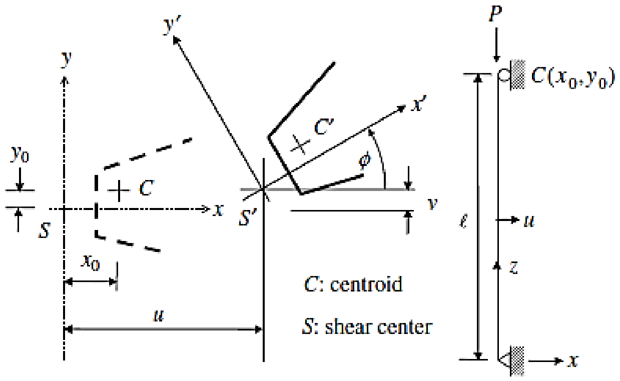
FLEXURAL-TORSIONAL BUCKLING OF COLUMNS

It is assumed that the cross section retains its original shape during buckling. For prismatic members having thin-walled open sections, there are two parallel longitudinal reference axes: One is the centroidal axis, and the other is the shear center axis. The column load P must be placed at the centroid to induce a uniform compressive stress over the entire cross section. Transverse loads for pure bending must be placed along the shear center axis in order to not induce unintended torsional response. Since the cross sectional rotation is measured by the rotation about the shear center axis, the only way not to generate unintended torsional moment by the transverse load is to place the transverse load directly on the shear center axis so as to eliminate the moment arm. It is assumed that the member ends are simply supported for simplicity so that displacements in the x- and y-directions and the moments about these axes vanish at the ends of the member. Hence,

$$\phi = \phi'' = 0$$



Flexural-torsional buckling deformation

In order to consider a meaningful warping restraint, the member ends must be welded (not bolted) thoroughly with thick end plates or embedded into heavy bulkhead with no gap at the ends. These types of torsional boundary conditions are not expected to be encountered in ordinary construction practice. Strain energy stored in the member in the adjacent equilibrium configuration consists of four parts, ignoring the small contribution of the bending shear strain energy and the warping shear strain energy: the energies due to bending in the x- and y-directions; the energy due to St. Venant shear stress; and the energy due to warping torsion. Thus

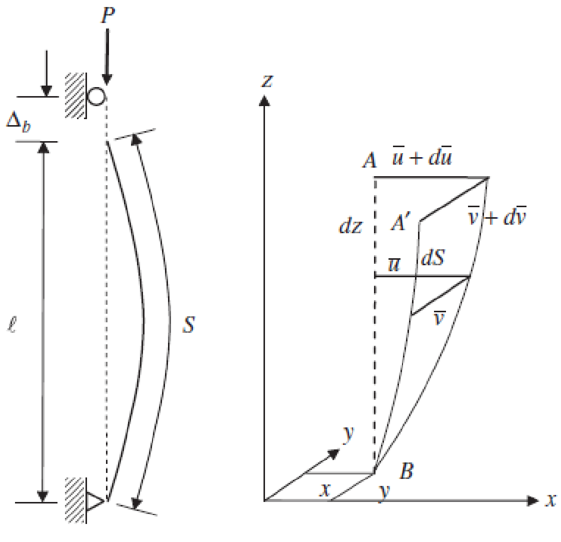
$$U = \frac{1}{2}\int_0^{\ell} EI_y(u'')^2\,dz + \frac{1}{2}\int_0^{\ell} EI_x(v'')^2\,dz + \frac{1}{2}\int_0^{\ell} GK_T(\phi')^2\,dz + \frac{1}{2}\int_0^{\ell} EI_w(\phi'')^2\,dz$$

For an open cross section consisting of a series of rectangular elements, the St. Venant torsional constant is evaluated by:

$$K_T = \frac{1}{3}\sum_{i=1}^{n} b_i t_i^3$$

The loss of potential energy of external loads is equal to the negative of the product of the loads and the distances they travel as the column takes an adjacent equilibrium position. Figure shows a longitudinal fiber whose ends get close to one another by an amount D_b . The distance D_b is equal to the difference between the arc length S and the chord length ‘ of the fiber. Thus

$$V = -\int_A \Delta_b \sigma\, dA$$



Fiber deformations due to buckling

As shown in Figure above when the x and y displacements of the lower end of a differential element *dz* of the column are designated as u and v, then the corresponding displacements at the upper end are *u*+ *du* and *v* + *dv*. From the Pythagorean theorem, the length of the deformed element is

$$dS = \sqrt{(d\bar{u})^2 + (d\bar{v})^2 + (dz)^2} = \sqrt{\left(\frac{d\bar{u}}{dz}\right)^2 + \left(\frac{d\bar{v}}{dz}\right)^2 + 1}\ dz$$

