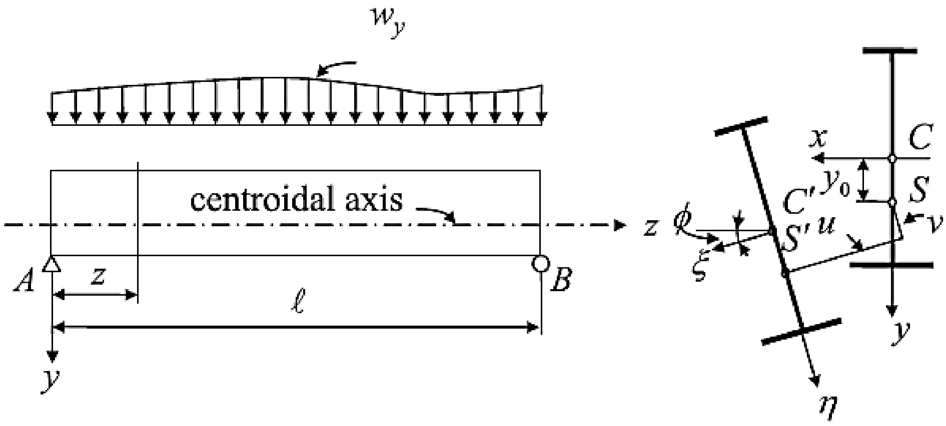
LATERAL-TORSIONAL BUCKLING

INTROUDACTION

A transversely (or combined transversely and axially) loaded member that is bent with respect to its major axis may buckle laterally if its compression flange is not sufficiently supported laterally. The reason buckling occurs in a beam at all is that the compression flange or the extreme edge of the compression side of a narrow rectangular beam, which behaves like a column resting on an elastic foundation, becomes unstable. If the flexural rigidity of the beam with respect to the plane of the bending is many times greater than the rigidity of the lateral bending, the beam may buckle and collapse long before the bending stresses reach the yield point. As long as the applied loads remain below the limit value, the beam remains stable; that is, the beam that is slightly twisted and/or bent laterally returns to its original configuration upon the removal of the disturbing force. With increasing load intensity, the restoring forces become smaller and smaller, until a loading is reached at which, in addition to the plane bending equilibrium configuration, an adjacent, deflected, and twisted, equilibrium position becomes equally possible. The original bending configuration is no longer stable, and the lowest load at which such an alternative equilibrium configuration becomes possible is the critical load of the beam. At the critical load, the compression flange tends to bend laterally, exceeding the restoring force provided by the remaining portion of the cross section to cause the section to twist. Lateral buckling is a misnomer, for no lateral deflection is possible without concurrent twisting of the section.

DIFFERENTIAL EQUATIONS FOR LATERAL-TORSIONAL BUCKLING

If transverse loads do not pass through the shear center, they will induce torsion. In order to avoid this additional torsional moment (thereby weakening the flexural capacity) in the flexural members, it is customary to use flexural members of at least singly symmetric sections so that the transverse loads will pass through the plane of the web as shown in Figure.



The section is symmetric about the *y-axis*, and *u* and *v* are the components of the displacement of the shear center parallel to the axes ξ and *η*. The rotation of the shear center *ϕ* is taken positive about the *z*-axis according to
the right-hand screw rule, and the z-axis is perpendicular to the *ξη* plane.
The following assumptions are employed:

1. The beam is prismatic.
2. The member cross section retains its original shape during buckling.
3. The externally applied loads are conservative.
4. The analysis is limited within the elastic limit.
5. The transverse load passes through the axis of symmetry in the plane of bending.

In the derivation of the governing differential equations of the lateral-torsional buckling of beams, it is necessary to define two coordinate systems:one for the undeformed configuration, *x*, *y*, *z*, and the other for the deformed configuration *ξ, η, ς* as shown in Figure above. Hence, the fixed coordinate axes, *x*, *y*, *z*, constitute a right-hand rectangular coordinate system, while the coordinate axes *ξ, η, ς* make a pointwise rectangular coordinate system as the *ς* axis is tangent to the centroidal axis of the deformed configuration. As the loading will constitute the conservative force system, it will become necessary to relate the applied load in the fixed coordinate system to those in the deformed configuration. This can be readily accomplished by considering the direction cosines of the angles between the axes shown in Figure. These cosines are summarized in Table below. The curvatures of the deflected axis of the beam in the *xz* and *yz* planes can be taken as $d^2u{=}dz^2$ and $d^2v{=}dz^2$, respectively for small deflections. M_x and M_y are assumed positive when they create positive curvatures; $EI_x\ \eta'' = M_x$ and $EI_y\ \xi'' = M_y$.

