BUCKLING OF FRAMES

BUCKLING OF FRAME

Consider an elastically constrained column AB shown in Fig. 1. The two members, *AB* and *BC*, are assumed to have identical member length and flexural rigidity for simplicity. The moments, m and *M*, are due to the rotation at point *B* and possibly due to the axial shortening of member *AB*. Since $Q = (M + m)/l <<< p_{cr}$, Q is set equal to zero and the effect of any axial shortening is neglected.

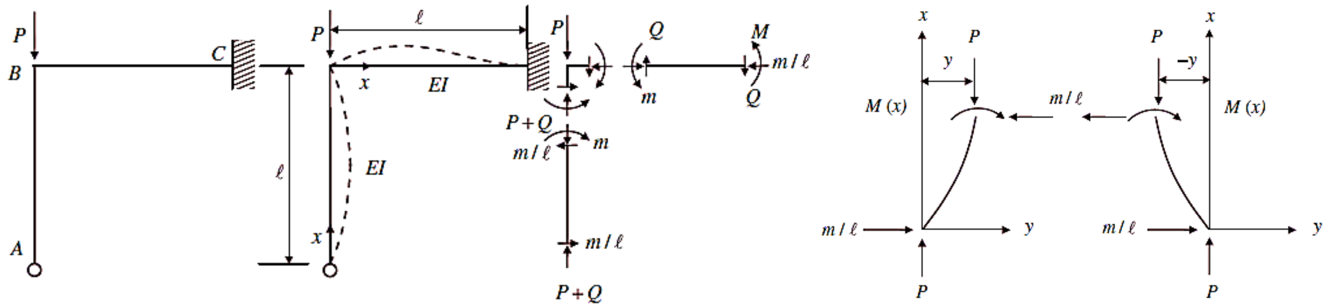

Figure 1: Buckling of simple frame

Summing moment at the top of the free body gives

(from the left free body)

$$M(x) + Py - \frac{mx}{\ell} = 0$$

$$EIy'' = -M(x) = -\left(Py - \frac{mx}{\ell}\right)$$

(from the right free body)

$$M(x) - P(-y) - \frac{mx}{\ell} = 0$$

$$EIy'' = M(x) = -Py + \frac{mx}{\ell}$$

As expected, the assumed deformed shape does not affect the Governing Differential Equation (GDE) of the behavior of member *AB*.

$$EIy'' + Py = \frac{mx}{\ell}$$

Let $k^2 = P/EI \Rightarrow y'' + k^2 y = (mx/\ell P)\, k^2$

The general solution to this *DE* is given

$$y = A \sin kx + B \cos kx + \frac{m}{\ell P} x$$

$$y = 0 \quad \text{at } x = 0 \Rightarrow B = 0$$

$$y = 0 \quad \text{at } x = \ell \Rightarrow A = -\frac{m}{P \sin k\ell}$$

$$y = \frac{m}{P}\left(\frac{x}{\ell} - \frac{\sin kx}{\sin k\ell}\right) \Leftarrow \text{buckling mode shape}$$

Since joint B is assumed to be rigid, continuity must be preserved. That is

$$\left.\frac{dy}{dx}\right|_{col} = \left.\frac{dy}{dx}\right|_{bm}$$

$$\text{for col } \left.\frac{dy}{dx}\right|_{x=\ell} = \frac{m}{P}\left(\frac{1}{\ell} - \frac{k \cos kx}{\sin k\ell}\right) = \frac{m}{P}\left(\frac{1}{\ell} - \frac{k}{\tan k\ell}\right)$$

$$= \frac{m}{kEI}\left(\frac{1}{k\ell} - \frac{1}{\tan k\ell}\right)$$

$$\text{for beam } \left.\frac{dy}{dx}\right|_{x=0} = \theta_N = \frac{m\ell}{4EI}$$

Recall the slope deflection equation: $m = (2EI/\ell)(2\theta_N + \cancel{\theta_F} - \cancel{3\varphi}) \Rightarrow$

$\theta_N = m\ell/4EI$

Equating the two slopes at joint B gives

$$\frac{m\ell}{4EI} = -\frac{m}{kEI}\left(\frac{1}{k\ell} - \frac{1}{\tan k\ell}\right) \Leftarrow \text{Note the direction of rotation at joint } B!$$

