APPROXIMATE CALCULATION OF CRITICAL LOADS

ELASTIC BUCKLING ANALYSIS BY ENERGY METHODS

It has been shown that energy methods provide a convenient means of formulating the governing differential equation and necessary natural boundary conditions. The solutions that are obtained by solving the governing equations are exact within the framework of the theory (for example, classical beam theory) computing unknown forces and displacements in elastic structures. Besides providing convenient methods for computing unknown displacements and forces in structures, the energy principles are fundamental to the study of structural stability and structural dynamics.
Table 1 summarizes the energy theorem derived here.

Table 1: Variation Principles of Energy Methods

Displacement Methods	Force Methods
Principle of Virtual Work	**Principle of Complementary Virtual Work**
$\delta W_E = \delta U$	$\delta W_E^* = \delta U^*$
$\delta W_E = \sum_{i=1}^{n} Q_i \delta \Delta_i$	$\delta W_E^* = \sum_{i=1}^{n} \Delta_i \delta Q_i$
$\delta U = \int_V \sigma_{ij} \delta e_{ij}\, dV$	$\delta U^* = \int_V e_{ij} \delta \sigma_{ij}\, dV$
Principle of Minimum Potential Energy	**Principle of Minimum Complementary Energy**
$\delta \Pi = \delta(U + V) = 0$	$\delta \Pi^* = \delta(U^* + V^*) = 0$
$U = \int_V \left(\int_0^{e_{ij}} \sigma_{ij} de_{ij}\right) dv$	$U^* = \int_V \left(\int_0^{\sigma_{ij}} e_{ij} d\sigma_{ij}\right) dv$
$U = \int_V \left(v e_{ij} e_{ij} + \frac{\lambda}{2} e_{kk}^2 \right) dV = U^*$	$U^* = \int_V \left(\frac{1+\mu}{2E} \sigma_{ij}\sigma_{ij} - \frac{\mu}{2E} \sigma_{kk}^2 \right) dV = U$
$V = -\sum_{i=1}^{n} Q_i \Delta_i$	$V^* = -\sum_{i=1}^{n} Q_i \Delta_i$
Castigliano Theorem, Part I	**Castigliano Theorem, Part II**
$Q_i = \frac{\partial U}{\partial \Delta_i}$	$\Delta_i = \frac{\partial U^*}{\partial Q_i} = \frac{\partial U}{\partial Q_i}$
$k_{ij} = \frac{\partial^2 U}{\partial \Delta_i \partial \Delta_j}$	$f_{ij} = \frac{\partial^2 U}{\partial Q_i \partial Q_j}$

Notes: The Lamé constants λ and ν in the table are given by

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)}$$

and

$$\nu = \frac{E}{2(1+\mu)}$$

Terms in "bold font" are valid for linearly elastic materials only.

It is noted that a duality exists between those principles and theorems involving generalized displacements as the varied quantities (displacement methods) and those involving variations in the generalized forces (force methods). Principles and theorems related to the principle of virtual work are grouped as displacement methods, and those related to the principle of the complementary virtual work are grouped as force methods. These equations apply to nonlinear as well as linearly elastic materials, except where noted otherwise in Table 1.

STABILITY CRITERIA

The stability criteria must be established in order to answer the question of whether a structure is in stable equilibrium under a given set of loadings. If upon releasing the structure from its virtually displaced state the structure returns to its previous configuration, then the structure is said to be in stable equilibrium. On the other hand if the structure does not return to its undisturbed state following the release of the virtual displacements, the condition is either neutral equilibrium or unstable equilibrium. Stability can also be defined in terms of the total potential energy $\prod$ of the structure. Recall that $\prod$ is the sum of the strain energy *U* stored in the deformed elastic body and the loss of the potential of the generalized external forces *V*. If the total potential energy increases during a virtual displacement, then the equilibrium configuration is defined to be stable; if $\prod$ decreases or remains unchanged, the configuration is unstable. The stability criteria can also be expressed in mathematical form. For simplicity it is assumed that the structure's deformation is characterized by a finite number of generalized displacements Δ_i. If the structure is given a virtual displacement $\delta\Delta_i$, then it is possible to write the total potential energy in a Taylor series expansion about Δ_i. Consider, for example, a two-degree-of-freedom system.

$$\Pi(\Delta_1+\delta\Delta_1,\Delta_2+\delta\Delta_2) = \Pi(\Delta_1,\Delta_2)+\frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1+\frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2$$
$$+\frac{1}{2!}\left[\frac{\partial^2\Pi}{\partial\Delta_1^2}(\delta\Delta_1)^2+2\frac{\partial^2\Pi}{\partial\Delta_1\partial\Delta_2}\delta\Delta_1\delta\Delta_2+\frac{\partial^2\Pi}{\partial\Delta_2^2}(\delta\Delta_2)^2\right]+\cdots$$

The change in potential energy is then

$$\Delta\Pi = \delta\Pi+\frac{1}{2!}\delta^2\Pi+\cdots$$

where the first variation is equal to zero by virtue of the principle of the minimum total potential energy.

$$\delta\Pi = \frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1+\frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2 = 0$$

and the second variation is

$$\delta^2\Pi=\delta(\delta\Pi)=\frac{\partial^2\Pi}{\partial\Delta_1^2}(\delta\Delta_1)^2+2\frac{\partial^2\Pi}{\partial\Delta_1\partial\Delta_2}\delta\Delta_1\delta\Delta_2+\frac{\partial^2\Pi}{\partial\Delta_2^2}(\delta\Delta_2)^2$$

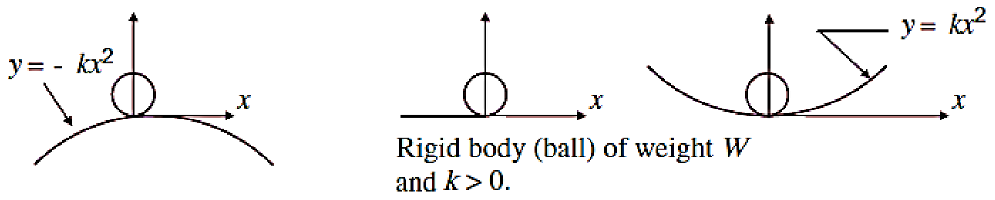
Since $\delta\prod = 0$, the second variation is the relevant term. If $\delta^2\prod$ is positive, then $\Delta\prod$ is positive, $\prod$ is a local minimum, and the equilibrium condition is stable. The special case in which the second variation is zero corresponds to a state known as neutral equilibrium. When a structure that is in neutral equilibrium is released from a virtual displacement, there is no net restoring force present, and the system remains in its virtual displaced state. Hence, by the first definition of stability, neutral equilibrium is a special case of unstable equilibrium. The criteria for stability are summarized as follows:

$\Delta\prod > 0$ stable equilibrium
$\Delta\prod = 0$ neutral equilibrium
$\Delta\prod < 0$ unstable equilibrium