Since column buckling due to the axial load and the lateral-torsional buckling of beams under the transverse loading are uncoupled in the linear elastic first-order analysis, only the transverse loading will be considered in the derivation of the governing differential equations. Excluding the strain energy of vertical bending prior to buckling, the strain energy in the neighboring equilibrium configuration is

$$U = \frac{1}{2}\int_0^\ell \left[EI_y(u'')^2 + EI_w(\phi'')^2 + GK_T(\phi')^2\right] dz$$

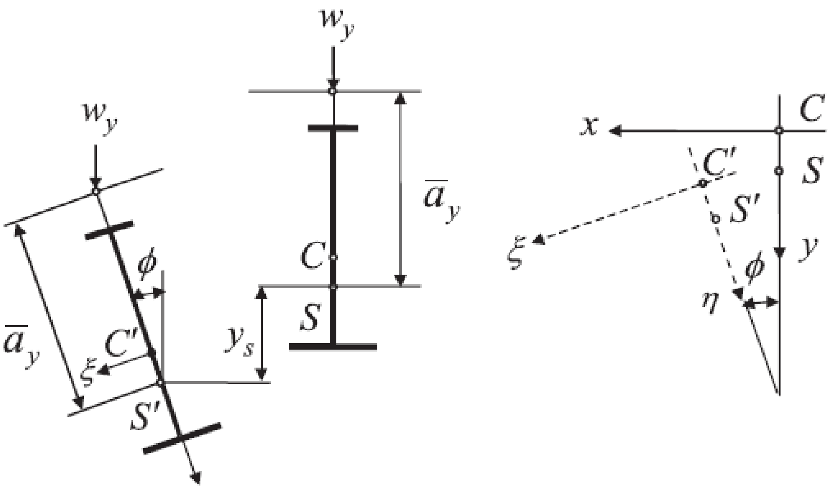
The load w_y, is lowered by a net distance of $y_s + |\bar{a}_y|(1 - \cos\phi)$. Since ϕ is small, $1 - \cos\phi = \phi^2/2$. The vector distance $\bar{a}_y$ is measured from the

	x	y	z
ξ	1	ϕ	$-du/dz$
η	$-\phi$	1	$-dv/dz$
ς	du/dz	dv/dz	1

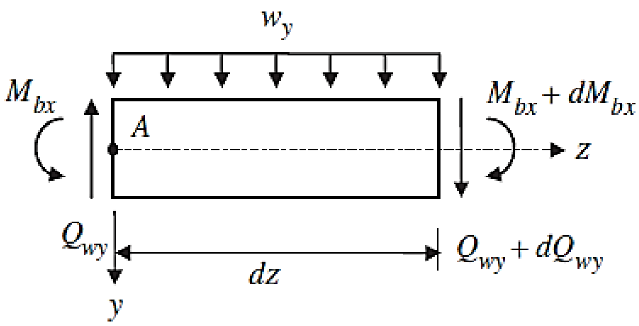
shear center to the transverse load application point. Hence, the loss of the potential energy of the transverse load w_y is

$$V_{wy} = -\int_0^\ell w_y y_s dz + \frac{\bar{a}_y}{2}\int_0^\ell w_y \phi^2 dz$$

It is noted that the sign of y_s is positive and ay is negative as shown in Figure.