If the frame is made of the same material, then

$$\frac{\ell}{4I_b} = -\frac{1}{kI_c}\left(\frac{1}{k\ell} - \frac{1}{\tan k\ell}\right) \quad \text{or}$$

$$\frac{k\ell}{4} = -\frac{I_b}{I_c}\left(\frac{1}{k\ell} - \frac{1}{\tan k\ell}\right) \Leftarrow \text{stability condition equation}$$

Rearranging the stability condition equation gives

$$\frac{k\ell I_c}{4I_b} = -\frac{1}{k\ell} + \frac{1}{\tan k\ell} \Rightarrow \frac{1}{\tan k\ell} = \frac{1}{k\ell} + \frac{k\ell I_c}{4I_b} = \frac{4I_b + (k\ell)^2 I_c}{k\ell 4I_b} \Rightarrow$$

$$\tan k\ell = \frac{4k\ell I_b}{4I_b + (k\ell)^2 I_c}$$

If $I_b = 0$, then $P_{cr} = \dfrac{\pi^2 EI_c}{\ell^2}$

If $I_b = \infty$, then $P_{cr} = \dfrac{2\pi^2 EI_c}{\ell^2}$

For $I_b = I_c$, then $\tan k\ell = 4k\ell/(4+(k\ell)^2)$, the smallest root of this equation is $k\ell = 3.8289$.

$$P_{cr} = \frac{14.66EI_c}{\ell^2} = \frac{1.485\pi^2 EI_c}{\ell^2} \Rightarrow \text{as expected } 1 < 1.485 < 2.$$

BUCKLING MODES OF FRAMES

Consider first the frame in which side sway is prevented by bracing either internally or externally. It is obvious that the upper end of each column is elastically restrained by the beam to which the column is rigidly framed, and that the critical load of the column depends not only on the column stiffness, but also on the stiffness of the beam. It would be very informative to assume the beam stiffness to be either infinitely stiff or infinitely flexible as these two conditions constitute the upper and lower bounds of the connection rigidities. When the beam is assumed to be infinitely stiff, the beam must then remain straight while the frame deforms as shown in part (a), (1) Sidesway prevented, Fig. 2. Under this condition, the columns behave as if they were fixed at both ends, and the critical load of the column is equal to four times the Euler load of the same column pinned at its both ends. As the other extreme case of the opposite side, the beam can be assumed to be infinitely flexible. The frame then deforms as shown in part (b), (1) Sidesway prevented, Fig. 2, and the columns behave as if they were pinned at the top, and the critical load is the same as that of the propped column: approximately twice that of the Euler load of the same column pinned at both ends.
For an actual frame, the stiffness of the beam must be somewhere between the two extreme cases examined above. The critical load on the column in such a frame can be bounded as follows:

$$4P_E > P_{cr} > 2P_E$$

where P_{cr} is the critical load of the column and PE is the Euler load of the same column pinned at both ends.
It is just as informative to apply the same logic to frames in which sidesway is permitted. If the beam is assumed to be infinitely stiff, the

frame buckles in the manner shown in part (a), (2) Sidesway permitted, Fig. 2. The upper ends of the columns are permitted to translate, but they cannot rotate by definition.

