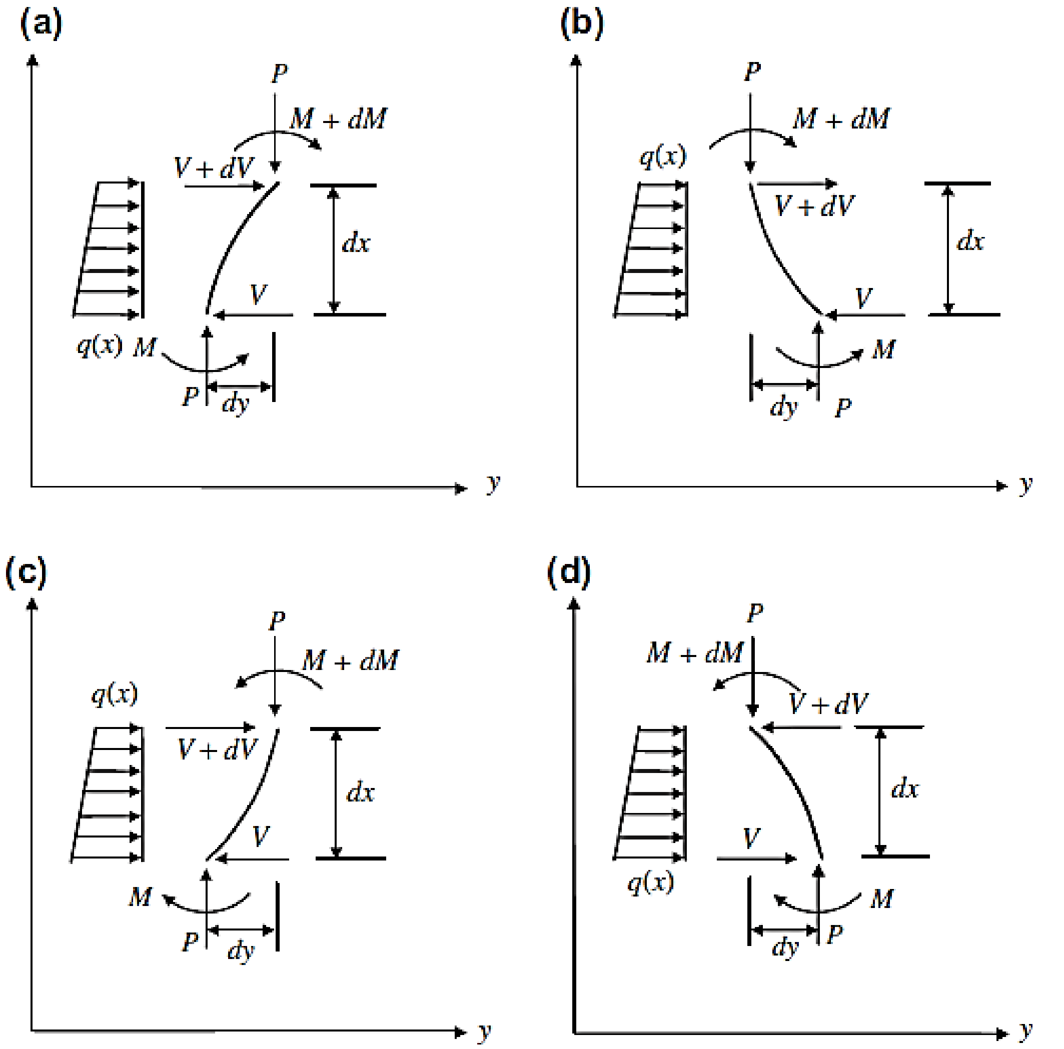


Figure 1-4 Free-body diagrams of a beam-column

Summing moments about the top of the free body yields

$$\sum M_{top} = 0$$

$$= -(M + dM) + M - Vdx - Pdy - \cancel{q\,dx\,dx/2} \Rightarrow$$

$$-\frac{dM}{dx} - P\frac{dy}{dx} = V$$

For the coordinate system shown in Fig. 1-4(d), the curve represents a decreasing function (negative slope) with the convex side to the positive y direction. Hence, *-EIy"=M(x)*. Thus,

$$-(-EIy'')' - (-Py') = V$$

which leads to

$$EIy''' + Py' = V \quad \text{or} \quad EIy^{iv} + Py'' = q(x)$$

It can be shown that the free-body diagrams shown in Figs. 1-4(b) and 1-4(c) will lead to Eq. (3). Hence, the governing differential equation is independent of the shape of the free-body diagram assumed. Rearranging Eq. (3) and if considered *q(x)*= 0 gives:

$$EIy^{iv} + Py'' = 0 \Rightarrow y^{iv} + k^2y'' = 0, \quad \text{where } k^2 = \frac{P}{EI}$$