It should be noted that the position of the transverse load ay affects the lateral-torsional buckling strength significantly. When the load is applied at the upper flange, it tends to increase the positive rotation of the cross section as shown in Figure, thereby lowering the critical load. This could result in a significantly lower critical value than that when the load is applied at or below the shear center. Although the difference in the critical values is gradually decreasing following the increase of the span length, the position of the transverse load should be properly reflected whenever it is not negligibly small. The first term of Eq. above can be expanded by integration by parts using the relationships that can be derived from Figure.



$$\sum F_y = 0 = -Q_{wy} + Q_{wy} + dQ_{wy} + w_y dz$$

$$\frac{dQ_{wy}}{dz} = -w_y$$

$$\sum M_A = 0 = +M_{bx} - w_y dz\frac{dz}{2} - (Q_{wy} + dQ_{wy})dz - M_{bx} - dM_{bx}$$

$$\frac{dM_{bx}}{dz} = -Q_{wy}$$

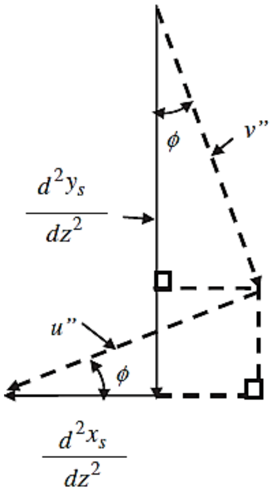
Hence,

$$-\int_0^\ell w_y y_s dz = \int_0^\ell \frac{dQ_{wy}}{dz} y_s dz = [Q_{wy} y_s]_0^\ell - \int_0^\ell Q_{wy}\frac{dy_s}{dz}dz$$

$$= +\left[M_{bx}\frac{dy_s}{dz}\right]_0^\ell - \int_0^\ell M_{bx}\frac{d^2y_s}{dz^2}dz$$

Reflecting any combination of the geometric and natural boundary conditions at the ends of the beam, the two terms in the above equation indicated by slashes must vanish. Therefore,

$$V_{wy} = -\int_0^\ell M_{bx}\frac{d^2y_s}{dz^2}dz + \frac{\bar{a}_y}{2}\int_0^\ell w_y \phi^2 dz$$



The term $d^2y_s=dz^2$ represents the curvature in the *yz* plane; all deformations being small, the curvatures in other planes may be related as a vectorial sum indicated in Figure. $y_s = v\cos\phi + u\sin\phi$

$$\frac{d^2y_s}{dz^2} = v''\cos\phi + u''\sin\phi \cong v'' + \phi u''$$

Therefore, the loss of potential energy is

$$V_{wy} = -\int_0^\ell M_{bx}(v'' + \phi u'')dz + \frac{\bar{a}_y}{2}\int_0^\ell w_y\phi^2 dz$$
$$= -\int_0^\ell M_{bx}v''dz - \int_0^\ell M_{bx}\phi u''dz + \frac{\bar{a}_y}{2}\int_0^\ell w_y\phi^2 dz$$

The above equation is the change of potential energy from unloaded to the buckled state. Just prior to buckling, $\phi = u'' = 0$ and the static potential energy is

$$-\int_0^\ell M_{bx}v''dz$$

Hence, the loss of potential energy due to buckling (in the neighboring equilibrium) is

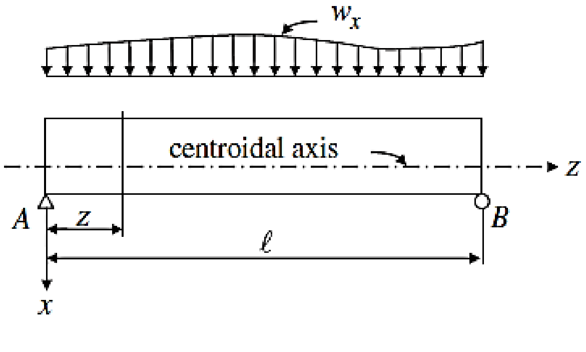
$$V_{wy} = -\int_0^\ell M_{bx}\phi u''dz + \frac{\bar{a}_y}{2}\int_0^\ell w_y\phi^2 dz$$

The total potential energy functional becomes

$$\Pi = U + V$$

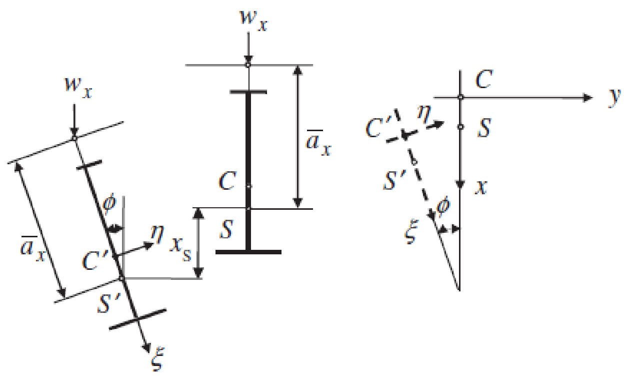
$$= \frac{1}{2}\int_0^\ell \left[EI_y(u'')^2 + EI_w(\phi'')^2 + GK_T(\phi')^2\right]dz$$
$$-\int_0^\ell M_{bx}\phi u'' + \frac{\bar{a}_y}{2}\int_0^\ell w_y\phi^2 dz$$