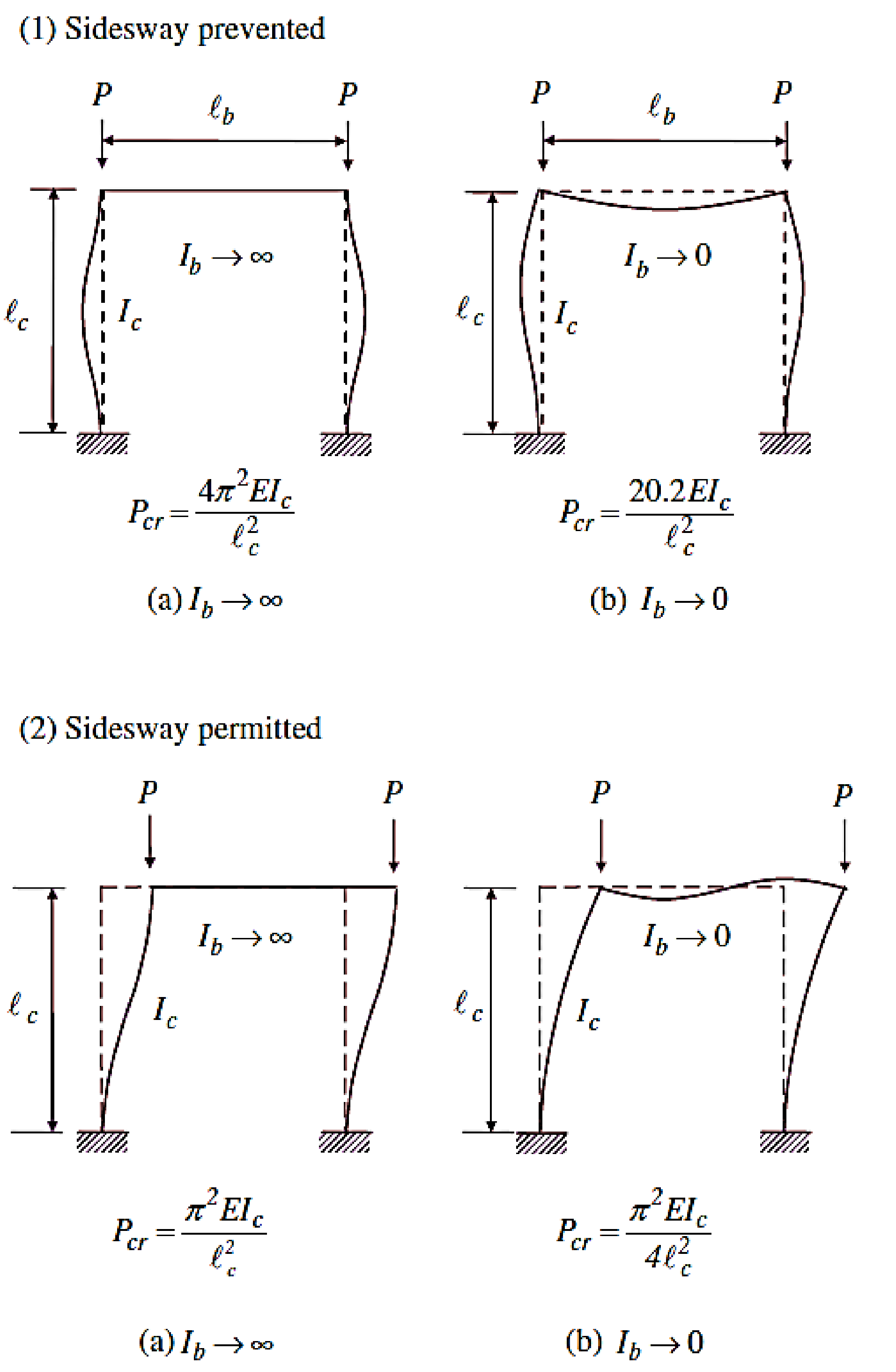

Figure 2: Modes of buckling

Hence, the critical load on each column in the frame is equal to the Euler load of the same column pinned at both ends. On the other extreme, if the beam is assumed infinitely flexible, the upper ends of the columns are both permitted to rotate and translate as shown in part (b), (2) Sidesway permitted, Fig. 2. In this extreme case, each column acts as if it were a cantilever column, and the critical load on each column is equal to one-fourth the Euler load of the same column pinned at both ends. The critical load on each column of the frame in which sidesway is permitted can be bounded as follows:

$$P_E > P_{cr} > 1/4\ P_E$$

$$P_{cr,\ braced\ frame} > P_{cr,\ unbraced\ frame}$$

A portal frame will always buckle in the sidesway permitted mode unless it is braced. Unlike the braced frame where sidesway is inherently prohibited, both the sidesway permitted and prevented modes are theoretically possible in the unbraced frame under the loading condition shown in Fig. 2. The unbraced frame, however, will buckle first at the smallest critical load, which is the one corresponding to the sidesway permitted mode. This conclusion is valid for multistory frames as well as for single-story frames. The reason appears to be obvious as the effective length of the compression member in an unbraced frame is always increased due to the frame action, while that in the braced frame is always reduced unless the beams in the frame are infinitely flexible.

