COLUMN BUCKLING ANALYSIS BY FINITE DIFFERENCE METHODS

The basic differential equation of beam -column is: $EIy^{iv} + Py'' = q(x)$

This can be written in difference equation form, if we know that

$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h}$ $\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

$\frac{d^3y}{dx^3} = \frac{y_{i+3} - y_{i+1} + y_{i-1} - y_{i-3}}{2h^3}$ $\frac{d^4y}{dx^4} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} - y_{i-2}}{2h^3}$

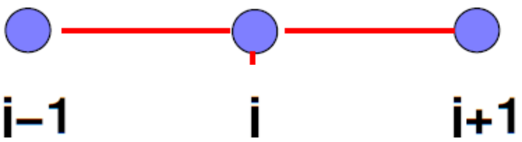


Figure 5: column division for finite difference method

Then: $y_{i+2} + (k_n - 4)y_{i+1} + (6 - 2k_n)y_i + (k_n - 4)y_{i-1} + y_{i-2} = 0$

$$k_n = \frac{pL^2}{n^2EI} \quad \& \quad h = \frac{L}{n}$$

Example: Use the F.D.M. to find P_{cr} for the column shown in the Fig 6.

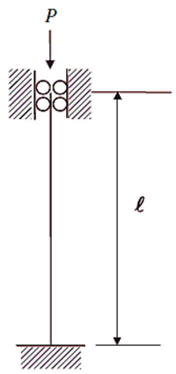


Figure 3:Hinge- fixed ends column

1. Assume $n=2$

$$y_{i+2} + (k_n - 4)y_{i+1} + (6 - 2k_n)y_i + (k_n - 4)y_{i-1} + y_{i-2} = 0$$

@ fixed end : $y_{i-2} = y_i$

@ hinge end : $y_{i+2} = -y_i$

$$y_i + 0 + (6 - 2k_n)y_i + 0 - y_i = 0$$

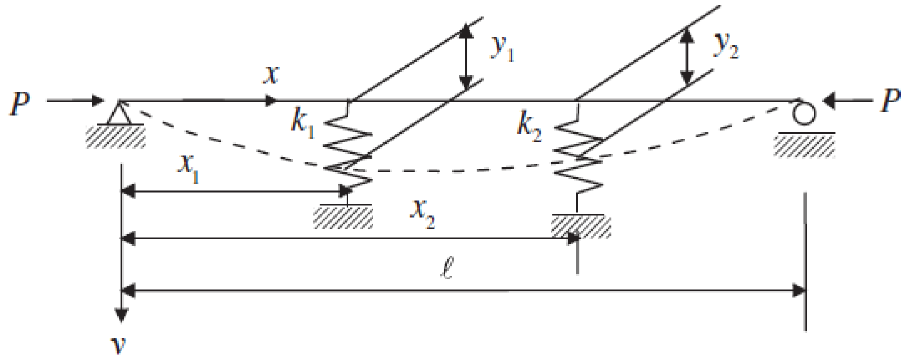
$$(6 - 2k_n)y_i = 0 \rightarrow k_{n=3} \quad k_n = \frac{PL^2}{n^2EI} \rightarrow 3 = \frac{P_{cr} \times L^2}{4EI} \rightarrow P_{cr} = \frac{12L^2}{EI}$$

H.W.: Resolve the previous example

1. Using advantage of B.C.

2. assume n=3 & n=4

H.W.: For the beam rest on elastic foundation knowing k is subgrade reaction along the beam (KN/m). calculate P_{cr} use n=2.



COLUMN BUCKLING ANALYSIS BY MATRIX METHODS

Consider a prismatic column shown in Figure 6. The axial strain of a point at a distance *y* from the neutral axis is:

$$\varepsilon_x = \frac{du}{dx} - y\frac{d^2v}{dx^2} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

where u and v are displacement components in the x and y directions, respectively, and

du/dx = axial strain;

$-y(d^2v)/(dx^2)$ = strain produced by curvature; and

$1/2[(dv)/(dx)]^2$ = nonlinear part of the axial strain.