Assuming the solution to be of a form $y = \alpha e^{mx}$, then $y' = \alpha m e^{mx}$, $y'' = \alpha m^2 e^{mx}$, $y''' = \alpha m^3 e^{mx}$, and $y^{iv} = \alpha m^4 e^{x}$. Substituting these derivatives back to the simplified homogeneous differential equation yields

$$\alpha m^4 e^{mx} + \alpha k^2 m^2 e^{mx} = 0 \Rightarrow \alpha e^{mx}(m^4 + k^2 m^2) = 0$$

Since $\alpha \neq 0$ *and* $e^{mx} \neq 0 \Rightarrow m^2(m^2 + k^2) = 0 \Rightarrow m = \pm 0, \ \pm ki$. Hence,

$$y_h = c_1 e^{kix} + c_2 e^{-kix} + c_3 x e^0 + c_4 e^0$$

Know the mathematical identities $\begin{cases} e^0 = 1 \\ e^{ikx} = \cos kx + i \sin kx \\ e^{-ikx} = \cos kx - i \sin kx \end{cases}$

Hence, $y_h = A \sin kx + B \cos kx + Cx + D$ where integral constants *A*, *B*, *C*, and *D* can be determined uniquely by applying proper boundary conditions of the structure.

TRANSVERSELY LOADED BEAM SUBJECTED TO AXIAL COMPRESSION

A slender member meeting the Euler-Bernoulli-Navier hypotheses under transverse loads and inplane compressive load (see Fig.1) is called a beam-column. An exact analysis of a beam-column can only be accomplished by solving the governing differential equation or its derivatives (for example, slope-deflection equations). Consider a very simple case of a beam-column shown in Fig. 1. The beam-column is subjected simultaneously to a transverse load Q at its mid-span and a concentric compressive force P. Since the response of a beam-column under these loads is no longer linear, the method of superposition does not apply even if the final results are within the elastic limit.

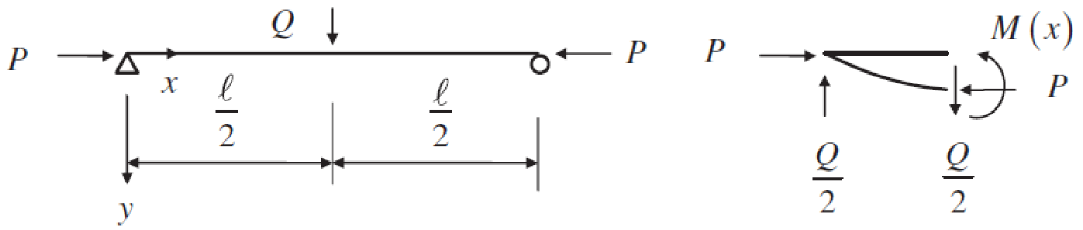


Figure 1: Simple beam-column

Summing moments at a point x from the origin gives

$$M(x) - Py - \frac{Q}{2}x = 0 \quad \text{for } 0 \leq x \leq \ell/2 \quad \text{with } M(x) = -EIy''$$

$$\text{or} \quad y'' + k^2 y = -\frac{Q}{2}\frac{x}{EI} = -\frac{Qx}{2P}k^2 \quad \text{with } k^2 = \frac{P}{EI}$$

The general solution to this differential equation is $y = y_h + y_P$. The homogeneous solution has been given earlier. The particular solution can be obtained by the method of undetermined coefficients. Assume the particular solution to be of the form

$$y_P = C + Dx \quad \text{with } y_P' = D, \; y_P'' = 0$$

Substituting these derivatives into the differential equation yields

$$0 + k^2(C + Dx) = -\frac{Qx}{2P}k^2$$

Hence,

$$C = 0 \quad \text{and} \quad D = -\frac{Q}{2P} \Rightarrow y_P = -\frac{Q}{2P}x$$