If the potential energy P is quadratic in the displacements Δ_i, which is the case when the structure is linearly elastic and the deformations are small, then all variations higher than the second are necessarily zero. In this case the type of equilibrium is governed by the following conditions:

$\delta^2\prod > 0$ stable equilibrium
$\delta^2\prod = 0$ neutral equilibrium
$\delta^2\prod < 0$ unstable equilibrium

these condition is called the sufficient condition. A rigid body stability concept can be illustrated as follows:



unstable equilibrium	neutral equilibrium	stable equilibrium
$U+V=0+Wy=-kx^2W$	$\frac{d(U+V)}{dx}=0$	$U+V=kx^2W$
$\frac{d(U+V)}{dx}=-2kxW=0 \ @x=0$	$\frac{d^2(U+V)}{dx^2}=0$	$\frac{d(U+V)}{dx}=2kxW=0 \ @x=0$
$\frac{d^2(U+V)}{dx^2}=-2kW<0$		$\frac{d^2(U+V)}{dx^2}=2kW>0$

Figure 1: Concept of rigid body equilibrium

RAYLEIGH-RITZ METHOD

The energy methods introduced in previous section are a convenient means of computing unknown forces and displacements in elastic structures. They can be the basis of deriving the governing differential equations and required boundary conditions of the problem. They are also the starting point of many modern matrix/finite element methods. The solutions that are obtained using these methods are exact within the framework of the theory (for example, classical beam theory). Energy methods are also used to derive approximate solutions in situations where exact solutions are difficult or nearly impossible to obtain. The most widely known and used approximate procedure is the Rayleigh-Ritz method,6 in which the structure’s displacement field is approximated by functions that include a finite number of independent coefficients (or natural coordinates; one for the Rayleigh method and more than one for the Rayleigh-Ritz method). The assumed solution functions must satisfy the kinematic boundary conditions (otherwise, the convergence is not guaranteed, no matter how many functions are assumed), but they need not satisfy the natural boundary conditions (if they satisfy the natural boundary condition, a fairly good solution accuracy can be expected). The unknown constants in the assumed functions are determined by invoking the principle of minimum potential energy. Suppose, for example, the assumed function has n independent constants a_i (i = 1, 2,.,n). Since the approximate state of deformation of the structure is characterized (amplitude as well as shape) by these n constants, the degrees of freedom of the structure have been reduced from ∞ to n. Invoking the principle of minimum potential energy, it follows that:

$$\delta \Pi = \frac{\partial \Pi}{\partial a_1}\delta a_1 + \frac{\partial \Pi}{\partial a_2}\delta a_2 + \cdots + \frac{\partial \Pi}{\partial a_n}\delta a_n = 0$$

Since δa_i are arbitrary, the Eq. above implies that:

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad i = 1, 2\ldots, n$$

This equation yields a system of n simultaneous equations that can be solved for the coefficients ai for static problems, and in the case of eigenvalue problems, the determinant (characteristic determinant) for the unknown constants is set equal to zero for the n eigenvalues.

Example 1 Consider a both-ends pinned column shown in Fig. 2. The strain energy stored in the deformed body is

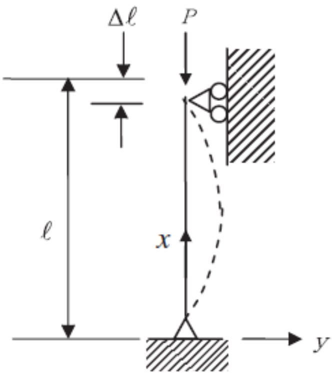


Figure2: Simple column model

$$U = \frac{1}{2}\int_0^{\ell}\frac{M^2}{EI}dx = \frac{1}{2}\int_0^{\ell}\frac{(-EIy'')^2}{EI}dx = \frac{EI}{2}\int_0^{\ell}(y'')^2 dx$$

The potential energy of the applied load is

$V = -P\Delta\ell$ (the reason for the negative sign: as $\Delta\ell$ increases, V decreases)

$$ds^2 = dx^2 + dy^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]dx^2 \Rightarrow ds = \sqrt{1 + (y')^2}dx$$

$$\Delta\ell = \int_0^{\ell} ds - \int_0^{\ell} dx = \int_0^{\ell}\sqrt{1 + (y')^2}dx - \int_0^{\ell} dx$$

$$= \int_0^{\ell}\left[1 + \frac{1}{2}(y')^2 + ..\right]dx - \int_0^{\ell} dx \doteq \frac{1}{2}\int_0^{\ell}(y')^2 dx$$

$$V = -\frac{P}{2}\int_0^{\ell}(y')^2 dx$$

$$\delta\Pi = \delta U + \delta V = \delta\left[\frac{EI}{2}\int_0^\ell (y'')^2 dx - \frac{P}{2}\int_0^\ell (y')^2 dx\right] = 0$$

Assume the solution function to be of the form:

$$y = \sum_{i=1}^{n} a_i\phi_i = \sum_{i=1}^{n} a_i \sin(i\pi x/\ell)$$

$$\Pi = \frac{EI}{2}\int_0^\ell (y'')^2 dx - \frac{P}{2}\int_0^\ell (y')^2 dx$$

$$= \frac{EI}{2}\int_0^\ell \left[\sum_{i=1}^{n}(-1)\frac{i^2\pi^2}{\ell^2} a_i \sin\frac{i\pi x}{\ell}\right]^2 dx - \frac{P}{2}\int_0^\ell \left(-\sum_{i=1}^{n}\frac{i\pi}{\ell} a_i \cos\frac{i\pi x}{\ell}\right)^2 dx$$

$$= \frac{EI}{2}\left(\frac{\pi^4}{2\ell^3}\sum_{i=1}^{n} i^4 a_i^2\right) - \frac{P}{2}\left(\frac{\pi^2}{2\ell}\sum_{i=1}^{n} i^2 a_i^2\right)$$

Recall the following orthogonality of finite integrals of trigonometric functions: $\int_0^\ell(\sin^2 ax)\,dx = (\ell/2)$, $\int_0^\ell(\cos^2 ax)\,dx = (\ell/2)$, $\int_0^\ell(\sin ix)(\sin jx)\,dx = 0\ (i \neq j)$, and $\int_0^\ell(\cos ix)(\cos jx)\,dx = 0\ (i \neq j)$

$$\frac{\partial\Pi}{\partial a_i} = 0 = \frac{EI\pi^4}{4\ell^3} i^4(2a_i) - \frac{P}{2}\frac{\pi^2}{2\ell} i^2(2a_i) = \left(\frac{EI\pi^4}{\ell^2} i^2 - P\pi^2\right) a_i = 0$$

$$\text{As } a_i \neq 0,\ P_i = \frac{i^2\pi^2 EI}{\ell^2} \text{ or } (P_{cr})_{i=1} = \frac{\pi^2 EI}{\ell^2} \Leftarrow \text{exact solution}$$