CRITICAL LOADS OF FRAMES

1. Review of the Differential Equation Method

Case 1: Antisymmetric buckling

It is assumed that a set of usual assumptions normally employed in the classical analysis of linear elastic structures under the small displacement theory is valid. The sidesway buckling mode shape assumed and the forces acting on each member are identified in Fig. 3(a) and (b), respectively. The moment of the left vertical member at a point x from the origin based on the coordinate shown in Fig. 3(c) is (moment produced by the continuity shear developed in BC is neglected)

$$M(x) = M_{ab} - Py = M_{int} = EI_1 y''$$

or

$$y'' + k_1^2 y = \frac{M_{ab}}{EI_1}$$

where $k_1^2 = \dfrac{P}{EI_1}$

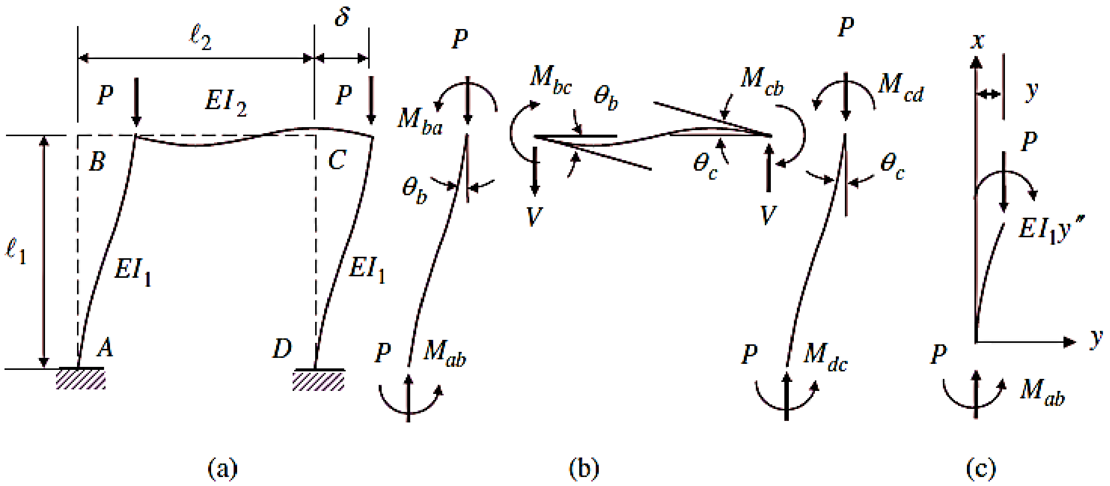

Figure 3: Buckling of unbraced frame

The general solution of Eq. above is given by

$$y = A \sin k_1x + B \cos k_1x \;\; M_{ab}/P$$

Two independent boundary conditions are needed to determine the integral constants, *A* and *B*. They are :

$$y=0 \text{ @ } x=0 \text{ …..} B=- M_{ab}/P \quad \text{and} \quad y'=0 \text{ @ } x=0 \text{ ….. } A=0$$

$$y = M_{ab}/P(1-\cos k_1 x)$$

Denoting the horizontal displacement at the top of the column ($x = l_1$) by δ, then

$$\delta= M_{ab}/P(1-\cos k_1 l_1)$$

Summing the moment of member AB at A gives:

$P\delta- M_{ab} - M_{ba}= 0$ ….. Substituting in Eq. above give $M_{ab} \cos k_1l_1+ M_{ba}= 0$

Since it is assumed that there is no axial compression presented in member *BC*, the slope-deflection equations without axial force apply. Hence,

$$M_{ba}=2EI/l_2(2\theta_b+\theta_c) \;\; \text{……. but } \theta_b=\theta_c \text{ ……. } M_{ba}=(6EI/l_2)\theta_b$$

The compatibility condition at joint *B* requires that θ_b equal to the slope at $x = l_1$. Hence,

$$\frac{M_{bc}\ell_2}{6EI_2} = \frac{M_{ab}}{k_1 EI_1} \sin k_1\ell_1 \quad \text{or} \quad \frac{6I_2}{k_1 I_1 \ell_2} M_{ab} \sin k_1\ell_1 - M_{bc} = 0$$

Setting the coefficient determinant equal to zero gives: $\frac{\tan k_1\ell_1}{k_1\ell_1} = -\frac{I_1\ell_1}{6I_2\ell_2}$

The critical load of the frame is the smallest root of this transcendental equation. For $I_2=I_1=I$, $l_1 = l_2 = l$ .... $\tan kl/kl=-1/6$
By using any transcendental equation: $kl=2.71646$ and $P_{cr}=7.38EI/l^2$

Case 2: Symmetric buckling ……..H.W.