The total solution is

$$y = A\cos kx + B\sin kx - \frac{Qx}{2P}$$

The two constants of integration can be determined from the following boundary conditions:

$$y = 0 \quad \text{at } x = 0 \Rightarrow A = 0$$

$$y' = 0 \quad \text{at } x = \ell/2$$

(Note : the boundary condition, $y = 0$ at $x = \ell$, cannot be used here as $0 \le x \le \ell/2$)

$$y' = Bk\cos kx - \frac{Q}{2P}, 0 = Bk\cos\frac{k\ell}{2} - \frac{Q}{2P} \Rightarrow B = \frac{Q}{2Pk\cos\frac{k\ell}{2}}$$

$$y = \frac{Q\sin kx}{2Pk\cos\frac{k\ell}{2}} - \frac{Qx}{2P} \quad \text{for } 0 \le x \le \frac{\ell}{2} \quad \text{with } P_{cr} = P_E = \frac{\pi^2 EI}{\ell^2}$$

By observation, the maximum lateral deflection occurs at the midspan.

$$y_{\max}\Big|_{x=\frac{\ell}{2}} = \frac{Q}{2Pk}\left(\tan\frac{k\ell}{2} - \frac{k\ell}{2}\right) \quad \text{with } u = \frac{k\ell}{2} = \frac{\ell}{2}\sqrt{\frac{P}{EI}}$$

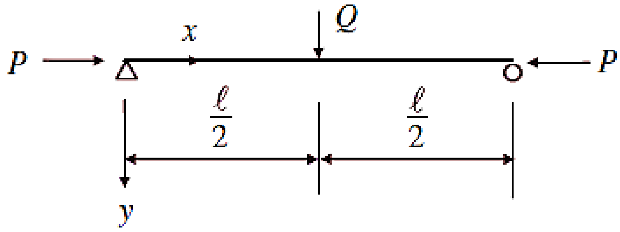
$$y_{\max}\Big|_{x=\frac{\ell}{2}} = \frac{Qk^3\ell^3}{16Pku^3}\left(\tan\frac{k\ell}{2} - \frac{k\ell}{2}\right) = \frac{Q\ell^3}{48EI}\left[\frac{3(\tan u - u)}{u^3}\right] = \frac{Q\ell^3}{48EI}X(u)$$

$$y_{max}\Big|_{x=\frac{\ell}{2}} = \delta_{max} = \frac{Q\ell^3}{48EI} \quad \text{when } P = 0$$

$$u^2 = \frac{\ell^2}{4}\frac{P}{EI} \Rightarrow P = \frac{4EIu^2}{\ell^2} \quad \text{and} \quad P_E = \frac{\pi^2 EI}{\ell^2}$$

$$\frac{P}{P_E} = \frac{4EIu^2}{\ell^2}\frac{\ell^2}{\pi^2 EI} = \frac{4u^2}{\pi^2}, \quad X(u) = \frac{3(\tan u - u)}{u^3}$$

The previous section showed that the deflection at the mid-span of a simple beam-column subjected to a lateral load shown in Fig. 3 is



$$\delta = y_{max} = \delta_0 \frac{3(\tan u - u)}{u^3}$$

Figure 3: simple beam-column subjected to a lateral load

where

$$\delta_0 = \frac{Q\ell^3}{48EI}, u = \frac{k\ell}{2}, \quad \text{and} \quad k = \sqrt{\frac{P}{EI}}$$

Recall the power series expansion of $\tan u$ given by

$$\tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \frac{17u^7}{315} + \ldots$$

Hence,

$$\delta = \delta_0\left(1 + \frac{2u^2}{5} + \frac{17u^4}{105} + \ldots\right)$$

Noting

$$u^2 = \frac{k^2\ell^2}{4} = \frac{P\ell^2}{4EI}\frac{\pi^2}{\pi^2} = 2.46\,\frac{P}{P_E}$$

$$\delta = \delta_0\left[1 + 0.984\,\frac{P}{P_e} + 0.998\left(\frac{P}{P_e}\right)^2 + \ldots\right]$$

$$\doteq \delta_0\left[1 + \frac{P}{P_E} + \left(\frac{P}{P_E}\right)^2 + \ldots\right]$$

$$= \delta_0\,\frac{1}{1 - \dfrac{P}{P_E}} \Leftarrow \text{from power series sum for } \frac{P}{P_E} < 1$$

where

$\dfrac{1}{1 - \dfrac{P}{P_E}}$ is called amplification factor or magnification factor.