In the case when the transverse load wx is considered for a similar derivation, the Figure below is used, and a parallel process can be applied. By virtue of assumption 5, the beam cross section must be doubly symmetric in order to accommodate both w_x and w_x simultaneously, and as a consequence, biaxial bending is uncoupled.

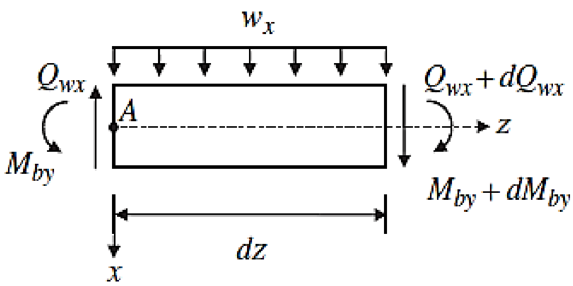


The load w_x is lowered by a distance $x_s + |\bar{a}_x|(1\text{-}cos\phi)$as shown in Figure. Since ϕ is small, $1\text{-}\cos\phi=\phi^2/2$. The vector distance $\bar{a}_x$ is measured from the shear center to the transverse load point. Hence,

$$V_{wx} = -\int_0^\ell w_x x_s dz + \frac{\bar{a}_x}{2}\int_0^\ell w_x \phi^2 dz$$



It is noted that the sign of x_s is positive and $\bar{a}_x$ is negative as is shown in Figure. The first term of Eq. can be expanded by integration by parts using the relationships that can be derived from Figure.



$$\sum F_x = -Q_{wx} + Q_{wx} + dQ_{wx} + w_x dz = 0$$

$$\frac{dQ_{wx}}{dz} = -w_x$$

$$\sum M_A = -M_{by} + w_x dz\frac{dz}{2} + (Q_{wx} + dQ_{wx})dz + M_{by} + dM_{by} = 0$$

$$\frac{dM_{by}}{dz} = -Q_{wy}$$

$$-\int_0^\ell w_x x_s dz = \int_0^\ell \frac{dQ_{wx}}{dz} x_s dz = [Q_{wx} x_s]_0^\ell - \int_0^\ell Q_{wx} \frac{dx_s}{dz} dz$$
$$= +\left[M_{by} \frac{dx_s}{dz}\right]_0^\ell - \int_0^\ell M_{by} \frac{d^2 x_s}{dz^2} dz$$

Reflecting any combination of the geometric and natural boundary conditions at the ends of the beam, the two terms in the above equation indicated by slashes must vanish. Therefore,

$$V_{wx} = -\int_0^\ell M_{by} \frac{d^2 x_s}{dz^2} dz + \frac{\bar{a}_x}{2} \int_0^\ell w_x \phi^2 dz$$

The term $d^2x_s{=}dz^2$ represents the curvature in the *xz* plane; all deformations being small, the curvatures in other planes may be related as a vectorial sum .

$$\frac{d^2 x_s}{dz^2} = u'' \cos\phi - v'' \sin\phi \cong u'' - \phi v''$$

Therefore, the loss of potential energy is

$$V_{wx} = -\int_0^\ell M_{by}(u'' - \phi v'') dz + \frac{\bar{a}_x}{2} \int_0^\ell w_x \phi^2 dz$$
$$= -\int_0^\ell M_{by} u'' dz + \int_0^\ell M_{by} \phi v'' dz + \frac{\bar{a}_x}{2} \int_0^\ell w_x \phi^2 dz$$

The above equation is the change of potential energy from unloaded to the buckled state. Just prior to buckling, $\phi = v'' = 0$, and the static potential energy is

$$-\int_0^\ell M_{by} u''$$

Hence, the loss of potential energy due to buckling (in the neighboring equilibrium) is

$$V_{wx} = \int_0^\ell M_{by} \phi v'' dz + \frac{\bar{a}_x}{2} \int_0^\ell w_x \phi^2 dz$$