THE RAYLEIGH QUOTIENT

The approximate solution of the eigenvalue problem usually reduces to the integration of a differential equation of the form

$$Lw - \lambda Mw = 0$$

where *w* is the displacement that satisfies not only the differential equation below, but also certain homogeneous boundary conditions (this condition may preclude the cantilevered end condition), *L* and *M* are certain differential operators, and λ is an unknown numerical parameter. For the stability of a column, the governing differential equation is

$$L \equiv \frac{d^2}{dx^2} EI \frac{d^2}{dx^2} \qquad \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = -P\frac{d^2w}{dx^2}$$

$$M \equiv -\frac{d^2}{dx^2}$$

$$\lambda = P$$

If a linear differential operator L has the following property, it is called a self-adjoint or symmetric operator: *(Lu, v) =(Lv, u)*

The inner product of two functions g and h over the domain *V* is defined as:

$$(g, h) \equiv \text{ inner product of } g \text{ and } h \equiv \int_V gh\, dv$$

An operator is said to be positive definite if the following inequality is valid for any function from its field of definition, $u(q) \neq 0$:

$$(Lu, u) > 0, (Lu, u) \equiv 0 \text{ for } u(q) \equiv 0$$

Multiplying both sides of Eq. $\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = -P\frac{d^2w}{dx^2}$ by *w* and integrating over the domain yields

$$\int_0^{\ell} w\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)dx = -P\int_0^{\ell} w\frac{d^2w}{dx^2}dx$$

Integrate the left-hand side of Eq. by parts twice, as follows:

$$\int_0^{\ell} w \frac{d^2}{dx^2}\left(EI \frac{d^2 w}{dx^2}\right) dx = \int_0^{\ell} EI \left(\frac{d^2 w}{dx^2}\right)^2 dx + w \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2}\right)\Bigg|_0^{\ell} - EI \frac{dw}{dx} \frac{d^2 w}{dx^2}\Bigg|_0^{\ell}$$

For simply supported, fixed, or cantilevered end conditions, the last two quantities are zero. Integrating the right-hand side of Eq. gives

$$-P \int_0^{\ell} w \frac{d^2 w}{dx^2} dx = P \int_0^{\ell} \left(\frac{dw}{dx}\right)^2 dx - Pw \frac{dw}{dx}\Bigg|_0^{\ell}$$

The last expression vanishes for fixed and simple supports (not for the cantilevered end). Substituting the expanded integrals back into Eq. gives:

$$P = \frac{EI \int_0^{\ell} \left(d^2 w / dx^2\right)^2 dx}{\int_0^{\ell} (dw/dx)^2 dx} \quad \text{(C1 method)}$$

It is noted that this Eq. works for cantilevered columns despite the fact that one of the concomitants is not zero. As mentioned earlier, the error involved in the approximate solution propagates much faster in the higher order derivatives. In order to improve the critical value computed from the Rayleigh quotient, $d^2w=dx^2$ in the numerator is replaced by *M/EI*. Then:

$$P_{cr} = \frac{\left(1/EI\right) \int_0^{\ell} M^2 dx}{\int_0^{\ell} (w')^2 dx} \quad \text{(C2 method)}$$

ENERGY METHOD APPLIED TO COLUMNS SUBJECTED TO DISTRIBUTED AXIAL LOADS

- **Cantilever Column**

The Rayleigh method can also be applied to the calculation of the critical value of the distributed compressive loads. As a first approximation of the deflection curve, the following equation may be tried:

$$y = \delta\left(1 - \cos\frac{\pi x}{2\ell}\right)$$

$$M = \int_x^\ell q(\eta - y)d\xi$$

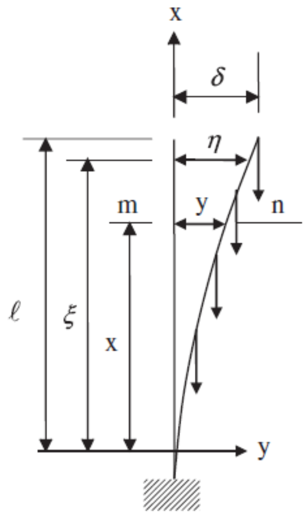


Figure 3: Cantilever column subjected to distributed axial load

The deflection η is also expressed as

$$\eta = \delta\left(1 - \cos\frac{\pi\xi}{2\ell}\right)$$

$$M = q\int_x^\ell (\eta - y)d\xi = q\left[\int_x^\ell \eta d\xi - y(\ell - x)\right]$$

$$\int_x^\ell \eta d\xi = \delta\int_x^\ell \left(1 - \cos\frac{\pi\xi}{2\ell}\right)d\xi = \delta\left[(\ell - x) - \frac{2\ell}{\pi}\sin\frac{\pi\xi}{2\ell}\Big|_x^\ell\right]$$

$$= \delta\left[(\ell - x) - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right]$$

$$M = q\delta\left[(\ell - x) - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right) - \left(1 - \cos\frac{\pi x}{2\ell}\right)(\ell - x)\right]$$

$$= q\delta\left[(\ell - x)\cos\frac{\pi x}{2\ell} - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right]$$

$$U = \frac{1}{2EI}\int_0^\ell M^2 dx = \frac{\delta^2 q^2 \ell^3(-192 + 54\pi + \pi^3)}{12EI\pi^3}$$

and the work done by the distributed load above the section mn is

$$\frac{1}{2}q(\ell - x)\left(\frac{dy}{dx}\right)^2 dx$$

The total loss of potential energy of the distributed load during

The total loss of potential energy of the distributed load during buckling is:

$$V = -\frac{1}{2}q\int_0^\ell (\ell - x)\left(\frac{dy}{dx}\right)^2 dx = -\frac{1}{2}q\delta^2\int_0^\ell (\ell - x)\left(\frac{\pi}{2\ell}\sin\frac{\pi x}{2\ell}\right)^2 dx$$

$$= -\frac{\delta^2 q}{32}(\pi^2 - 4)$$

$$\frac{\partial \Pi}{\partial \delta} = \frac{\partial U}{\partial \delta} + \frac{\partial V}{\partial \delta} = 0 = \frac{\delta q^2\ell^3\left(-192 + 54\pi + \pi^3\right)}{6EI\pi^3} - \frac{\delta q}{16}(\pi^2 - 4) = 0$$

$$q = \frac{\left(\pi^2 - 4\right)}{16}\,\frac{6EI\pi^3}{(-192 + 54\pi + \pi^3)\ell^3} = 7.888\,\frac{EI}{\ell^3}$$

