SLOPE-DEFLECTION EQUATION WITHOUT AXIAL FORCE

A typical derivation process will be traced here as it will be used again in the development of the slope-deflection equations that include the effect of axial compression on the bending stiffness From the deformations of a beam shown in Fig.7, the moment at a distance x from the origin is expressed as:

$$M_x = M_{ab} - (M_{ab} + M_{ba})\frac{x}{\ell}$$

Know $y'' = -\frac{M_x}{EI}$

Taking successive derivatives of the above equation gives

$$EIy^{iv} = 0$$

The general solution of the differential equation is

$$y = A + Bx + Cx^2 + Dx^3$$

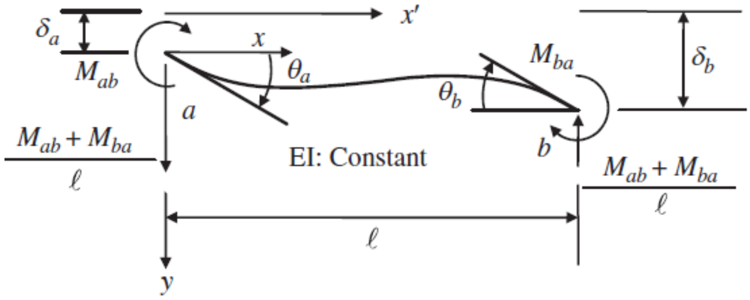

Figure 7: Deformations of beam

$$y' = B + 2Cx + 3Dx^2$$

$$y'' = 2C + 6Dx$$

The four kinematic boundary conditions available are

$$y = \delta_a \quad \text{at } x = 0 \quad \text{and} \quad y = \delta_b \quad \text{at } x = \ell$$

$$y' = \theta_a \quad \text{at } x = 0 \quad \text{and} \quad y' = \theta_b \quad \text{at } x = \ell$$

$$\delta_a = A, \quad \theta_a = B$$

$$\delta_b = \delta_a + \theta_a \ell + C\ell^2 + D\ell^3$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$2\delta_b = 2\theta_a \ell + 2C\ell^2 + 2D\ell^3 + 2\delta_a$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$2\delta_b - \theta_b \ell = 2\delta_a + \theta_a \ell - D\ell^3$$

from which

$$D = \frac{1}{\ell^3}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]$$

$$3\delta_b = 3\theta_a \ell + 3C\ell^2 + 3D\ell^3 + 3\delta_a$$

$$\theta_b \ell = \theta_a \ell + 2C\ell^2 + 3D\ell^3$$

$$3\delta_b - \theta_b \ell = 3\delta_a + 2\theta_a \ell + C\ell^2$$

from which

$$C = \frac{1}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell]$$

$$y'' = \frac{2}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell] + \frac{6}{\ell^3}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]x$$

$$y''(0) = \frac{2}{\ell^2}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell] = -\frac{M_{ab}}{EI}$$

$$y''(\ell) = \frac{2}{\ell^3}[3(\delta_b - \delta_a) - (2\theta_a + \theta_b)\ell]\ell + \frac{6}{\ell^2}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell]\ell$$

$$= -\frac{M_{ab}}{EI} + \frac{6}{\ell^2}[-2(\delta_b - \delta_a) + (\theta_a + \theta_b)\ell] = \frac{M_{ba}}{EI}$$

$$M_{ab} = \frac{2EI}{\ell}\left[2\theta_a + \theta_b - \frac{3}{\ell}(\delta_b - \delta_a)\right]$$

$$M_{ba} = \frac{2EI}{\ell}\left[2\theta_b + \theta_a - \frac{3}{\ell}(\delta_b - \delta_a)\right]$$

If any fixed end moments exist prior to releasing the joint constraints such as M_{ab} fixed and M_{ba} fixed, then final member end moments become

$$M_{AB} = \frac{2EI}{\ell}\left[2\theta_a + \theta_b - \frac{3}{\ell}(\delta_b - \delta_a)\right] + M_{ab\ fixed}$$

$$M_{ba} = \frac{2EI}{\ell}\left[2\theta_b + \theta_a - \frac{3}{\ell}(\delta_b - \delta_a)\right] + M_{ba\ fixed}$$

EFFECTS OF AXIAL LOADS ON BENDING STIFFNESS

The classical slope-deflections equations that are introduced in any standard text on indeterminate structures give the moments, M_{ab} and M_{ba}, induced at the ends of member AB as a function of end rotations θ_a and θ_b and by a displacement Δ of one end to the other. In conventional linear structural analysis (first-order analysis), it is customary to ignore the effect of axial forces on the bending stiffness of flexural members. It can be shown that the effect of amplification is negligibly small as long as the axial load remains small in comparison with the critical load of the member. When the ratio of the axial load to the critical load becomes sizable, however, the bending stiffness is reduced markedly due to the axial compression, and it is no longer acceptable to neglect this reduction. As the first-order analysis results may become dangerously unconservative, modern design specifications call for a mandatory second-order analysis (AISC 2005).