For biaxial bending, the total energy functional given by:

$$\begin{aligned}\Pi &= \frac{1}{2}\int_0^{\ell}\Big[EI_y\left(u''\right)^2 + EI_x\left(v''\right)^2 + EI_w\left(\phi''\right)^2 + GK_T\left(\phi'\right)^2 \\ &\quad - \int_0^{\ell} M_{bx}\phi u'' dz + \int_0^{\ell} M_{by}\phi v'' dz + \frac{1}{2}\int_0^{\ell}\left(\bar{a}_x w_x + \bar{a}_y w_y\right)\phi^2\Big] dz \\ &= \int_0^{\ell} F\left(u'', v'', \phi, \phi', \phi''\right) dz\end{aligned}$$

doubly symmetric sections by virtue of assumption 3. Π will be stationary (minimum) if the following Euler-Lagrange equations are satisfied:

$$\frac{\partial F}{\partial u} - \frac{d}{dz}\frac{\partial F}{\partial u'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial u''} = 0 \qquad EI_y u^{iv} - \frac{d^2}{dz^2}\left(M_{bx}\phi\right) = 0$$

$$\frac{\partial F}{\partial v} - \frac{d}{dz}\frac{\partial F}{\partial v'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial v''} = 0 \qquad EI_x v^{iv} + \frac{d^2}{dz^2}\left(M_{by}\phi\right) = 0$$

$$\frac{\partial F}{\partial \phi} - \frac{d}{dz}\frac{\partial F}{\partial \phi'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial \phi''} = 0$$

Noting that

$$\frac{\partial F}{\partial u} = 0, \quad \frac{\partial F}{\partial u'} = 0, \quad \frac{\partial F}{\partial u''} = EI_y u'' - M_{bx}\phi$$

Substituting the followings into third Eq.

$$\frac{\partial F}{\partial \phi} = -M_{bx}u'' + M_{by}v'' + \left(\bar{a}_x w_x + \bar{a}_y w_y\right)\phi$$

$$\frac{\partial F}{\partial \phi'} = GK_T\phi' \quad \frac{\partial F}{\partial \phi''} = EI_w\phi''$$

one obtains

$$EI_w\phi^{iv} - GK_T\phi'' - M_{bx}u'' + M_{by}v'' + \left(\bar{a}_x w_x + \bar{a}_y w_y\right)\phi = 0$$

These Equations are general differential equations describing the lateral-torsional buckling behavior of prismatic straight beams.

GENERALIZATION OF GOVERNING DIFFERENTIAL EQUATIONS

If a wide flange beam is subjected to constant bending moment M_{bx} only, the three general governing differential equations are reduced to:

$$EI_y u^{iv} - \frac{d^2}{dz^2}(M_{bx}\phi) = 0 \qquad \ldots\ldots\ldots(1)$$

$$EI_y u'' - M_{bx}\phi = 0 \qquad \ldots\ldots.(2)$$

$$EI_w \phi''' - GK_T\phi' + M_{bx}u' - M_{bx}'u + \int_0^\ell M_{bx}''u dz = 0$$

Integrating the first equation of Eqs. (1) twice, the second equation once, and applying in the second equation integration by parts ($\int M_{bx}u''dz = M_{bx}u' - \int u'M_{bx}'dz = M_{bx}u' - M_{bx}'u + \int M_{bx}''udz$), one obtains

$$EI_y u'' + M_{bx}\phi = Az + B$$

$$EI_w \phi''' - GK_T\phi' + M_{bx}u' - M_{bx}'u + \int_0^\ell M_{bx}''udz = C \quad \ldots(3)$$

where A, B, and C are arbitrary integral constants. These integral constants, as evident from the statically meaning of the transformation of Eqs. (1) into Eqs. (3), are respectively equal to the variations of the transverse shear force Q_x acting in the initial section $z = 0$ in the direction of the axis x, of the bending moment M_y with respect to the axis y, and of the torsional moment M_z with respect to the axis z. If the variations of the statical factors, Q_x, M_y, and Mz vanish in the initial section $z = 0$, which is the case in a cantilever at the free end, then the integration constants, *A*, *B*, and *C* are equal to zero and Eqs. (3) reduce to Eqs. (2). If the beam has at the ends a rigid or elastic fixing to restrain translation and rotation, the integration constants, *A*, *B*, and *C* will not vanish and the general Eqs. (1) must be used.