Although this Eq. is only 0.65% greater than the exact solution, it would seem interesting to see how much the accuracy can be improved by taking one more term in the assumed displacement function. Consider the following function for the deflection of the cantilever shown in Fig. 3:

$$y = a\left(1 - \cos\frac{\pi x}{2\ell}\right) + b\left(1 - \cos\frac{3\pi x}{2\ell}\right)$$

$$\eta = a\left(1 - \cos\frac{\pi \xi}{2\ell}\right) + b\left(1 - \cos\frac{3\pi \xi}{2\ell}\right)$$

$$\int_x^\ell \eta\, d\xi = \int_x^\ell \left[a\left(1 - \cos\frac{\pi \xi}{2\ell}\right) + b\left(1 - \cos\frac{3\pi \xi}{2\ell}\right)\right] d\xi$$

$$= a\left[(\ell - x) - \frac{2\ell}{\pi}\sin\frac{\pi\xi}{2\ell}\bigg|_x^\ell\right] + b\left[(\ell - x) - \frac{2\ell}{3\pi}\sin\frac{3\pi\xi}{2\ell}\bigg|_x^\ell\right]$$

$$= a\left[(\ell - x) - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right] + b\left[(\ell - x) + \frac{2\ell}{3\pi}\left(1 + \sin\frac{3\pi x}{2\ell}\right)\right]$$

$$M = q\begin{Bmatrix} a\left[(\ell - x) - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right) - \left(1 - \cos\frac{\pi x}{2\ell}\right)(\ell - x)\right] \\ +b\left[(\ell - x) + \frac{2\ell}{3\pi}\left(1 + \sin\frac{3\pi x}{2\ell}\right) - \left(1 - \cos\frac{3\pi x}{2\ell}\right)(\ell - x)\right] \end{Bmatrix}$$

$$= q\left\{a\left[(\ell - x)\cos\frac{\pi x}{2\ell} - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right] + b\left[(\ell - x)\cos\frac{3\pi x}{2\ell} + \frac{2\ell}{3\pi}\left(1 + \sin\frac{3\pi x}{2\ell}\right)\right]\right\}$$

$$U = \frac{1}{2EI}\int_0^\ell M^2 dx$$

$$= \frac{q^2\ell^3}{108\pi^3 EI}\begin{bmatrix} \left(-1728 + 486\pi + 9\pi^3\right)a^2 + \left(64 + 54\pi + 9\pi^3\right)b^2 \\ + (384 - 9\pi)ab \end{bmatrix}$$

The total loss of potential energy of the distributed load during buckling is:

$$V = -\frac{1}{2}q\int_0^\ell (\ell - x)\left(\frac{dy}{dx}\right)^2 dx$$

$$= -\frac{1}{2}q\int_0^\ell (\ell - x)\left(\frac{\pi}{2\ell}\sin\frac{\pi x}{2\ell}a + \frac{3\pi}{2\ell}\sin\frac{3\pi x}{2\ell}b\right)^2 dx$$

$$= -\frac{1}{32}q\left[\left(\pi^2 - 4\right)a^2 + \left(9\pi^2 - 4\right)b^2 + 24ab\right]$$

$$\Pi = U + V$$

$$\frac{\partial \Pi}{\partial a} = \frac{\partial U}{\partial a} + \frac{\partial V}{\partial a} = \frac{q^2\ell^3}{108\pi^3 EI}\left[\left(-3456 + 972\pi + 18\pi^3\right)a + (384 - 9\pi)b\right]$$

$$-\frac{1}{32}q\left[\left(2\pi^2 - 8\right)a + 24b\right] = 0$$

$$\frac{\partial \Pi}{\partial b} = \frac{\partial U}{\partial b} + \frac{\partial V}{\partial b} = \frac{q^2\ell^3}{108\pi^3 EI}\left[\left(128 + 108\pi + 18\pi^3\right)b + (384 - 9\pi)a\right]$$

$$-\frac{1}{32}q\left[\left(18\pi^2 - 8\right)b + 24a\right] = 0$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant for the coefficient matrix for a and b must be equal to zero. Solving the resulting polynomial for the critical value yields:

$$q_{cr} = 7.888 \frac{EI}{\ell^3}$$

The uniform load *ql* reduces the critical buckling load *P* applied at the cantilever tip. It is written in the form:

$$P_{cr} = \frac{mEI}{\ell^2}$$

where the factor m is equal to $\pi^2/4$ when *ql* is equal to zero and it approaches zero when q‘ approaches the value given by Eq. of q_{cr} .Using the notation

$$n = \frac{4q\ell^3}{\pi^2 EI}$$

The following illustration is an example case of using the energy method to compute values of n and m interactively. The moment due to the concentrated load *P* is:

$$M_P = P(\delta - y) = \delta P \cos\frac{\pi x}{2\ell}$$

$$M_q = q\delta\left[(\ell - x)\cos\frac{\pi x}{2\ell} - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right]$$

$$M = M_P + M_q = \delta\left\{P\cos\frac{\pi x}{2\ell} + q\left[(\ell - x)\cos\frac{\pi x}{2\ell} - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right]\right\}$$

$$\begin{aligned}
U &= \frac{1}{2EI}\int_0^\ell M^2 dx \\
&= \frac{\delta^2}{2EI}\int_0^\ell \left\{P\cos\frac{\pi x}{2\ell} + q\left[(\ell - x)\cos\frac{\pi x}{2\ell} - \frac{2\ell}{\pi}\left(1 - \sin\frac{\pi x}{2\ell}\right)\right]\right\}^2 dx \\
&= \delta^2\ell\big(-12\pi\ell\pi qP + 54\ell^2\pi q^2 - 192\ell^2 q^2 + \ell^2\pi^3 q^2 + 3\pi^3 P^2 \\
&\quad + 3\ell\pi^3 qP\big)/\big(12EI\pi^3\big) \\
&= \frac{\delta^2\ell}{12EI\pi^3}\big(93.01883P^2 + 55.3197182\ell qP + 8.65228\ell^2 q^2\big)
\end{aligned}$$

$$V_q = -\frac{1}{2}q\int_0^{\ell}(\ell - x)\left(\frac{dy}{dx}\right)^2 dx = -\frac{1}{2}q\delta^2\int_0^{\ell}(\ell - x)\left(\frac{\pi}{2\ell}\sin\frac{\pi x}{2\ell}\right)^2 dx$$
$$= -\frac{\delta^2 q}{32}(\pi^2 - 4)$$

$$V = V_P + V_q = -\frac{\delta^2 P}{2}\left(\frac{\pi}{2\ell}\right)^2\frac{2\ell}{2} - \frac{\delta^2 q}{32}(\pi^2 - 4)$$
$$= -\delta^2\left(0.6168503\,\frac{P}{\ell} + 0.183425138q\right)$$

$$\frac{\partial U}{\partial \delta} + \frac{\partial V}{\partial \delta} = 0.25\,\frac{\ell}{EI}P^2 + 0.148678816\,\frac{\ell^2}{EI}qP + 0.0232541088\,\frac{\ell^3}{EI}q^2$$
$$- 0.6168503\,\frac{P}{\ell} - 0.183425138q = 0$$

If $P = 0$, then

$$q_{cr} = \frac{7.88786EI}{\ell^3}$$

If $n = 1(q = \pi^2 EI/4\ell^3)$, then

$P_{cr} = 1.7223\ EI/\ell^2 \Leftarrow 0.13\%$ greater than the exact solution.