The maximum bending moment is

$$M_{\max} = \frac{Q\ell}{4} + P\delta = \frac{Q\ell}{4} + \frac{PQ\ell^3}{48EI}\frac{1}{1 - \dfrac{P}{P_E}} = \frac{Q\ell}{4}\left(1 + \frac{P\ell^2}{12EI}\frac{1}{1 - \dfrac{P}{P_E}}\right)$$

$$= \frac{Q\ell}{4}\left(1 + 0.82\,\frac{P}{P_E}\frac{1}{1 - \dfrac{P}{P_E}}\right)$$

or

$$M_{\max} = \frac{Q\ell}{4}\left(\frac{1 - 0.18\,\dfrac{P}{P_E}}{1 - \dfrac{P}{P_E}}\right)$$

where

$$\left(\frac{1 - 0.18\dfrac{P}{P_E}}{1 - \dfrac{P}{P_E}}\right)$$

is amplification factor for bending moment due to a concentrated load.

The variation of δ with Q as given by the amplification factor is plotted on the left side of Fig. 4 for P = 0, P = 0.4P_{cr}, and P = 0.7Pcr. The curves show that the relation between Q and δ is linear even when P≠0, provided P is a constant. However, if P is allowed to vary, as is the case on the right side of Figure 4, the load-deflection relation is not linear. This is true regardless of whether Q remains constant (dashed curve) or increases

as P increases (solid curve). The deflection of a beam-column is thus a linear function of Q but a nonlinear function of P.

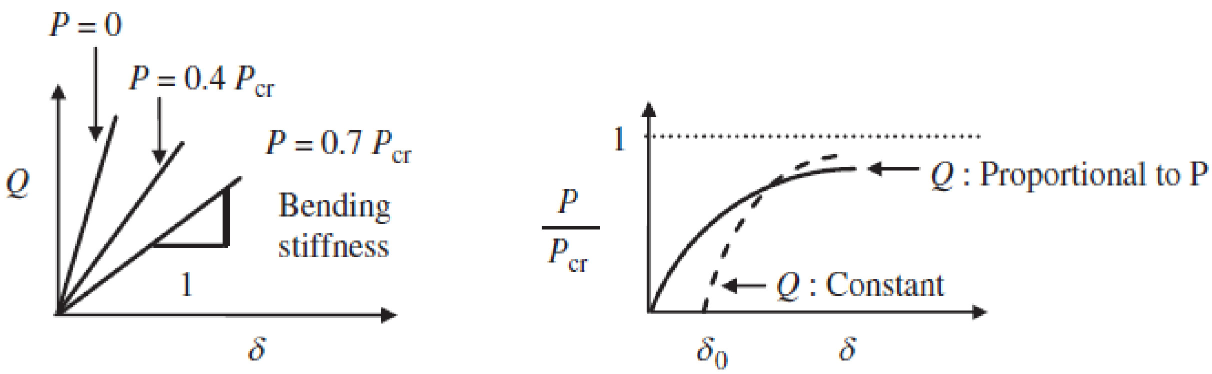
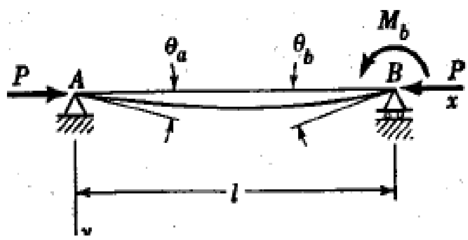


Figure 4: Lateral displacements of beam-column

BENDING OF BEAM-COLUMNS BY COUPLES

- Case 1: one end is subjected to moment



the deflection curve is obtained by:

$$y = \frac{M_b}{P}\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right)$$

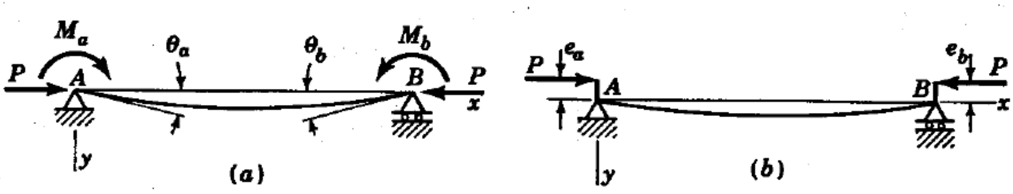
$$\theta_a = \left(\frac{dy}{dx}\right)_{x=0} = \frac{M_b}{P}\left(\frac{k}{\sin kl} - \frac{1}{l}\right) = \frac{M_b l}{6EI}\frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$

$$\theta_b = -\left(\frac{dy}{dx}\right)_{x=l} = -\frac{M_b}{P}\left(\frac{k\cos kl}{\sin kl} - \frac{1}{l}\right) = \frac{M_b l}{3EI}\frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

to simplify these expressions let:

$$\phi(u) = \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$
$$\psi(u) = \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