LATERAL-TORSIONAL BUCKLING FOR VARIOUS LOADING AND BOUNDARY CONDITIONS

If the external load consists of a couple of end moments so that the moment remains constant along the beam length, then Eqs. (1) become

$$EI_y u^{iv} - M\phi'' = 0$$
$$EI_w \phi^{iv} - GK_T\phi'' - Mu'' = 0 \qquad \text{.................(4)}$$

Equations (4) are a pair of differential equations with constant coefficients. Assume $u = A \sin \pi z/l$ and $\phi = B \sin \pi z/l$. It should be notedthat the assumed displacement functions are indeed the correct eigen functions. Therefore, one expects to have the exact solution. Differentiating the assumed functions, one obtains

$$u' = A\frac{\pi}{\ell}\cos\frac{\pi z}{\ell},\ u'' = -A\left(\frac{\pi}{\ell}\right)^2\sin\frac{\pi z}{\ell},\ u''' = -A\left(\frac{\pi}{\ell}\right)^3\cos\frac{\pi z}{\ell},$$

$$u^{iv} = A\left(\frac{\pi}{\ell}\right)^4\sin\frac{\pi z}{\ell}$$

$$\phi' = B\frac{\pi}{\ell}\cos\frac{\pi z}{\ell},\ \varepsilon'' = -B\left(\frac{\pi}{\ell}\right)^2\sin\frac{\pi z}{\ell},\ \phi''' = -B\left(\frac{\pi}{\ell}\right)^3\cos\frac{\pi z}{\ell},$$

$$\phi^{iv} = B\left(\frac{\pi}{\ell}\right)^4\sin\frac{\pi z}{\ell}$$

Substituting these derivatives into Equations (4) yields:

$$\begin{vmatrix} \left(\frac{\pi}{\ell}\right)^2 EI_y & M \\ -M & \left[\left(\frac{\pi}{\ell}\right)^2 EI_w + GK_T\right] \end{vmatrix} = 0$$

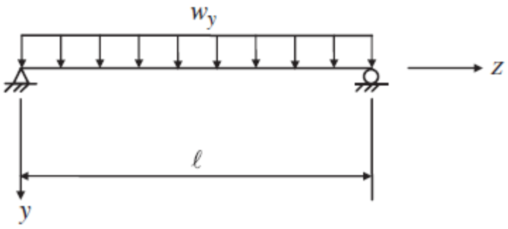
Solving this characteristic equation for the critical moment gives

$$M_{cr} = \frac{\pi}{\ell}\sqrt{EI_y(EI_w\pi^2/\ell^2 + GK_T)} \qquad \text{........(5)}$$

In the case of a uniformly distributed load w_y, the bending moment in a simple beam as shown in Figure below becomes $M_x(z)= w_y z(l-z)/2$. For this load, Eqs. (1) become:

$$EI_y u^{iv} + \frac{w_y}{2}[z(\ell - z)\phi]'' = 0$$

$$EI_w \phi^{iv} - GK_T \phi'' + \frac{w_y}{2} z(\ell - z) u'' = 0 \qquad \text{.........(6)}$$



Equations (6) are coupled differential equations with variable coefficients. integrated Eqs. (6) by the method of infinite series. The critical load $(w_y l)_{cr}$ is given by:

$$(w_y \ell)_{cr} = \frac{\gamma_1 \sqrt{EI_y GK_T}}{\ell^2} \qquad \text{........(7)}$$

The coefficient γ_1 depends on the parameter

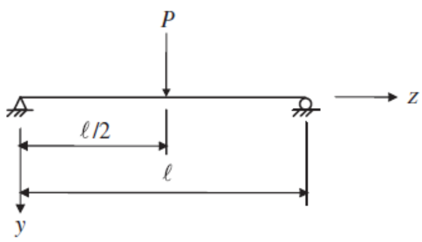
$$m = \frac{GK_T \ell^2}{EI_w} \qquad \text{........(8)}$$

Table 1 gives a series of values of γ_1 for a wide range of combination of the load positions and m for beams with doubly symmetric sections.