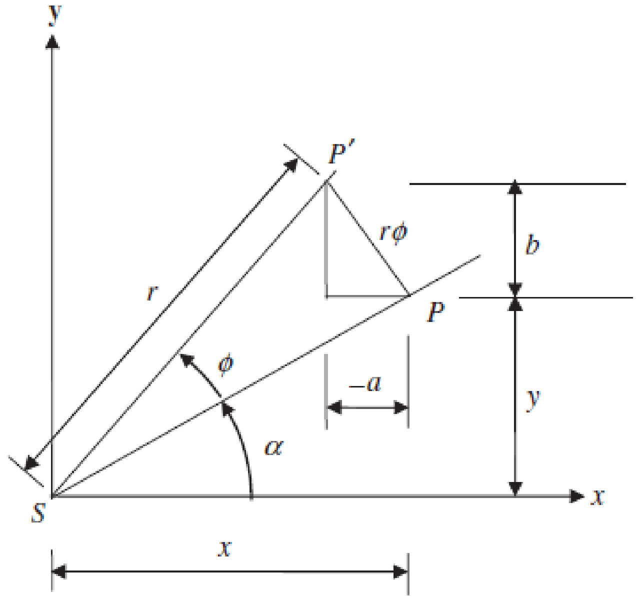
if the magnitude of the derivatives is small compared to unity. Hence,

$$\sqrt{\left(\frac{d\bar{u}}{dz}\right)^2 + \left(\frac{d\bar{v}}{dz}\right)^2 + 1}\ dz \doteq \left[\frac{1}{2}\left(\frac{d\bar{u}}{dz}\right)^2 + \frac{1}{2}\left(\frac{d\bar{v}}{dz}\right)^2 + 1\right] dz$$

$$S = \int_0^\ell \left[\frac{1}{2}\left(\frac{d\bar{u}}{dz}\right)^2 + \frac{1}{2}\left(\frac{d\bar{v}}{dz}\right)^2 + 1\right] dz$$

from which

$$\Delta_b = S - \ell = \frac{1}{2}\int_0^\ell \left[\left(\frac{d\bar{u}}{dz}\right)^2 + \left(\frac{d\bar{v}}{dz}\right)^2\right] dz$$



Lateral translation of longitudinal fiber due to rotation about shear center

where $\bar{u}$ *and* $\bar{v}$ are the translation of the shear center *u and v* plus additional translation due to rotation of the cross section about the shear center. The additional translations $d\bar{u}$ *and* $d\bar{v}$, in the *x*- *and y*-directions, are denoted, as that $PP' = r\phi$, $a = r\phi \sin\alpha$, and $b = r\phi \cos\alpha$.

Since $x = r\cos\alpha$ and $y = r\sin\alpha$, one may also write $-a = -y\phi$ and $b = x\phi$. Hence, the total displacements of the fiber are

$$\bar{u} = u - y\phi$$

$$\bar{v} = v + x\phi$$

$$\Delta_b = \frac{1}{2}\int_0^{\ell}\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + (x^2+y^2)\left(\frac{d\theta}{dz}\right)^2 - 2y\left(\frac{du}{dz}\right)\left(\frac{d\theta}{dz}\right) + 2x\left(\frac{dv}{dz}\right)\left(\frac{d\theta}{dz}\right)\right]dz$$

$$V = -\frac{1}{2}\int_0^{\ell}\int_A \sigma\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + (x^2+y^2)\left(\frac{d\theta}{dz}\right)^2 - 2y\left(\frac{du}{dz}\right)\left(\frac{d\theta}{dz}\right) + 2x\left(\frac{dv}{dz}\right)\left(\frac{d\theta}{dz}\right)\right]dA\,dz$$