2. Application of Slope-Deflection Equations to Frame Stability

It is assumed again that the axial compression in member BC would be negligibly small.

Since $\theta_a \equiv 0$, the moment at the top joint of member AB is

$$M_{ba} = (S_1\bar{k})_1\theta_b$$

where $\bar{k}_i = [(EI)/\ell]_i$
The moment in the horizontal member is

$$M_{bc} = (S_1\bar{k})_2\theta_b + (S_2\bar{k})_2\theta_c$$

but $\theta_b=-\theta_c$ …….. $M_{bc} = [(S_1\bar{k})_2 - (S_2\bar{k})_2]\theta_b$

Case 1: Antisymmetric buckling
Since there is no axial force in member BC, $(S_1)_2 = 4$ and $(S_2)_2 = 2$. For joint equilibrium M_{ba} and M_{bc} are the same in magnitude and opposite in sign. Thus

$$\sum M_b = 0 \Rightarrow M_{ba} + M_{bc} = 0$$

For $I_2 = I_1 = I$ and $\ell_2 = \ell_1 = \ell$, Eq. (4.4.26) reduces to

$$S_1\frac{EI}{\ell}\theta_b = \left[(4-2)\frac{EI}{\ell}\right]\theta_b$$

For which $S_1=2$ will lead to the critical load of $P_{cr}=25.18EI/l^2$

Case 2: Symmetric buckling ……..H.W.

SECOND-ORDER ANALYSIS OF A FRAME BY SLOPE-DEFLECTION EQUATIONS

The current AISC (2005) specification stipulates that "any second-order elastic analysis method that considers both *P - Δ* and *P-δ* effects may be used." Since both the joint rotation (*P - δ* effect) and joint translation (*P - Δ* effect) are reflected by the slope-deflection equations with axial force by a means of stability functions, *S1* and *S2*, an elastic analysis using the slope deflection equations is considered to be acceptable second-order analysis.

Example: Buckling of a rigidly connected equilateral triangle shown in Fig. 4. Take the counterclockwise moment and rotation as positive quantities as adopted in the derivation of the slope-deflection equations in previous Chapter.

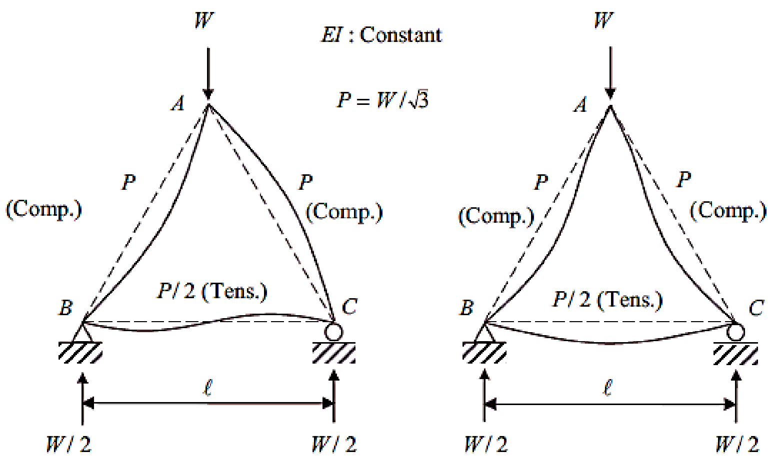

Figure 4: Equilateral triangle

The moment at each end of each member is then given by:

$$M_{ab} = \bar{k}(S_1\theta_a + S_2\theta_b), \quad M_{ac} = \bar{k}(S_1\theta_a + S_2\theta_c)$$

$$M_{ba} = \bar{k}(S_1\theta_b + S_2\theta_a), \quad M_{bc} = \bar{k}(S_1'\theta_b + S_2'\theta_c)$$

$$M_{ca} = \bar{k}(S_1\theta_c + S_2\theta_a), \quad M_{cb} = \bar{k}(S_1'\theta_c + S_2'\theta_b)$$

where $\bar{k} = EI/\ell$, and S_1' and S_2' reflect the tensile force in member BC.