- Case 2: both ends are subjected to moments



By substituting M_a by M_b and x by $(l-x)$ in the same equation of case one. Adding the two results together, then the deflection curve for this case:

$$y = \frac{M_b}{P}\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right) + \frac{M_a}{P}\left[\frac{\sin k(l-x)}{\sin kl} - \frac{l-x}{l}\right]$$

Substituting $M_a=Pe_a$ & $M_b=Pe_b$ we obtain:

$$y = e_b\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right) + e_a\left[\frac{\sin k(l-x)}{\sin kl} - \frac{l-x}{l}\right]$$

$$\theta_a = \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u)$$
$$\theta_b = \frac{M_b l}{3EI}\psi(u) + \frac{M_a l}{6EI}\phi(u)$$

Case 3: both ends are subjected to equal moments ($M_a= M_b= M_o$)

$$y = \frac{M_0}{P\cos(kl/2)}\left[\cos\left(\frac{kl}{2} - kx\right) - \cos\frac{kl}{2}\right]$$
$$= \frac{M_0 l^2}{8EI}\frac{2}{u^2\cos u}\left[\cos\left(u - \frac{2ux}{l}\right) - \cos u\right]$$

The deflection at the center of the beam is obtained by substituting $x=l/2$

$$\delta = (y)_{x=l/2} = \frac{M_0 l^2}{8EI}\frac{2(1-\cos u)}{u^2\cos u} = \frac{M_0 l^2}{8EI}\lambda(u)$$

The slope at the ends are:

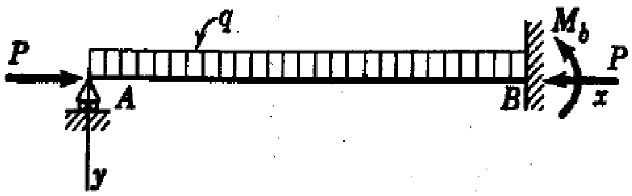
$$\theta_a = \theta_b = \left(\frac{dy}{dx}\right)_{x=0} = \frac{M_0 l}{2EI}\frac{\tan u}{u}$$

The max. bending moment which obtained at the middle of span:

$$M_{max} = -EI\left(\frac{d^2y}{dx^2}\right)_{x=l/2} = M_0 \sec u$$

BEAM-COLUMNS WITH BUILT UP ENDS

Case 1: one end is fixed



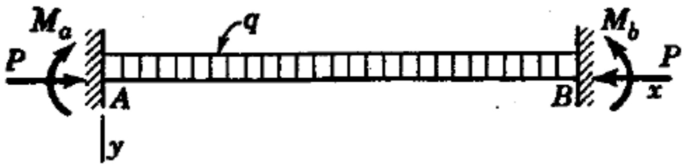
The rotation at the fixed due to the uniform load and the moment equal to zero

$$\frac{ql^3}{24EI}\chi(u) + \frac{M_0 l}{2EI}\frac{\tan u}{u} = 0$$

$$M_0 = -\frac{ql^2}{12}\frac{\chi(u)}{(\tan u)/u}$$

$$\chi(u) = \frac{3(\tan u - u)}{u^3}$$

Case 2: both ends are fixed



The deflection curve is symmetric and the moment at fixed ends are equals ($M_a = M_b = M_o$)

$$\frac{ql^3}{24EI}\chi(u) + \frac{M_0 l}{2EI}\frac{\tan u}{u} = 0$$

$$M_0 = -\frac{ql^2}{12}\frac{\chi(u)}{(\tan u)/u}$$

Case 3: unsymmetrical loaded beam

$$\theta_a = \theta_{0a} + \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u) = 0$$

$$\theta_b = \theta_{0b} + \frac{M_a l}{6EI}\phi(u) + \frac{M_b l}{3EI}\psi(u) = 0$$