In order to simplify Eq above, the following geometric relations can be used:

$$\int_A dA = A,\quad \int_A y\,dA = y_0 A,\quad \int_A x\,dA = x_0 A$$

$$\int_A (x^2+y^2)\,dA = I_x + I_y = r_0^2 A$$

where r_0 is polar radius of gyration of the cross section with respect to the shear center. It should be noted that the shear center is the origin of the coordinate system shown in Figure above. Hence,

$$V = -\frac{P}{2}\int_0^{\ell}\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + r_0^2\left(\frac{d\phi}{dz}\right)^2 - 2y_0\left(\frac{du}{dz}\right)\left(\frac{d\phi}{dz}\right) + 2x_0\left(\frac{dv}{dz}\right)\left(\frac{d\phi}{dz}\right)\right]dz$$

The total potential energy functional **Π** is given by the sum of *U* and *V*

$$= \int_{\ell} F(z, u', v', \phi', u'', v'', \phi'')\,dz$$

Π= U+V

According to the rules of the calculus of variations, P will be stationary (minimum) if the following three Euler-Lagrange differential equations are satisfied:

$$\frac{\partial F}{\partial u} - \frac{d}{dz}\frac{\partial F}{\partial u'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial u''} = 0$$
$$\frac{\partial F}{\partial v} - \frac{d}{dz}\frac{\partial F}{\partial v'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial v''} = 0$$
$$\frac{\partial F}{\partial \phi} - \frac{d}{dz}\frac{\partial F}{\partial \phi'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial \phi''} = 0$$

$\longrightarrow$

$$EI_y u^{iv} + Pu'' - Py_0\phi'' = 0$$
$$EI_x v^{iv} + Pv'' + Px_0\phi'' = 0$$
$$EI_w\phi^{iv} + (r_0^2P - GK_T)\phi'' - y_0Pu'' + x_0Pv'' = 0$$

These three differential equations are the simultaneous differential equations of torsional and flexural-torsional buckling for centrally applied loads only. Their general solution in the most general case can be obtained by means of the characteristic polynomial approach. Assume the be of the form

$$u = A\sin\frac{\pi z}{\ell}, \quad v = B\sin\frac{\pi z}{\ell}, \quad \phi = C\sin\frac{\pi z}{\ell}$$

where A, B, and C are arbitrary constants. Substituting derivatives of these functions into the differential equations and reducing by the common factor *sin(πz/l)*, one obtains

$$(EI_yk^2 - P)A + y_0PC = 0$$
$$(EI_xk^2 - P)B - x_0PC = 0$$
$$y_0PA - x_0PB + (EI_wk^2 + GK_T - r_0^2P)C = 0$$

where $k^2 = \pi^2/\ell^2$.

$$\begin{vmatrix} P_y - P & 0 & y_0P \\ 0 & P_x - P & -x_0P \\ y_0P & -x_0P & r_0^2(P_\phi - P) \end{vmatrix} = 0$$

where

$$P_x = \frac{\pi^2EI_x}{\ell^2}, P_y = \frac{\pi^2EI_y}{\ell^2}, P_\phi = \frac{1}{r_0^2}\left(EI_w\frac{\pi^2}{\ell^2} + GK_T\right)z$$

$$(P_y - P)(P_x - P)(P_\phi - P) - (P_y - P)\frac{P^2x_0^2}{r_0^2} - (P_x - P)\frac{P^2y_0^2}{r_0^2} = 0$$

The solution of the above cubic equation gives the critical load of the column.

Case 1: If the cross section is doubly symmetrical, then $x_0 = y_0 = 0$.

$$(P_y - P)(P_x - P)(P_\phi - P) = 0$$

The three roots and corresponding mode shapes are:

$$P_{cr} = P_y = \frac{\pi^2 EI_y}{\ell^2}: \quad A \neq 0, B = C = 0 \Rightarrow \text{pure flexural buckling}$$