With $dV = dA dx$, the element strain energy is

$$U = \frac{1}{2}\int_V \varepsilon\sigma dV = \frac{1}{2}\int_\ell \int_A E\, \varepsilon_x^2 dA dx$$

where E = modulus of elasticity.

$$\int_A dA = A, \int_A y\, dA = 0, \int_A y^2\, dA = I, \quad \text{and} \quad \int_A E\frac{du}{dx}dA = P$$

where P is the axial force, positive in tension, leads the strain energy to be written:

$$U = \frac{1}{2}\int_0^\ell EA\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_0^\ell EI\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^\ell P\left(\frac{dv}{dx}\right)^2 dx$$

The first integral in Eq. above yields the stiffness matrix for a bar element associated with the kinematic degrees of freedom u1 and u2. The second integral yields the stiffness matrix for a beam element. The third integral sums the work done by the external load *P* when differential elements dx are stretched by an amount $[(dv/dx)^2 \times dx/2]$ (there exists another interpretation of the third integral: a change in the potential energy of the applied load during buckling). The third integral leads to the derivation of the element geometric stiffness matrix *K*. The lateral displacement field *v* of the beam and its derivative *dv/dx* are:

$$v = \lfloor N \rfloor \{\Delta\}$$

$$\frac{dv}{dx} = \frac{d\lfloor N \rfloor}{dx}\{\Delta\} = \lfloor G \rfloor\{\Delta\}$$

where

$$\lfloor \Delta \rfloor = \lfloor\, v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \,\rfloor$$

$$\lfloor N \rfloor = \left\lfloor 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3} \quad x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2} \quad \frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3} \quad -\frac{x^2}{\ell} + \frac{x^3}{\ell^2} \right\rfloor$$

$$\lfloor G \rfloor = \left\lfloor -\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \quad 1 - \frac{4x}{\ell} + \frac{3x^2}{\ell^2} \quad \frac{6x}{\ell^2} - \frac{6x^2}{\ell^3} \quad -\frac{2x}{\ell} + \frac{3x^2}{\ell^2} \right\rfloor$$

The third integral is expanded as

$$\frac{1}{2}\lfloor \Delta \rfloor [K_G]\{\Delta\} = \frac{1}{2}\lfloor \Delta \rfloor \left[P \int_0^{\ell} \{G\}\lfloor G \rfloor dx \right] \{\Delta\}$$

Hence,

$$K_{G11} = P \int_0^{\ell} \left(-\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \right)^2 dx = \frac{6P}{5}$$

$$K_{G12} = P \int_0^{\ell} \left(-\frac{6x}{\ell^2} + \frac{6x^2}{\ell^3} \right)\left(1 - \frac{4x}{\ell} + \frac{3x^2}{\ell^2} \right) dx = \frac{P}{10}$$

Other elements are evaluated likewise.

$$K_G = \frac{P}{30\ell} \begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

Example: Consider a propped (fixed-pinned) column shown in Fig. 7. The prismatic column length is L. Using the numbering scheme, one obtains the following stiffness relationship: As the global coordinate system and the local coordinate system are identical, there is no need for coordinate transformation. Let $\phi = Al^2/I$.

Superimposing element stiffness matrices of bar element and beam element, one obtains an element stiffness matrix for a two-dimensional frame element.

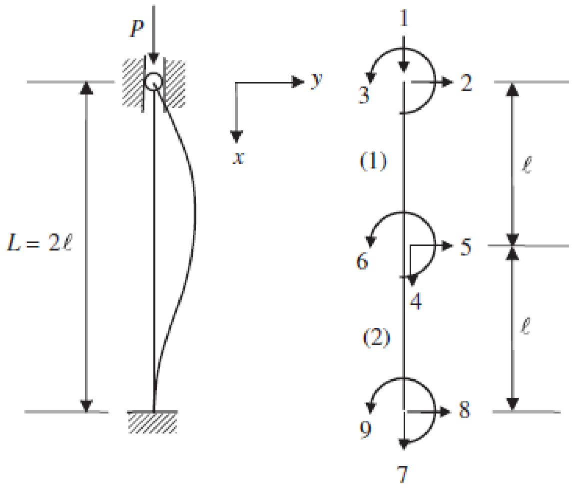


Figure 7: Column model, degrees-of-freedom

$$K_E^{(1)} = \frac{EI}{\ell^3} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} \phi & & & & & \\ 0 & 12 & & & & \\ 0 & 6\ell & 4\ell^2 & & & \\ -\phi & 0 & 0 & \phi & & \\ 0 & -12 & -6\ell & 0 & 12 & \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}$$