- H.W.: Using energy method for find critical load for column subjected to distributed load axial load *q* and concentrated load *p* at the top of column for following B.C.:

1. Simply –supported column if $y=\delta \sin\pi x/l$
2. Pinned –clamed column if $y= a(l^3x - 3lx^3 + 2x^4)$
3. Clamed – Pinned column if $y= a(3l^2x^2 - 5lx^3 + 2x^4)$
4. Both – ends clamped column if $y=a(1 - \cos 2\pi x/l)$

ELASTICALLY SUPPORTED BEAM-COLUMNS

As an example of the stability of a bar on elastic supports, consider a prismatic continuous beam simply supported at the ends on rigid supports and having several intermediate elastic supports.
Let *q= force* developed in the spring=*ky*. Then the work done by the spring is $(1/2)qy=(1/2)\ ky^2$. Rotational spring can also be considered at any support. Total potential energy function of the system becomes:

$$\Pi = U + V = \frac{EI}{2}\int_0^{\ell}(y'')^2 dx - \frac{P}{2}\int_0^{\ell}(y')^2 dx + \frac{1}{2}\sum_{i=1}^{n} k_i y_i^2$$

Let $k_1 = k_2 = k$ and $x_1\ \ l/3;\ x_2 = 2l/3$ to simplify the computation effort. It appears that at least three sine functions need to be considered for the three-span configuration shown in Fig. 4.

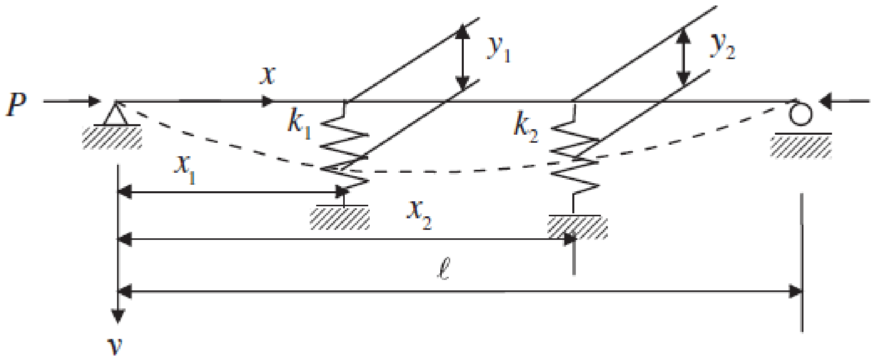


Figure 4: Column resting on elastic supports

Assume $y = a_1 \sin\frac{\pi x}{\ell} + a_2 \sin\frac{2\pi x}{\ell} + a_3 \sin\frac{3\pi x}{\ell}$

$$\Pi = \frac{EI}{2}\int_0^{\ell}\left[-a_1\left(\frac{\pi}{\ell}\right)^2\sin\frac{\pi x}{\ell} - a_2\left(\frac{2\pi}{\ell}\right)^2\sin\frac{2\pi x}{\ell} - a_3\left(\frac{3\pi}{\ell}\right)^2\sin\frac{3\pi x}{\ell}\right]^2 dx$$

$$-\frac{P}{2}\int_0^{\ell}\left[a_1\left(\frac{\pi}{\ell}\right)\cos\frac{\pi x}{\ell} + a_2\left(\frac{2\pi}{\ell}\right)\cos\frac{2\pi x}{\ell} + a_3\left(\frac{3\pi}{\ell}\right)\cos\frac{3\pi x}{\ell}\right]^2 dx$$

$$+\frac{1}{2}k\left[\left(a_1\sin\frac{\pi}{3} + a_2\sin\frac{2\pi}{3}\right)^2 + \left(a_1\sin\frac{2\pi}{3} + a_2\sin\frac{4\pi}{3}\right)^2\right]$$

Noting that

$$\int_0^\ell \sin\frac{i\pi x}{\ell}\sin\frac{j\pi x}{\ell}\,dx = \begin{cases} 0 \text{ for } i\neq j \\ \dfrac{\ell}{2} \text{ for } i = j \end{cases} \quad \text{and}$$

$$\int_0^\ell \cos\frac{i\pi x}{\ell}\cos\frac{j\pi x}{\ell}\,dx = \begin{cases} 0 \text{ for } i\neq j \\ \dfrac{\ell}{2} \text{ for } i = j \end{cases}$$

$$\Pi = \frac{EI\pi^4}{4\ell^3}\left(a_1^2 + 16a_2^2 + 81a_3^2\right) - \frac{P\pi^2}{4\ell}\left(a_1^2 + 4a_2^2 + 9a_3^2\right) + \frac{3}{4}k\left(a_1^2 + a_2^2\right)$$

$$\frac{\partial\Pi}{\partial a_1} = 0 = \frac{EI\pi^4}{4\ell^3}(2a_1) - \frac{P\pi^2}{4\ell}(2a_1) + \frac{3}{4}k(2a_1) = \left(\frac{EI\pi^4}{\ell^3} - \frac{P\pi^2}{\ell} + 3k\right)a_1$$

$$P_{cr} = \frac{4\ell}{\pi^2}\left(\frac{EI\pi^4}{4\ell^3} + \frac{3}{4}k\right) = \frac{\pi^2 EI}{\ell^2} + \frac{3k\ell}{\pi^2} = P_E\left(1 + \frac{3}{\pi^2}\frac{k\ell}{P_E}\right)$$

$$\frac{P_{cr}}{P_E} = 1 + \frac{3k\ell}{\pi^2 P_E}$$

$$\frac{\partial\Pi}{\partial a_2} = 0 = \frac{EI\pi^4}{4\ell^3}(32a_2) - \frac{P\pi^2}{4\ell}(8a_2) + \frac{3}{4}k(2a_2) = \left(\frac{4EI\pi^4}{\ell^3} - \frac{P\pi^2}{\ell} + \frac{3}{4}k\right)a_2$$

$$P_{cr} = \frac{\ell}{\pi^2}\left(\frac{4EI\pi^4}{\ell^3} + \frac{3}{4}k\right) = \frac{4\pi^2 EI}{\ell^2} + \frac{3k\ell}{4\pi^2} = P_E\left(4 + \frac{3}{4\pi^2}\frac{k\ell}{P_E}\right)$$

$$\frac{P_{cr}}{P_E} = 4 + \frac{3k\ell}{4\pi^2 P_E}$$

$$\frac{\partial\Pi}{\partial a_3} = 0 = \frac{EI\pi^4}{4\ell^3}(162a_3) - \frac{P\pi^2}{4\ell}(18a_3) = \left(\frac{9EI\pi^2}{\ell^3} - \frac{P}{\ell}\right)a_3$$

$$P_{cr} = \frac{\ell}{\pi^2}\left(\frac{9EI\pi^4}{\ell^3}\right) = \frac{9\pi^2 EI}{\ell^2}, \quad \frac{P_{cr}}{P_E} = 9$$

$$1 + \frac{3}{\pi^2}\frac{k\ell}{P_E} = 4 + \frac{3}{4\pi^2}\frac{k\ell}{P_E} \Rightarrow \frac{3}{\pi^2}\left(1 - \frac{1}{4}\right)\frac{k\ell}{P_E} = 3 \Rightarrow \frac{k\ell}{P_E} = 13.16$$

$$4 + \frac{3k\ell}{4\pi^2 P_E} = 9 \Rightarrow \frac{k\ell}{P_E} = 5\,\frac{4\pi^2}{3} = 65.8$$

Assuming $l = 3L$ and $P_{cr} = 9P_E$ for three equal spans,

$$k = 65.8P_E/(3L) = 65.8P_E/9/(3L) = 2.437P_{cr} = \beta\, P_{cr}/L$$

This equivalent β value of 2.437 is slightly less than that (β = 2.437) obtained for three equal spans rigid body system, which is logical as the elastic strain energy stored in the deformed body shares a portion of the energy provided by the spring system.