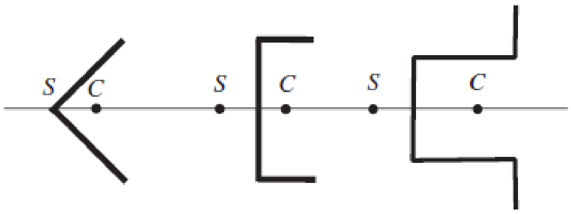
$$P_{cr} = P_x = \frac{\pi^2 EI_x}{\ell^2}: \quad B \neq 0, A = C = 0 \Rightarrow \text{pure flexural buckling}$$

$$P_{cr} = P_\phi = \frac{1}{r_0^2}\left(\frac{\pi^2 EI_w}{\ell^2} + GK_T\right):$$

$$C \neq 0, A = B = 0 \Rightarrow \text{pure torsional buckling}$$

Coupled flexural-torsional buckling does not occur in a column with a cross section where the shear center coincides with the center of gravity. Doubly symmetric sections and the Z purlin section have the shear center and the center of gravity at the same location.

Case 2: If there is only one axis of symmetry as shown in Figure, say the x axis, then shear center lies on the x axis and $y_0 = 0$. Then:



Singly symmetric sections

$$(P_y - P)\left[(P_x - P)(P_\phi - P) - \frac{P^2x_0^2}{r_0^2}\right] = 0$$

This equation is satisfied either if

$$P_{cr} = P_y$$

or if

$$(P_x - P)(P_\phi - P) - \frac{P^2x_0^2}{r_0^2} = 0$$

The first expression corresponds to pure flexural buckling with respect to the y axis. The second is a quadratic equation in P and its solutions correspond to buckling by a combination of flexure and twisting, that is, flexural-torsional buckling. The smaller root of the second equation is

$$P_{F-T} = \frac{1}{2K}\left[P_\phi + P_x - \sqrt{(P_\phi + P_x)^2 - 4KP_xP_\phi}\right]$$

$$K = \left[1 - \left(\frac{x_0}{r_0}\right)^2\right]$$

Case 3: If there is no axis of symmetry, then $x_0 \neq 0$; $y_0 \neq 0$ and Eq. of buckling torsion cannot be simplified. In such cases, bending about either principal axis is coupled with both twisting and bending about the other principal axis. All the three roots to Eq. correspond to torsional-flexural buckling and are lower than all the separable critical loads. Hence, if $P_y < P_x < P\phi$, then

$$P_{cr} < P_y < P_x < P_\phi$$

TORSIONAL AND FLEXURAL-TORSIONAL BUCKLING UNDER THRUST AND END MOMENTS

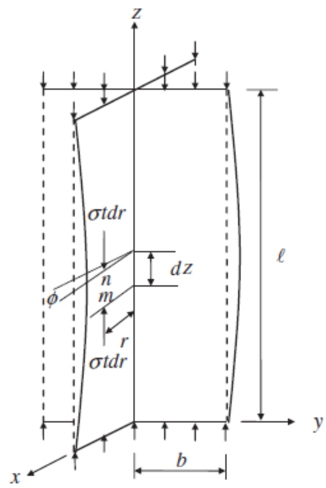
In the previous section, the total potential energy functional of a column for torsional and flexural-torsional buckling expressed with respect to the shear center was derived. When it is desired to express the same in the centroidal coordinate system, it can be done readily, provided that the sign of x_0 and y_0 needs to be reversed as they are defined in two separate coordinate systems (this time they are measured from the centroid). Hence,

$$U = \frac{1}{2}\int_0^\ell EI_y\left(\frac{d^2u}{dz^2}\right)^2 dz + \frac{1}{2}\int_0^\ell EI_x\left(\frac{d^2v}{dz^2}\right)^2 dz + \frac{1}{2}\int_0^\ell GK_T\left(\frac{d\phi}{dz}\right)^2 dz + \frac{1}{2}\int_0^\ell EI_w\left(\frac{d^2\phi}{dz^2}\right)^2 dz$$

and
$$V = -\frac{P}{2}\int_0^\ell \left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2 + r_0^2\left(\frac{d\phi}{dz}\right)^2 + 2y_0\left(\frac{du}{dz}\right)\left(\frac{d\phi}{dz}\right) - 2x_0\left(\frac{dv}{dz}\right)\left(\frac{d\phi}{dz}\right)\right] dz$$

It should be noted that r_0 is the polar radius of gyration of the cross section with respect to the shear center.