$$K_E^{(2)} = \frac{EI}{\ell^3} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} \phi & & & & & \\ 0 & 12 & & & & \\ 0 & 6\ell & 4\ell^2 & & & \\ -\phi & 0 & 0 & \phi & & \\ 0 & -12 & -6\ell & 0 & 12 & \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}$$

$$K_G^{(1)} = -\frac{P}{\ell} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{bmatrix} 0 & & & & & \\ 0 & 6/5 & & & & \\ 0 & \ell/10 & 2\ell^2/15 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -6/5 & -\ell/10 & 0 & 6/5 & \\ 0 & \ell/10 & -\ell^2/30 & 0 & -\ell/10 & 2\ell^2/15 \end{bmatrix}$$

$$K_G^{(2)} = -\frac{P}{\ell} \begin{matrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \begin{bmatrix} 0 & & & & & \\ 0 & 6/5 & & & & \\ 0 & \ell/10 & 2\ell^2/15 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -6/5 & -\ell/10 & 0 & 6/5 & \\ 0 & \ell/10 & -\ell^2/30 & 0 & -\ell/10 & 2\ell^2/15 \end{bmatrix}$$

The elastic stiffness matrices K_E and the stability matrices K_G can now be assembled, reduced, and rearranged, separating the degrees of freedom associated with the axial deformations and the flexural deformations, respectively. Assembling the element stiffness matrices to construct the structural stiffness matrix is of course to combine the element contribution to the global stiffness. Reducing the assembled stiffness matrix is necessary to eliminate the rigid body motion, thereby making the structural stiffness matrix nonsingular.

$$K_G = -\frac{P}{\ell} \begin{matrix} 1 \\ 4 \\ 3 \\ 5 \\ 6 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\ell^2/15 & -\ell/10 & -\ell^2/30 \\ 0 & 0 & -\ell/10 & 12/5 & 0 \\ 0 & 0 & -\ell^2/30 & 0 & 4\ell^2/15 \end{bmatrix}$$

Noting that K^*_G is equal to K_G for $P = 1$, one can set up the stability determinant $|K_E + \lambda K^*_G| = 0$. This leads to

$$\begin{vmatrix} \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 4\ell^2 - \frac{2}{15}\frac{\lambda\ell^2}{EI} & -6\ell + \frac{1}{10}\frac{\lambda\ell^3}{EI} & 2\ell^2 + \frac{1}{30}\frac{\lambda\ell^4}{EI} \\ 0 & 0 & -6\ell + \frac{1}{10}\frac{\lambda\ell^3}{EI} & 24 - \frac{12}{5}\frac{\lambda\ell^2}{EI} & 0 \\ 0 & 0 & 2\ell^2 + \frac{1}{30}\frac{\lambda\ell^4}{EI} & 0 & 8\ell^2 - \frac{4}{15}\frac{\lambda\ell^4}{EI} \end{vmatrix} = 0$$

Let $\mu = \lambda\ell^2/EI$

$$\begin{vmatrix} \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 2\left(2 - \frac{\mu}{15}\right) & -6 + \frac{\mu}{10} & 2 + \frac{\mu}{30} \\ 0 & 0 & -6 + \frac{\mu}{10} & 12\left(2 - \frac{\mu}{5}\right) & 0 \\ 0 & 0 & 2 + \frac{\mu}{30} & 0 & 4\left(2 - \frac{\mu}{15}\right) \end{vmatrix} = 0$$

Expanding this determinant, one obtains a cubic equation in μ

$$3\mu^3 - 220\mu^2 + 3,840\mu - 14,400 = 0$$

The lowest root of this equation is $\mu = 5.1772 \Rightarrow 5.1772 = \lambda\ell^2/EI$

Hence,

$$P_{cr} = \frac{5.1772EI}{\ell^2} = \frac{5.1772EI}{(0.5L)^2} = \frac{20.7088EI}{L^2} = \frac{2.098\pi^2 EI}{L^2}$$

$$= 1.026P_{exact} = 1.026\left(\frac{20.19EI}{L^2}\right)$$

Considering the fact that only two elements were used to model the column, this (2.6% difference) is a fairly good performance.