COLUMN BUCKLING ANALYSIS BY FINITE DIFFERENCE METHODS

The basic differential equation of beam -column is: $EIy^{iv} + Py'' = q(x)$

This can be written in difference equation form, if we know that

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$\frac{d^3y}{dx^3} = \frac{y_{i+3} - y_{i+1} + y_{i-1} - y_{i-3}}{2h^3} \qquad \frac{d^4y}{dx^4} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4\,y_{i-1} - y_{i-2}}{2h^3}$$

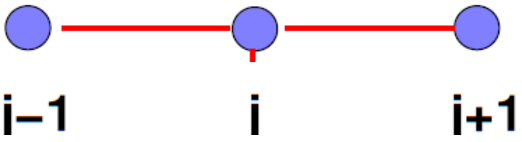


Figure 5: column division for finite difference method

Then: $y_{i+2} + (k_n - 4)y_{i+1} + (6 - 2k_n)y_i + (k_n - 4)y_{i-1} + y_{i-2} = 0$

$$k_n = \frac{pL^2}{n^2 EI} \qquad \& \qquad h = \frac{L}{n}$$

Example: Use the F.D.M. to find P_{cr} for the column shown in the Fig 6.

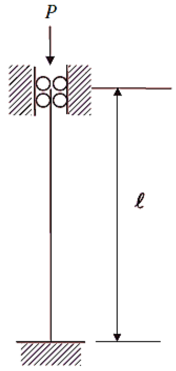


Figure 3:Hinge- fixed ends column

1. Assume *n=2*

$$y_{i+2} + (k_n - 4)y_{i+1} + (6 - 2k_n)y_i + (k_n - 4)y_{i-1} + y_{i-2} = 0$$

@ fixed end : $y_{i-2} = y_i$

@ hinge end : $y_{i+2} = -y_i$

$$y_i + 0 + (6 - 2k_n)y_i + 0 - y_i = 0$$

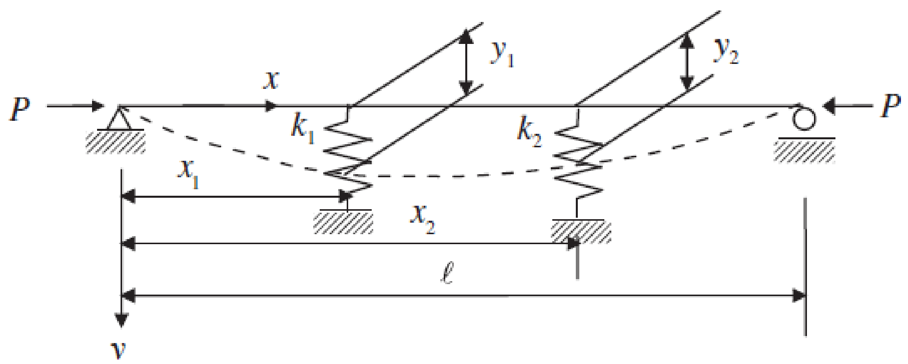
$$(6 - 2k_n)y_i = 0 \rightarrow k_{n=3} \qquad k_n = \frac{PL^2}{n^2 EI} \rightarrow 3 = \frac{P_{cr} \times L^2}{4EI} \rightarrow P_{cr} = \frac{12L^2}{EI}$$

H.W.: Resolve the previous example

1. Using advantage of B.C.

2. assume n=3 & n=4

H.W.: For the beam rest on elastic foundation knowing k is subgrade reaction along the beam (KN/m). calculate P_{cr} use n=2.



COLUMN BUCKLING ANALYSIS BY MATRIX METHODS

Consider a prismatic column shown in Figure 6. The axial strain of a point at a distance *y* from the neutral axis is:

$$\varepsilon_x = \frac{du}{dx} - y\frac{d^2v}{dx^2} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

where u and v are displacement components in the x and y directions, respectively, and
du/dx = axial strain;
$-y(d^2v)/(dx^2)$ = strain produced by curvature; and
$1/2[(dv)/(dx)]^2$ = nonlinear part of the axial strain.
With $dV = dAdx$, the element strain energy is

$$U = \frac{1}{2}\int_V \varepsilon\sigma dV = \frac{1}{2}\int_\ell \int_A E\,\varepsilon_x^2 dAdx$$

where E = modulus of elasticity.

$$\int_A dA = A, \int_A y\,dA = 0, \int_A y^2\,dA = I, \quad \text{and} \quad \int_A E\frac{du}{dx}dA = P$$

where P is the axial force, positive in tension, leads the strain energy to be written:

$$U = \frac{1}{2}\int_0^\ell EA\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_0^\ell EI\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^\ell P\left(\frac{dv}{dx}\right)^2 dx$$

The first integral in Eq. above yields the stiffness matrix for a bar element associated with the kinematic degrees of freedom u1 and u2. The second integral yields the stiffness matrix for a beam element. The third integral sums the work done by the external load *P* when differential elements dx are stretched by an amount $[(dv/dx)^2 \times dx/2]$ (there exists another interpretation of the third integral: a change in the potential energy of the applied load during buckling). The third integral leads to the derivation of the element geometric stiffness matrix *K*. The lateral displacement field *v* of the beam and its derivative *dv/dx* are:

$$v = \lfloor N \rfloor \{\Delta\}$$

$$\frac{dv}{dx} = \frac{d\lfloor N \rfloor}{dx}\{\Delta\} = \lfloor G \rfloor \{\Delta\}$$

where

$$\lfloor \Delta \rfloor = \lfloor v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \rfloor$$

$$\lfloor N \rfloor = \left\lfloor 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3} \quad x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2} \quad \frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3} \quad -\frac{x^2}{\ell} + \frac{x^3}{\ell^2} \right\rfloor$$

$$\lfloor G \rfloor = \left\lfloor -\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \quad 1 - \frac{4x}{\ell} + \frac{3x^2}{\ell^2} \quad \frac{6x}{\ell^2} - \frac{6x^2}{\ell^3} \quad -\frac{2x}{\ell} + \frac{3x^2}{\ell^2} \right\rfloor$$

The third integral is expanded as

$$\frac{1}{2}\lfloor \Delta \rfloor [K_G]\{\Delta\} = \frac{1}{2}\lfloor \Delta \rfloor \left[P \int_0^{\ell} \{G\} \lfloor G \rfloor dx \right] \{\Delta\}$$

Hence,

$$K_{G11} = P \int_0^{\ell} \left(-\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \right)^2 dx = \frac{6P}{5}$$

$$K_{G12} = P \int_0^{\ell} \left(-\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \right) \left(1 - \frac{4x}{\ell} + \frac{3x^2}{\ell^2} \right) dx = \frac{P}{10}$$

Other elements are evaluated likewise.

$$K_G = \frac{P}{30\ell} \begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

Example: Consider a propped (fixed-pinned) column shown in Fig. 7. The prismatic column length is L. Using the numbering scheme, one obtains the following stiffness relationship: As the global coordinate system and the local coordinate system are identical, there is no need for coordinate transformation. Let $\phi = Al^2/I$.