PURE TORSIONAL BUCKLING



In order to show how a compressive load may cause purely torsional buckling, consider a column of a cruciform with four identical thin-walled flanges of width b and thickness *t* as shown in Figure. As demonstrated by Case 1 in the previous section, the torsional buckling load will be the lowest for the cruciform column unless the column length is longer than 40 times the flange width where the thickness is 5% of the width. It is imperative to draw a slightly deformed configuration of the column corresponding to the type of buckling to be examined (in this case, torsional buckling). The controidal axis z (which coincides with the shear center axis in this case) does not bend but twists slightly such that *mn* becomes part of a curve with a displacement component of *v* in the *y*-direction. Consider an element *mn* shown in Figure in the form of a strip of length *dz* located at a distance *r* from the z-axis and having a cross sectional area *tdr*. The displacement of this element in the y-direction becomes: $v=r\phi$

The compressive forces acting on the ends of the element mn are $-\sigma tdr$, where $\sigma = P/A$. The statically equivalent fictitious lateral load is then $-(\sigma tdr)(d^2v/dz^2)$ or $-(\sigma trdr)(d^2\phi/dz^2)$. The twisting moment about the z-axis due to this fictitious lateral load acting on the element mn is then $(-\sigma)(d^2\phi/dz^2)(dz)(t)(r^2dr)$. Summing up the twisting moments for the entire cross section yields

$$-\sigma\frac{d^2\phi}{dz^2}dz\int_A tr^2dr = -\sigma\frac{d^2\phi}{dz^2}dz\int_A r^2dA = -\sigma\frac{d^2\phi}{dz^2}dzI_0$$

where I_0 is the polar moment of inertia of the cross section with respect to the shear center S, coinciding in this case with the centroid. Recalling the notation for the distributed torque, one obtains

$$m_z = -\sigma\frac{d^2\phi}{dz^2}I_0$$

For a distributed torque

$$m_z = EI_w\phi^{iv} - GK_T\phi''$$

$$EI_w\phi^{v} - (GK_T - \sigma I_0)\phi'' = 0$$

For column cross sections in which all elements meet at a point such as that shown in Figure above, angles and tees, the warping constant vanishes. Hence, in the case of torsional buckling, Eq. above is satisfied if

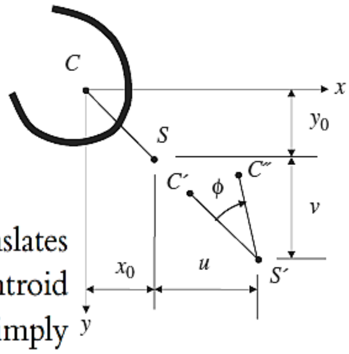
$$GK_T - \sigma I_0 = 0$$

which yields

$$\sigma_{cr} = \frac{GK_T}{I_0} = \frac{(4/3)bt^3 G}{(4/3)tb^3} = \frac{Gt^2}{b^2}$$

Introduce $k^2 = (\sigma I_0 - GK_T)/(EI_w)$ then $\phi^{iv} + k^2\phi'' = 0$, a similar form to a beam-column equation. The general solution of this equation is given by $\phi = a \sin kz + b \cos kz + cz + d$. Applying boundary conditions of a simply supported column gives $\phi = a \sin k\ell = 0$, from which $k\ell = n\pi$. Substituting for k yields

$$\sigma_{cr} = \frac{1}{I_0}\left(GK_T + \frac{n^2\pi^2}{\ell^2}EI_w\right)$$



FLEXURAL-TORSIONAL BUCKLING

Consider an unsymmetrical section shown in Figure. The x and y are principal axes, and x_0 and y_0 are the coordinate of the shear center S measured from the centroid C. During buckling the centroid translates to C' and then rotates to C''. Therefore, the final position of the centroid is $u + y_0\phi$ and $v - x_0\phi$. If only central load P is applied on a simply supported column, the bending moments with respect to the principal axes are

$$M_x = -P(v - x_0\phi) \quad \text{and} \quad M_y = -P(u + y_0\phi)$$

The sign convention for M_x and M_y is such that they are considered positive when they create positive curvature