The compatibility of the rigid joint requires the following moment-equilibrium condition at each joint:

$$M_{ab} + M_{ac} = 0 \qquad (S_1\theta_a + S_2\theta_b) + (S_1\theta_a + S_2\theta_c) = 0$$
$$M_{ba} + M_{bc} = 0 \qquad (S_1\theta_b + S_2\theta_a) + (S_1\theta_b + S_2\theta_c)' = 0$$
$$M_{ca} + M_{cb} = 0 \qquad (S_1\theta_c + S_2\theta_a) + (S_1\theta_c + S_2\theta_b)' = 0$$

$$2S_1\theta_a + S_2\theta_b + S_2\theta_c = 0$$

$$S_2\theta_a + (S_1 + S_1')\theta_b + S_2'\theta_c = 0$$

$$S_2\theta_a + S_2'\theta_b + (S_1 + S_1')\theta_c = 0$$

$$\begin{bmatrix} 2S_1 & S_2 & S_2 \\ S_2 & S_1 + S_1' & S_2' \\ S_2 & S_2' & S_1 + S_1' \end{bmatrix} \begin{Bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the augmented matrix equal to zero for the stability condition (a nontrivial solution) gives:

$$\det = 0 = (S_1 + S_1' - S_2')\left[S_1(S_1 + S_1' + S_2') - S_2^2\right] = 0$$

Two buckling modes are indicated by this Eq.:

$$S_1 + S_1' - S_2' = 0 \quad \text{or} \quad S_1(S_1 + S_1' + S_2') - S_2^2 = 0$$

$S_1(S_1+ S_1'+ S_2')- S_2^2=0$ give $kl=4.0122$ & $P_{cr} =16.1EI/l^2$

$S_1+ S_1'- S_2'=0$ give $kl=5.3217$ & $P_{cr} =28.32EI/l^2$

- For $k\ell = 4.0122,\ S_1 = 1.1490,\ S_2 = 3.0150,\ S_1' = 4.9763,\ S_2' = 1.7861$

$$\begin{bmatrix} 2.298 & 3.015 & 3.015 \\ 3.015 & 6.1253 & 1.7861 \\ 3.015 & 1.7861 & 6.1253 \end{bmatrix} \begin{Bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let $\theta_a = 1$ and expand the first and second rows of the matrix equation

$$2.298 + 3.015\,\theta_b + 3.015\,\theta_c = 0$$

from which

$$\theta_b = \frac{1}{3.015}(-2.298 - 3.015\,\theta_c)$$

and

$$3.015 + 6.1253\,\theta_b + 1.7861\,\theta_c = 0$$

$$3.015 + \frac{6.1253}{3.015}(-2.298 - 3.015\,\theta_c) + 1.7861\,\theta_c = 0$$

from which

$$\theta_c = -0.381 \qquad \theta_b = -0.381$$

The buckling mode shape is given in Fig. 5.

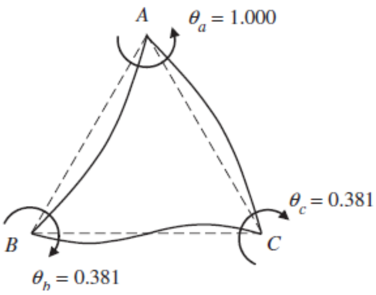

Figure 5: Equilateral triangle antisymmetric buckling mode

- For $k\ell = 5.3217$, $S_1 = -3.9419$, $S_2 = 6.2624$, $S_1{}' = 5.6170$, $S_2{}' = 1.6751$

$$\begin{bmatrix} -7.8838 & 6.2624 & 6.2624 \\ 6.2624 & 1.6751 & 1.6751 \\ 6.2624 & 1.6751 & 1.6751 \end{bmatrix} \begin{Bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let $\theta_a = 0 \qquad \theta_b = -\theta_c = 1.0$

The buckling mode shape is given graphically in Fig. 6.

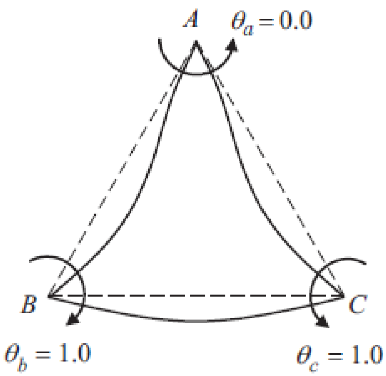

Figure 6: Equilateral triangle symmetric buckling mode