It is expedient to introduce $\Delta=\delta_b - \delta_a$ with $\delta_a= 0$ to avoid the rigid body translation. The moment of the beam-column shown in Fig. 9 at a distance *x* from the origin is

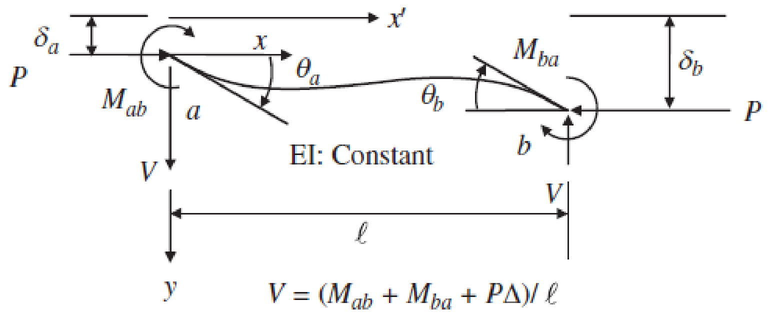

Figure 9: Deformations of beam-column

$$M_x = M_{ab} + Py - (M_b + M_{ba} + P\Delta)\frac{x}{\ell}$$

$$y'' = -\frac{M_x}{EI}$$

$$EIy'' + Py = -M_{ab} + (M_{ab} + M_{ba} + P\Delta)\frac{x}{\ell}$$

Taking successive derivatives on both sides yields

$$EIy^{iv} + Py'' = 0$$

Let $k^2 = \dfrac{P}{EI}$

The simplified differential equation is

$$y^{iv} + k^2 y'' = 0$$

for which the general solution is

$$y = A \sin kx + B \cos kx + Cx + D$$

The proper geometric boundary conditions are

$$y(0) = 0, \quad y(\ell) = \Delta, \quad y'(0) = \theta_a, \quad \text{and} \quad y'(\ell) = \theta_b$$

The proper natural boundary conditions are

$$y''(0) = -\frac{M_{ab}}{EI}, \quad \text{and} \quad y''(\ell) = \frac{M_{ba}}{EI}$$

Applying the geometric boundary conditions to eliminate the integral constants, A, B, C, D, and solving for M_{ab} and M_{ba} gives

$$0 = B + D$$

Let $\beta = k\ell$

$$\Delta = A \sin \beta + B \cos \beta + C\ell + D$$

$$\theta_a = Ak + C$$

The matrix equation for the integral constants becomes

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin \beta & \cos \beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k \cos \beta & -k \sin \beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ \theta_a \\ \theta_b \end{Bmatrix}$$

Applying Cramer's rule yields

$$A = \frac{\begin{vmatrix} 0 & 1 & 0 & 1 \\ \Delta & \cos \beta & \ell & 1 \\ \theta_a & 0 & 1 & 0 \\ \theta_b & -k \sin \beta & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 & 1 \\ \sin \beta & \cos \beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k \cos \beta & -k \sin \beta & 1 & 0 \end{vmatrix}} = \frac{D_a}{D_d}$$

$$D_a = \theta_a \begin{vmatrix} 1 & 0 & 1 \\ \cos\beta & \ell & 1 \\ -k\sin\beta & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ \Delta & \cos\beta & 1 \\ \theta_b & -k\sin\beta & 0 \end{vmatrix}$$

$$= \theta_a(\cos\beta + \beta\sin\beta - 1) + \theta_b - k\sin\beta\,\Delta - \theta_b\cos\beta$$

$$= \theta_a(\cos\beta + \beta\sin\beta - 1) + \theta_b(1 - \cos\beta) - k\sin\beta\;\Delta$$

$$D_d = -\begin{vmatrix} \sin\beta & \ell & 1 \\ k & 1 & 0 \\ k\cos\beta & 1 & 0 \end{vmatrix} - \begin{vmatrix} \sin\beta & \cos\beta & \ell \\ k & 0 & 1 \\ k\cos\beta & -k\sin\beta & 1 \end{vmatrix}$$

$$= -k + k\cos\beta - k(\cos^2\beta + \sin^2\beta) + k\,\beta\sin\beta + k\cos\beta$$