$$EI_x v'' = +M_x = -P(v - x_0\phi)$$

$$EI_y u'' = +M_y = -P(u + y_0\phi)$$

Consider a small longitudinal strip of cross section tds defined by coordinate x and y as was done in the case of pure torsional buckling. The components of its displacements in the - x and y directions during buckling the products of the compressive force acting on the slightly bend element, σtds and the second derivative of the displacements give a fictitious lateral load in the *x*- and *y*-directions of intensity

$$-(\sigma tds)\frac{d^2}{dz^2}[u+(y_0-y)\phi]$$

$$-(\sigma tds)\frac{d^2}{dz^2}[v-(x_0-x)\phi]$$

These fictitious lateral loads produce twisting moment about the shear center per unit length of the column of intensity

$$dm_z = -(\sigma tds)\frac{d^2}{dz^2}[u+(y_0-y)\phi](y_0-y) + (\sigma tds)\frac{d^2}{dz^2}[v-(x_0-x)\phi](x_0-x)$$

Integrating over the entire cross-sectional area and realizing that

$$\sigma\int_A tds = P, \quad \int_A xtds = \int_A ytds = 0, \quad \int_A y^2 tds = I_x$$

$$\int_A x^2 tds = I_y, \quad I_0 = I_x + I_y + A(x_0^2 + y_0^2)$$

one obtains

$$m_z = \int_A dm_z = P(x_0 v'' - y_0 u'') - r_0^2 P\phi''$$

where $r_0^2 = I_0/A$.

$$EI_w\phi^{iv} - (GK_T - r_0^2)\phi'' - x_0 P v'' + y_0 P u'' = 0$$

TORSIONAL AND FLEXURAL-TORSIONAL BUCKLING UNDER THRUST AND END MOMENTS

Consider the case when the column is subjected to bending moments M_x and M_y applied at the ends in addition to the concentric load P. The bending moments M_x and M_y are taken positive when they produce positive curvatures in the plane of bending. It is assumed that the effect of P on the bending stresses can be neglected and the initial deflection of the column due to the moments is considered to be small. Under this assumption, the normal stress at any point on the cross section of the column is independent of z and is given by

$$\sigma = -\frac{P}{A} - \frac{M_x y}{I_x} - \frac{M_y x}{I_y}$$

As is customarily done in the elastic buckling analysis, any prebuckling deformations are not considered in the adjacent equilibrium condition. Additional deflections u and v of the shear center and rotation ϕ with respect to the shear center axis are produced during buckling, and examination is being conducted on this new slightly deformed configuration. Thus, the components of deflection of any longitudinal fiber of the column are $u + (y_0 - y)\phi$ and $v - (x_0 - x)\phi$. Hence, the fictitious lateral loads and distributed twisting moment resulting from the initial compressive force in the fibers acting on their slightly bent and rotated cross sections are obtained in a manner used earlier.

$$q_x = -\int_A (\sigma t ds)\frac{d^2}{dz^2}[u + (y_0 - y)\phi]$$

$$q_y = -\int_A (\sigma t ds)\frac{d^2}{dz^2}[v - (x_0 - x)\phi]$$

$$m_z = -\int_A (\sigma t ds)\frac{d^2}{dz^2}[u + (y_0 - y)\phi](y_0 - y)$$

$$+\int_A (\sigma t ds)\frac{d^2}{dz^2}[v - (x_0 - x)\phi](x_0 - x)$$

$$q_x = -P\frac{d^2u}{dz^2} - (Py_0 - M_x)\frac{d^2\phi}{dz^2}$$

$$q_y = -P\frac{d^2v}{dz^2} + (Px_0 - M_y)\frac{d^2\phi}{dz^2}$$

$$m_z = -(Py_0 - M_x)\frac{d^2u}{dz^2} + (Px_0 - M_y)\frac{d^2v}{dz^2} - (M_x\beta_x + M_y\beta_y + r_0^2P)\frac{d^2\phi}{dz^2}$$

where the following new cross-sectional properties are introduced:

$$\beta_x = \frac{1}{I_x}\left(\int_A y^3 dA + \int_A x^2 y dA\right) - 2y_0$$

$$\beta_y = \frac{1}{I_y}\left(\int_A x^3 dA + \int_A xy^2 dA\right) - 2x_0$$

The three equations for bending and torsion of the column are

$$EI_y u^{iv} + Pu'' + (Py_0 - M_x)\phi'' = 0$$

$$EI_x v^{iv} + Pv'' - (Px_0 - M_y)\phi'' = 0$$

$$EI_w \phi^{iv} - (GK_T - M_x\beta_x - M_y\beta_y - r_0^2 P)\phi'' + (Py_0 - M_x)u'' - (Px_0 - M_y)v'' = 0$$

