FREE VIBRATION OF COLUMNS UNDER COMPRESSIVE LOADS

In previous lectures, deflection-amplification-type buckling and bifurcation-type buckling were discussed. In order to reach the solution of the critical load of the column problem, three different approaches were applied. In the deflection amplification-type problem, the concern is: What is the value of the compressive load for which the static deflections of a slightly crooked column become excessive? In the bifurcation-type buckling problem, two general approaches were taken: eigenvalue method and energy method. In the eigenvalue method, the concern is: What is the value of the compressive load for which a perfect column bifurcates into a nontrivial equilibrium configuration? In the energy method, the concern is: What is the value of the compressive load for which the potential energy of the column ceases to be positive definite? The body will return to its un deformed position upon release of the disturbing action if the potential energy is positive and the system is in stable equilibrium. On the other hand, if the potential energy of the system is not positive, the disturbed body will remain at the displaced position or be displaced further upon the release of the disturbing action.
All of these approaches are based on static concepts. The fourth approach is based on the dynamic concept. In this approach the concern is:
What is the value of the compressive load for which the free vibration of the perfect column ceases to occur?
It will be demonstrated that the natural frequency of the column is altered depending on the presence of the axial compressive load on the column. The governing differential equation of a prismatic column is given by

$$EI\frac{\partial^{iv}y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} = -m\frac{\partial^2 y}{\partial t^2}$$

where *m* is the mass per unit length of the column and the right-hand side of Eq. above is the inertia force per unit length of the column. Note that the inertia force always develops in the opposite direction of the positive acceleration. Invoking the method of separation of variables, the deflection as a function of the position coordinate *x* and time *t* is given by

$$y(x,t) = Y(x)T(t)$$

Substituting into governing differential equation gives

Dividing both sides ($EIY^{iv}T + PY''T$ $EI\frac{Y^{iv}}{Y} + P\frac{Y''}{Y} = -m\frac{T''}{T}$

The left-hand side of this Eq. is independent of *t*, and the right-hand side of Eq. is independent of *x* and is equal to the expression on the left. Being independent of both x and t, and yet identically equal to each other, each side of Eq. must be a constant. Let this constant be α so that

$$EI\frac{Y^{iv}}{Y} + P\frac{Y''}{Y} = -m\frac{T''}{T} = \alpha$$

This Eq. will be separated into two homogeneous ordinary differential equations as:

$$Y^{iv} + k^2 Y'' - \alpha Y = 0$$

$$T'' + \omega^2 T = 0$$

where

$$k^2 = \frac{P}{EI}$$

$$\omega^2 = \frac{\alpha EI}{m}$$

It is seen that α is a nonzero, positive constant. Following the procedure of the characteristic equation, the general solutions for the two ordinary linear differential equations with constant coefficients, the two Eqs. are obtained. The general solution for these two Eqs. are:

$$Y(x) = A_1 \cos \alpha_1 x + A_2 \sin \alpha_1 x + A_3 \cosh \alpha_2 x + A_4 \sinh \alpha_2 x$$

$$\alpha_1^2, \alpha_2^2 = \frac{k^2 + \sqrt{k^4 + 4\alpha}}{2}, \frac{-k^2 + \sqrt{k^4 + 4\alpha}}{2}$$

$$T(t) = B_1 \cos \omega t + B_2 \sin \omega t$$

For a simply supported column, the boundary conditions to determine the integral constants are

$$Y(0) = 0 \quad Y''(0) = 0$$

$$Y(\ell) = 0 \quad Y''(\ell) = 0$$

The first and second conditions yield

$$A_1 + A_3 = 0$$

$$-\alpha_1^2 A_1 + \alpha_2^2 A_3 = 0$$

By virtue of *Y(x)*, Eqs. above can only be satisfied when: $A_1 = A_2 = 0$

unless $\alpha_1 = \alpha_2 = 0$, which corresponds to the case of $P = 0$, which is a trivial case. The third and fourth conditions give

$$A_2 \sin \alpha_1 \ell + A_4 \sinh \alpha_2 \ell = 0$$

$$-\alpha_1^2 A_2 \sin \alpha_1 \ell + \alpha_2^2 A_4 \sinh \alpha_2 \ell = 0$$

For a nontrivial solution for A_2 and A_4, the coefficient determinant must vanish.

$$\begin{vmatrix} \sin \alpha_1 \ell & \sinh \alpha_2 \ell \\ -\alpha_1^2 \sin \alpha_1 \ell & \alpha_2^2 \sinh \alpha_2 \ell \end{vmatrix} = 0$$

Expanding the determinant gives

$$\left(\alpha_1^2 + \alpha_2^2\right) \sin \alpha_1 \ell \sinh \alpha_2 \ell = 0$$

Except for the case, α=0, (α_2=0), which is a trivial case, this Eq. is satisfied only when:

$$\sin \alpha_1 \ell = 0 \qquad \alpha_1 \ell = n\pi \qquad A_4 = 0$$

$$\alpha = \left(\frac{n\pi}{\ell}\right)^4 \left(1 - \frac{k^2 \ell^2}{n^2 \pi^2}\right) \qquad \omega_n = \sqrt{\frac{EI}{m}} \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\left(1 - \frac{k^2 \ell^2}{n^2 \pi^2}\right)}$$

$$m\omega_n^2 = \frac{n^2 \pi^2}{\ell^2} \left(\frac{n^2 \pi^2}{\ell^2} EI - P\right) (n = 1, 2, \ldots)$$

$$Y_n(x) = A_2 \sin \frac{n\pi x}{\ell}$$

Two initial conditions determine the other integral constants, B_1 and B_2 in

T(t). Assume the vibration is initiated by an initial displacement such that:

$$y(x,0) = w(x) \quad \text{and} \quad \frac{\partial y(x,0)}{\partial t} = 0$$

Then

$$Y(x)(B_1 \cos \omega t + B_2 \sin \omega t)|_{t=0} = w(x)$$

$$Y(x)(-B_1 \sin \omega t + B_2 \cos \omega t)|_{t=0} = 0$$

from which one obtains the following:

$$B_1 Y(x) = w(x) \quad \text{and} \quad B_2 = 0$$

Hence, the general solution of Eq. *y(x,y)* for the simply supported column is given by an infinite sum of natural vibration modes:

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\ell} \cos \omega_n t$$

Where $C_n = A_2 B_2$. The coefficient C_n can be determined from: $B_1 Y(x) = w(x)$

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\ell} = w(x)$$

Since this Eq. is a Fourier series expansion for the given initial deflection, the coefficient can be readily determined by use of the orthogonally condition:

$$C_n = \frac{2}{\ell} \int_0^{\ell} w(x) \sin \frac{n\pi x}{\ell} dx \quad (n = 1, 2, \ldots)$$

As the initial deflection *ω(x)* is assumed to be known, this Eq. can be evaluated. Note that $\int_0^l \sin^2 nx dx = {}^{l}/_{2}$. The general solution of the free vibration of a simply supported column is:

$$y(x,t) = \frac{2}{\ell} \sum_{n=1}^{\infty} \left[\int_0^{\ell} w(\xi) \sin \frac{n\pi \xi}{\ell} d\xi \right] \sin \frac{n\pi x}{\ell} \cos \omega_n t$$

It is of interest to note in $m\omega^2_n$ Eq. at the frequency of the vibration of the compressed column is reduced due to the presence of the compressive load. Once the load P reaches P_E, the frequency becomes equal to zero and the column vibrates with an infinitely long period.