Superimposing element stiffness matrices of bar element and beam element, one obtains an element stiffness matrix for a two-dimensional frame element.

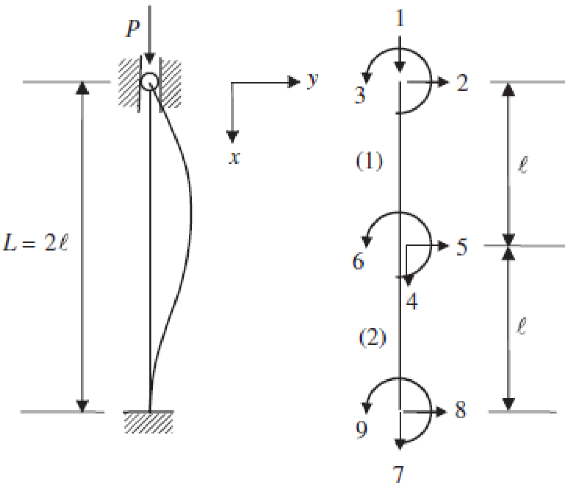


Figure 7: Column model, degrees-of-freedom

$$K_E^{(1)} = \frac{EI}{\ell^3} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} \phi & & & & & \\ 0 & 12 & & & & \\ 0 & 6\ell & 4\ell^2 & & & \\ -\phi & 0 & 0 & \phi & & \\ 0 & -12 & -6\ell & 0 & 12 & \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}$$

$$K_E^{(2)} = \frac{EI}{\ell^3} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} \phi & & & & & \\ 0 & 12 & & & & \\ 0 & 6\ell & 4\ell^2 & & & \\ -\phi & 0 & 0 & \phi & & \\ 0 & -12 & -6\ell & 0 & 12 & \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}$$

$$K_G^{(1)} = -\frac{P}{\ell} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 & & & & & \\ 0 & 6/5 & & & & \\ 0 & \ell/10 & 2\ell^2/15 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -6/5 & -\ell/10 & 0 & 6/5 & \\ 0 & \ell/10 & -\ell^2/30 & 0 & -\ell/10 & 2\ell^2/15 \end{bmatrix}$$

$$K_G^{(2)} = -\frac{P}{\ell} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} 0 & & & & & \\ 0 & 6/5 & & & & \\ 0 & \ell/10 & 2\ell^2/15 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -6/5 & -\ell/10 & 0 & 6/5 & \\ 0 & \ell/10 & -\ell^2/30 & 0 & -\ell/10 & 2\ell^2/15 \end{bmatrix}$$

The elastic stiffness matrices K_E and the stability matrices K_G can now be assembled, reduced, and rearranged, separating the degrees of freedom associated with the axial deformations and the flexural deformations, respectively. Assembling the element stiffness matrices to construct the structural stiffness matrix is of course to combine the element contribution to the global stiffness. Reducing the assembled stiffness matrix is necessary to eliminate the rigid body motion, thereby making the structural stiffness matrix nonsingular.

$$K_G = -\frac{P}{\ell} \begin{array}{c} 1 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\ell^2/15 & -\ell/10 & -\ell^2/30 \\ 0 & 0 & -\ell/10 & 12/5 & 0 \\ 0 & 0 & -\ell^2/30 & 0 & 4\ell^2/15 \end{bmatrix}$$

Noting that K^*_G is equal to K_G for $P = 1$, one can set up the stability determinant $|K_E + \lambda K^*_G| = 0$. This leads to

$$\begin{vmatrix} \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 4\ell^2 - \frac{2}{15}\frac{\lambda\ell^2}{EI} & -6\ell + \frac{1}{10}\frac{\lambda\ell^3}{EI} & 2\ell^2 + \frac{1}{30}\frac{\lambda\ell^4}{EI} \\ 0 & 0 & -6\ell + \frac{1}{10}\frac{\lambda\ell^3}{EI} & 24 - \frac{12}{5}\frac{\lambda\ell^2}{EI} & 0 \\ 0 & 0 & 2\ell^2 + \frac{1}{30}\frac{\lambda\ell^4}{EI} & 0 & 8\ell^2 - \frac{4}{15}\frac{\lambda\ell^4}{EI} \end{vmatrix} = 0$$

Let $\mu = \lambda\ell^2/EI$

$$\begin{vmatrix} \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 2\left(2 - \frac{\mu}{15}\right) & -6 + \frac{\mu}{10} & 2 + \frac{\mu}{30} \\ 0 & 0 & -6 + \frac{\mu}{10} & 12\left(2 - \frac{\mu}{5}\right) & 0 \\ 0 & 0 & 2 + \frac{\mu}{30} & 0 & 4\left(2 - \frac{\mu}{15}\right) \end{vmatrix} = 0$$

Expanding this determinant, one obtains a cubic equation in μ

$$3\mu^3 - 220\mu^2 + 3,840\mu - 14,400 = 0$$

The lowest root of this equation is $\mu = 5.1772 \Rightarrow 5.1772 = \lambda\ell^2/EI$

Hence,

$$P_{cr} = \frac{5.1772EI}{\ell^2} = \frac{5.1772EI}{(0.5L)^2} = \frac{20.7088EI}{L^2} = \frac{2.098\pi^2 EI}{L^2}$$

$$= 1.026P_{exact} = 1.026\left(\frac{20.19EI}{L^2}\right)$$

Considering the fact that only two elements were used to model the column, this (2.6% difference) is a fairly good performance.