T constant coefficients. Hence, the critical values of the external forces can be computed for any combinations of end conditions. If the load *P* is applied eccentrically with the coordinate of the point of application of *P* by e_x and e_y measured from the centroid, the end moments become $M_x = Pe_y$ and $M_y = Pe_x$. Then

$$EI_y u^{iv} + Pu'' + P(y_0 - e_y)\phi'' = 0$$

$$EI_x v^{iv} + Pv'' - P(x_0 - e_x)\phi'' = 0$$

$$EI_w \phi^{iv} - (GK_T - Pe_y\beta_x - Pe_x\beta_y - r_0^2 P)\phi'' + P(y_0 - e_y)u'' - P(x_0 - e_x)v'' = 0$$

If the thrust *P* acts along the shear center axis ($x_0 = e_x$ *and* $y_0 = e_y$), Eqs. above become very simple as they become independent of each other. The first two equations yield the Euler loads, and the third equation gives the critical load corresponding to pure torsional buckling of the column. If the thrust becomes zero, one obtains the case of pure bending of a beam by couples M_x and M_y at the ends. Equations above became:

$$EI_y u^{iv} - M_x\phi'' = 0$$

$$EI_x v^{iv} + M_y\phi'' = 0$$

$$EI_w \phi^{iv} - (GK_T - M_x\beta_x - M_y\beta_y)\phi'' - M_x u'' + M_y v'' = 0$$

Assume the *x*-axis is the strong axis. If $M_y = 0$, then the critical lateral torsional buckling moment can be computed from

$$EI_y u^{iv} - M_x\phi'' = 0$$

$$EI_w \phi^{iv} - (GK_T - M_x\beta_x)\phi'' - M_x u'' = 0$$

If the ends of the beam are simply supported, the displacement functions for u and ϕ can be taken in the form

$$u = A\sin\frac{\pi z}{\ell} \quad \phi = B\sin\frac{\pi z}{\ell}$$

Substituting derivatives of the displacement functions, one obtains the following characteristic polynomial for the critical moment:

$$\frac{\pi^2 EI_y}{\ell^2}\left(GK_T + EI_w\frac{\pi^2}{\ell^2} - M_x\beta_x\right) - M_x^2 = 0$$

Incorporating the following notations

$$P_y = \frac{\pi^2 EI_y}{\ell^2}, \quad P_\phi = \frac{1}{r_0^2}\left(GK_T + EI_w\frac{\pi^2}{\ell^2}\right)$$

$$M_x^2 + P_y\beta_x M_x - r_0^2 P_y P_\phi = 0$$

$$M_{xcr} = -\frac{P_y\beta_x}{2} \pm \sqrt{\left(\frac{P_y\beta_x}{2}\right)^2 + r_0^2 P_y P_\phi}$$

If the beam has two axes of symmetry, β_x vanishes and the critical moment becomes

$$M_{xcr} = \pm\sqrt{r_0^2 P_y P_\phi} = \pm\sqrt{r_0^2\frac{EI_y\pi^2}{\ell^2}\frac{1}{r_0^2}\left(GK_T + EI_w\frac{\pi^2}{\ell^2}\right)}$$

$$= \pm\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + EI_w\frac{\pi^2}{\ell^2}\right)}$$

where ± sign implies that a pair of end moments equal in magnitude but opposite in direction can cause lateral-torsional buckling of a doubly symmetrical beam. In this discussion, considerations have been given for the bending of a beam by couples applied at the ends so that the normal stresses caused by these moments remain constant, thereby maintaining the governing differential equations with constant coefficients. If a beam is subjected to lateral loads, the bending stresses vary with z and the resulting differential equations will have variable coefficients, for which there are no general closed-form solutions available and a variety of numerical integration schemes are used.