$$= -2k + 2k\cos\beta + k\;\beta\sin\beta = k(2\cos\beta + \beta\sin\beta - 2)$$

$$B = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ \sin\beta & \Delta & \ell & 1 \\ k & \theta_a & 1 & 0 \\ k\cos\beta & \theta_b & 1 & 0 \end{vmatrix}}{D_d} = \frac{D_b}{D_d}$$

$$D_b = -\begin{vmatrix} \sin\beta & \Delta & \ell \\ k & \theta_a & 1 \\ k\cos\beta & \theta_b & 1 \end{vmatrix}$$

$$= -\theta_a\sin\beta - \theta_b\beta - k\cos\beta\;\Delta + \theta_a\beta\cos\beta + k\;\Delta + \theta_b\sin\beta$$

$$= \theta_a(\beta\cos\beta - \sin\beta) + \theta_b(\sin\beta - \beta) + \Delta(k - k\cos\beta)$$

$$y' = Ak\cos kx - Bk\sin kx + C$$

$$y'' = -Ak^2\sin kx - Bk^2\cos kx$$

$$M_{ab} = -EIy''(0) = EIBk^2$$

$$= \left[\frac{EIk^2}{k(2\cos\beta + \beta\sin\beta - 2)}\right][(\beta\cos\beta - \sin\beta)\theta_a + (\sin\beta - \beta)\theta_b$$
$$+ (k - k\cos\beta)\Delta]$$

$$= \left[\frac{EI\beta}{\ell(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\theta_a + (\sin\beta - \beta)\theta_b\right.$$
$$\left. + (\beta - \beta\cos\beta)\frac{\Delta}{\ell}\right]$$

Let

$$S_1 = S = \frac{\beta(\beta \cos \beta - \sin \beta)}{2 \cos \beta + \beta \sin \beta - 2}$$

$$S_2 = \frac{(\sin \beta - \beta)}{2 \cos \beta + \beta \sin \beta - 2}$$

Recall identities

$$\sin \beta = 2 \sin(\beta/2) \cos(\beta/2)$$

$$\cos \beta = \cos^2(\beta/2) - \sin^2(\beta/2) = 1 - 2 \sin^2(\beta/2)$$

Dividing the numerator and denominator of S_1 by $\sin \beta$ gives

$$S_1 = S = \frac{\beta(\beta \cot \beta - 1)}{2 \cot \beta - \dfrac{2}{\sin \beta} + \beta} = \frac{\beta(\beta \cot \beta - 1)}{den1 + \beta}$$

where

$$den1 = 2 \cot \beta - \frac{2}{\sin \beta} = \frac{2 \cos \beta - 2}{\sin \beta} = \frac{2[1 - 2 \sin^2(\beta/2) - 1]}{2 \sin(\beta/2) \cos(\beta/2)}$$
$$= -2 \tan(\beta/2)$$

$$S_1 = S = \frac{\beta(\beta \cot \beta - 1)}{-2 \tan(\beta/2) + \beta}$$

$$S_1 = S = \frac{1 - \beta \cot \beta}{\dfrac{2 \tan(\beta/2)}{\beta} - 1}$$

Let $S_2 = C = \dfrac{\beta(\sin \beta - \beta)}{2 \cos \beta + \beta \sin \beta - 2}$

Taking the same procedure used above gives

$$S_2 = C = \frac{\beta(1 - \beta \operatorname{cosec} \beta)}{2 \cot \beta - \dfrac{2}{\sin \beta} + \beta} = \frac{\beta(1 - \beta \operatorname{cosec} \beta)}{-2 \tan(\beta/2) + \beta}$$

$$S_2 = C = \frac{\beta \operatorname{cosec} \beta - 1}{\dfrac{2 \tan(\beta/2)}{\beta} - 1}$$

$$\text{Let } S_3 = SC = \frac{\beta(\beta - \beta \cos \beta)/\ell}{2 \cos \beta + \beta \sin \beta - 2}$$

Again dividing the numerator and denominator of S_3 by $\sin \beta$ gives:

$$S_3 = SC = \frac{\beta(\beta \operatorname{cosec} \beta - \beta \cot \beta)/\ell}{2 \cot \beta - \dfrac{2}{\sin \beta} + \beta} = \frac{\beta(\beta \operatorname{cosec} \beta - \beta \cot \beta)/\ell}{-2 \tan(\beta/2) + \beta}$$

$$= \frac{(\beta \cot \beta - \beta \operatorname{cosec} \beta)/\ell}{\dfrac{2 \tan(\beta/2)}{\beta} - 1} = \frac{[-(1 - \beta \cot \beta) - (\beta \operatorname{cosec} \beta - 1)]/\ell}{\dfrac{2 \tan(\beta/2)}{\beta} - 1}$$

$$S_3 = SC = -\frac{S_1 + S_2}{\ell} = -\frac{S + C}{\ell}$$

Recall $M_{ab} = M(0) = -EIy''(0)$.