Table 1: Values of γ_1 for simply supported I-beam under uniformly distributed load

	m								
Load at	**0.4**	**4**	**8**	**16**	**32**	**64**	**128**	**256**	**512**
TF	92.1	35.9	30.1	27.1	25.9	25.7	26.0	26.4	26.9
SC	144.2	52.9	42.5	36.1	32.5	30.5	29.4	28.9	28.6
BF	226.0	78.2	60.0	48.2	40.8	36.3	33.4	31.6	30.5

Notes: TF = Top flange, SC = Shear center, BF = Bottom flange.



If the beam is loaded by a concentrated load at its mid-span as shown in Figure, the bending moment becomes $M_x(z) = Pz/2$ For this load, Eqs. (1) become

$$EI_y u^{iv} + \frac{P}{2}(z\phi'') = 0$$

$$EI_w \phi^{iv} - GK_T \phi'' + \frac{P}{2} z u'' = 0 \qquad \ldots\ldots\ldots(9)$$

Integrated Eqs. (9) by the method of infinite series. The critical load P_{cr} is given by:

$$P_{cr} = \frac{\gamma_2 \sqrt{EI_y GK_T}}{\ell^2}$$

The stability coefficient γ_2 depends on the parameter m defined by Eq. (8). Table 2 gives a series of values for a wide range of combination of γ_2 and m for beams with doubly symmetric section.

Table 2: Values of γ_2 for simply supported I-beam under concentrated load at the mid-span

Load at	m: 0.4	4	8	16	32	64	128	256	512
TF	50.7	19.9	16.8	15.3	14.7	14.8	15.0	15.4	15.7
SC	86.8	31.9	25.6	21.8	19.5	18.3	17.7	17.3	17.1
BF	148.8	50.9	38.7	30.8	25.8	22.7	20.7	19.4	18.6

Notes: TF = Top flange, SC = Shear center, BF = Bottom flange.

If both ends fixed beams are subjected to a uniformly distributed load, the critical loads may be expressed by:

$$(w_y \ell)_{cr} = \frac{\gamma_3 \sqrt{EI_y GK_T}}{\ell^2}$$

The stability coefficient γ_3 depends on the parameter m defined by Eq. (8). Table 3 gives a series of values for a wide range of combinations of γ_3 and m for beams with doubly symmetric sections.

Table 3: Values of γ_3 for both ends fixed I-beam under uniformly distributed load

Load at	m: 0.4	4	8	16	32	64	128	256	512
TF	610.6	206.8	156.7	125.0	107.0	98.9	97.1	98.7	101.6
SC	1316.8	434.1	320.4	244.4	195.4	165.1	146.8	135.8	128.8
BF	2802.0	900.3	647.2	482.0	352.6	272.7	220.0	185.4	162.4

Notes: TF = Top flange, SC = Shear center, BF = Bottom flange.

If beams with simple-fixed end conditions are loaded by a concentrated load, the critical load may be expressed by

$$P_{cr} = \frac{\gamma_4 \sqrt{EI_y GK_T}}{\ell^2}$$

The stability coefficient γ_4 depends on the parameter m defined by Eq. (8). Table 4 gives a series of values for a wide range of combinations of γ_4 and m for beams with doubly symmetric sections.

For beams with simple-fixed end conditions subjected to a uniformly distributed load, the critical load may be expressed by:

$$(w_y \ell)_{cr} = \frac{\gamma_5 \sqrt{EI_y GK_T}}{\ell^2}$$

Table 5: Values of γ_5 for simple-fixed I-beam under a uniformly distributed load

	m								
Load at	**0.4**	**4**	**8**	**16**	**32**	**64**	**128**	**256**	**512**
TF	259.0	92.4	73.0	61.6	56.0	54.2	54.3	55.3	56.5
SC	468.3	160.4	122.3	97.8	82.8	74.0	69.0	66.1	64.3
BF	838.8	275.9	203.0	153.8	121.4	100.6	87.3	78.7	73.0

Notes: TF = Top flange, SC = Shear center, BF = Bottom flange.

If beams with simple-fixed end conditions are loaded by a concentrated load, the critical load may be expressed by:

$$P_{cr} = \frac{\gamma_6 \sqrt{EI_y GK_T}}{\ell^2}$$

The stability coefficient γ_6 depends on the parameter m defined by Eq. (8). Table 6 gives a series of values for a wide range of combinations of γ_6 and m for beams with doubly symmetric sections.