FREE VIBRATION OF COLUMNS UNDER COMPRESSIVE LOADS

In previous lectures, deflection-amplification-type buckling and bifurcation-type buckling were discussed. In order to reach the solution of the critical load of the column problem, three different approaches were applied. In the deflection amplification-type problem, the concern is: What is the value of the compressive load for which the static deflections of a slightly crooked column become excessive? In the bifurcation-type buckling problem, two general approaches were taken: eigenvalue method and energy method. In the eigenvalue method, the concern is: What is the value of the compressive load for which a perfect column bifurcates into a nontrivial equilibrium configuration? In the energy method, the concern is: What is the value of the compressive load for which the potential energy of the column ceases to be positive definite? The body will return to its un deformed position upon release of the disturbing action if the potential energy is positive and the system is in stable equilibrium. On the other hand, if the potential energy of the system is not positive, the disturbed body will remain at the displaced position or be displaced further upon the release of the disturbing action.
All of these approaches are based on static concepts. The fourth approach is based on the dynamic concept. In this approach the concern is:
What is the value of the compressive load for which the free vibration of the perfect column ceases to occur?
It will be demonstrated that the natural frequency of the column is altered depending on the presence of the axial compressive load on the column. The governing differential equation of a prismatic column is given by

$$EI\frac{\partial^{iv} y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} = -m\frac{\partial^2 y}{\partial t^2}$$

where *m* is the mass per unit length of the column and the right-hand side of Eq. above is the inertia force per unit length of the column. Note that the inertia force always develops in the opposite direction of the positive acceleration. Invoking the method of separation of variables, the deflection as a function of the position coordinate *x* and time *t* is given by

$$y(x,t) = Y(x)T(t)$$

Substituting into governing differential equation gives

$$EIY^{iv}T + PY''T = -mYT''$$

Dividing both sides of Eq. by *YT* yields: $EI\frac{Y^{iv}}{Y} + P\frac{Y''}{Y} = -m\frac{T''}{T}$

The left-hand side of this Eq. is independent of *t*, and the right-hand side of Eq. is independent of *x* and is equal to the expression on the left. Being independent of both x and t, and yet identically equal to each other, each side of Eq. must be a constant. Let this constant be α so that

$$EI\frac{Y^{iv}}{Y} + P\frac{Y''}{Y} = -m\frac{T''}{T} = \alpha$$

This Eq. will be separated into two homogeneous ordinary differential equations as:

$$Y^{iv} + k^2 Y'' - \alpha Y = 0$$

$$T'' + \omega^2 T = 0$$

where

$$k^2 = \frac{P}{EI}$$

$$\omega^2 = \frac{\alpha EI}{m}$$

It is seen that α is a nonzero, positive constant. Following the procedure of the characteristic equation, the general solutions for the two ordinary linear differential equations with constant coefficients, the two Eqs. are obtained. The general solution for these two Eqs. are:

$$Y(x) = A_1 \cos \alpha_1 x + A_2 \sin \alpha_1 x + A_3 \cosh \alpha_2 x + A_4 \sinh \alpha_2 x$$

$$\alpha_1^2, \alpha_2^2 = \frac{k^2 + \sqrt{k^4 + 4\alpha}}{2}, \frac{-k^2 + \sqrt{k^4 + 4\alpha}}{2}$$

$$T(t) = B_1 \cos \omega t + B_2 \sin \omega t$$

For a simply supported column, the boundary conditions to determine the integral constants are

$$Y(0) = 0 \quad Y''(0) = 0$$

$$Y(\ell) = 0 \quad Y''(\ell) = 0$$

The first and second conditions yield

$$A_1 + A_3 = 0$$
$$-\alpha_1^2 A_1 + \alpha_2^2 A_3 = 0$$

By virtue of $Y(x)$, Eqs. above can only be satisfied when: $A_1 = A_2 = 0$

unless $\alpha_1 = \alpha_2 = 0$, which corresponds to the case of $P = 0$, which is a trivial case. The third and fourth conditions give

$$A_2 \sin \alpha_1 \ell + A_4 \sinh \alpha_2 \ell = 0$$
$$-\alpha_1^2 A_2 \sin \alpha_1 \ell + \alpha_2^2 A_4 \sinh \alpha_2 \ell = 0$$

For a nontrivial solution for A_2 and A_4, the coefficient determinant must vanish.

$$\begin{vmatrix} \sin \alpha_1 \ell & \sinh \alpha_2 \ell \\ -\alpha_1^2 \sin \alpha_1 \ell & \alpha_2^2 \sinh \alpha_2 \ell \end{vmatrix} = 0$$

Expanding the determinant gives

$$\left(\alpha_1^2 + \alpha_2^2\right) \sin \alpha_1 \ell \sinh \alpha_2 \ell = 0$$

Except for the case, α=0, (α_2=0), which is a trivial case, this Eq. is satisfied only when:

$$\sin \alpha_1 \ell = 0 \qquad \alpha_1 \ell = n\pi \qquad A_4 = 0$$

$$\alpha = \left(\frac{n\pi}{\ell}\right)^4 \left(1 - \frac{k^2 \ell^2}{n^2 \pi^2}\right) \qquad \omega_n = \sqrt{\frac{EI}{m}} \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\left(1 - \frac{k^2 \ell^2}{n^2 \pi^2}\right)}$$

$$m\omega_n^2 = \frac{n^2 \pi^2}{\ell^2} \left(\frac{n^2 \pi^2}{\ell^2} EI - P\right) (n = 1, 2, \ldots)$$

$$Y_n(x) = A_2 \sin \frac{n\pi x}{\ell}$$

Two initial conditions determine the other integral constants, B_1 and B_2 in $T(t)$. Assume the vibration is initiated by an initial displacement such that:

$$y(x, 0) = w(x) \quad \text{and} \quad \frac{\partial y(x, 0)}{\partial t} = 0$$

Then

$$Y(x)(B_1 \cos \omega t + B_2 \sin \omega t)|_{t=0} = w(x)$$
$$Y(x)(-B_1 \sin \omega t + B_2 \cos \omega t)|_{t=0} = 0$$

from which one obtains the following:

$$B_1 Y(x) = w(x) \quad \text{and} \quad B_2 = 0$$

Hence, the general solution of Eq. *y(x,y)* for the simply supported column is given by an infinite sum of natural vibration modes:

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\ell} \cos \omega_n t$$

Where $C_n = A_2 B_2$. The coefficient C_n can be determined from: $B_1 Y(x) = w(x)$

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\ell} = w(x)$$

Since this Eq. is a Fourier series expansion for the given initial deflection, the coefficient can be readily determined by use of the orthogonally condition:

$$C_n = \frac{2}{\ell} \int_0^{\ell} w(x) \sin \frac{n\pi x}{\ell} dx \quad (n = 1, 2, \ldots)$$

As the initial deflection $\omega(x)$ is assumed to be known, this Eq. can be evaluated. Note that $\int_0^l \sin^2 nx dx = l/2$. The general solution of the free vibration of a simply supported column is:

$$y(x,t) = \frac{2}{\ell} \sum_{n=1}^{\infty} \left[\int_0^{\ell} w(\xi) \sin \frac{n\pi \xi}{\ell} d\xi \right] \sin \frac{n\pi x}{\ell} \cos \omega_n t$$

It is of interest to note in $m\omega^2_n$ Eq. at the frequency of the vibration of the compressed column is reduced due to the presence of the compressive load. Once the load P reaches P_E, the frequency becomes equal to zero and the column vibrates with an infinitely long period.