But $M_{ba} = -M(\ell) = EIy''(\ell)$ (note the negative sign!)

$$y'' = -Ak^2 \sin kx - Bk^2 \cos kx$$

$$M_{ba} = +EIy''(\ell)$$

$$= \left[\frac{-EIk^2}{k(2 \cos \beta + \beta \sin \beta - 2)}\right] \left\{ \begin{aligned} &\sin \beta[\theta_a(\cos \beta + \beta \sin \beta - 1) + \theta_b(1 - \cos \beta) - \Delta \ k \sin \beta] \\ &\quad + \cos \beta[\theta_a(\beta \cos \beta - \sin \beta) + \theta_b(\sin \beta - \beta) \\ &\quad + \Delta(k - k \cos \beta)] \end{aligned} \right\}$$

$$= \left(\frac{-EIk}{2 \cos \beta + \beta \sin \beta - 2}\right) \left[\begin{aligned} &\theta_a(\cos \beta \sin \beta + \beta \sin^2 \beta - \sin \beta + \beta \cos^2 - \cos \beta \sin \beta) \\ &\quad + \theta_b(\sin \beta - \cos \beta \sin \beta + \cos \beta \sin \beta - \beta \cos \beta) \\ &\quad + \Delta(k \cos \beta - k \cos^2 \beta - k \sin^2 \beta) \end{aligned} \right]$$

$$= \left(-\frac{EI\beta}{\ell}\right)\frac{[\theta_a(\beta - \sin\beta) + \theta_b(\sin\beta - \beta\cos\beta) + \Delta(k\cos\beta - k)]}{(2\cos\beta + \beta\sin\beta - 2)}$$

$$= \left(\frac{EI}{\ell}\right)\frac{[\theta_a\beta(\sin\beta - \beta) + \theta_b\beta(\beta\cos\beta - \sin\beta) + \Delta\beta(\beta - \beta\cos\beta)/\ell]}{(2\cos\beta + \beta\sin\beta - 2)}$$

$$M_{ab} = \frac{EI}{\ell}\left[S_1\theta_a + S_2\theta_b - (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

$$M_{ba} = \frac{EI}{\ell}\left[S_2\theta_a + S_1\theta_b - (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

If $M_{ab} = 0$ (when the support A is either pinned or roller), then

$$M_{ab} = \frac{EI}{\ell}\left[S_1\theta_a + S_2\theta_b - \left(S_1 + S_2\right)\frac{\Delta}{\ell}\right] = 0$$

$$\theta_a = \frac{1}{S_1}\left[-S_2\theta_b + (S_1 + S_2)\frac{\Delta}{\ell}\right]$$

Substituting θ_a into M_{ba} yields

$$M_{ba} = \frac{EI}{\ell}\left[\left(S_1 - \frac{S_2^2}{S_1}\right)\theta_b - \left(S_1 + S_2\right)\left(1 - \frac{S_2}{S_1}\right)\frac{\Delta}{\ell}\right]$$

Let $\overline{S} = \frac{1}{S_1}(S_1^2 - S_2^2)$, then

$$\overline{M}_{ba} = \frac{EI}{\ell}\left[\overline{S}\theta_b - \overline{S}\frac{\Delta}{\ell}\right]$$

$$\overline{S} = \frac{1}{S_1}\left(S_1^2 - S_2^2\right)$$

$$= \left[\frac{-2\tan(\beta/2) + \beta}{\beta(\beta\cot\beta - 1)}\right]\left[\frac{\beta^2(\beta\cot\beta - 1)^2}{(-2\tan(\beta/2) + \beta)^2} - \frac{\beta^2(1 - \operatorname{cosec}\beta)^2}{(-2\tan(\beta/2) + \beta)^2}\right]$$

$$= \frac{\beta}{(\beta\cot\beta - 1)[-2\tan(\beta/2) + \beta]}[(\beta\cot\beta - 1)^2 - (1 - \operatorname{cosec}\beta)^2]$$

$$= \frac{\beta^2}{(\beta\cot\beta - 1)[-2\tan(\beta/2) + \beta]}[-\beta + 2\tan(\beta/2)] = \frac{\beta^2}{1 - \beta\cot\beta}$$