Table 6: Values of γ_6 for simple-fixed I-beam under concentrated load at the mid-span

	m								
Load at	**0.4**	**4**	**8**	**16**	**32**	**64**	**128**	**256**	**512**
TF	129.1	46.1	36.5	30.9	28.2	27.4	27.7	28.5	29.4
SC	257.4	88.0	67.0	53.5	45.1	40.2	37.3	35.6	34.5
BF	499.6	160.6	118.1	89.2	70.0	57.4	49.2	43.8	40.2

Notes: TF = Top flange, SC = Shear center, BF = Bottom flange.

LATERAL-TORSIONAL BUCKLING BY ENERGY METHOD

The determination of the critical lateral-torsional buckling loads by longhand classical methods is very complex and tedious, particularly for non-uniform bending, as this will result in a system of differential equations with variable coefficients. In this section, the Rayleigh-Ritz method will be used to determine approximately the critical lateral-torsional buckling loads. In any energy method, it is required to establish expressions for the strain energy stored in the elastic body and the loss of potential energy of the externally applied loads. It is relatively simple to come up with the expression for the strain energy by:

$$U = (1/2)\int_v \sigma^T \varepsilon dv$$

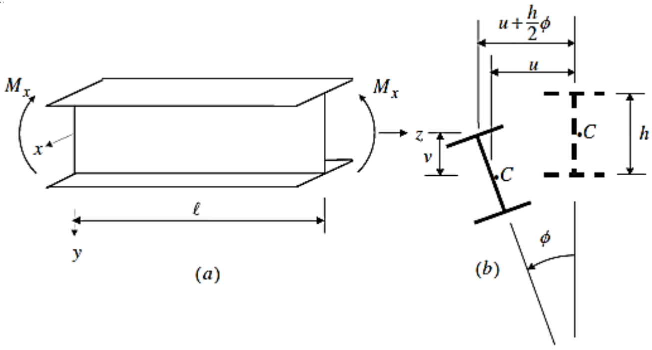
where σ^T = transpose of the stress vector, ε = strain vector, and v = volume of the body. Although the loss of the potential energy of the applied loads is simple in concept as being the negative product of the generalized force and the corresponding deformation during buckling, the expression for the corresponding deformation usually requires considerable geometric analyses.

1. Uniform Bending

Consider a prismatic, simply supported doubly symmetric (for simplicity) I-beam subjected to a uniform bending moment M_x as shown in Figure below. The strain energy stored in the beam during buckling consists of two parts: the energy associated with bending about the y-axis and the ener

$$U = \frac{1}{2}\int_0^\ell EI_y(u'')^2 dz + \frac{1}{2}\int_0^\ell GK_T(\phi')^2 dz + \frac{1}{2}\int_0^\ell EI_w(\phi'')^2 dz$$

For a beam subjected to pure bending, the loss of potential energy V is equal to the negative product of the ap $V = -2M_x\theta$ s and the corresponding angles due to buckling. Hence



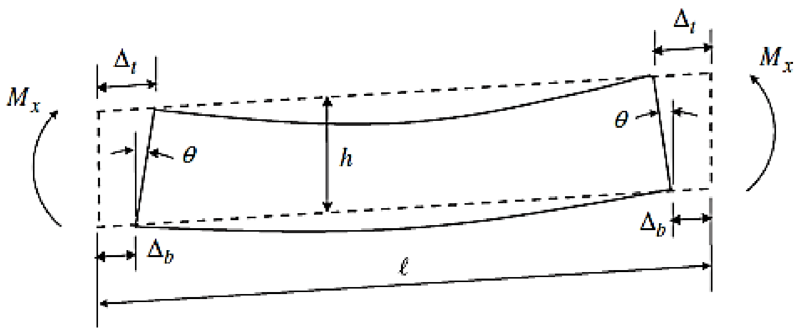
where θ is the angle of rotation about the x-axis at each end of the beam as shown in Figure. By the definition of the simple support, neither twisting of the beam nor lateral deformations of the flanges is allowed at the support. Hence, the top flange deflects more than the bottom flange, as illustrated in Figure below. Thus, the angle θ is:

$$\Delta_t = \frac{1}{4}\int_0^{\ell} (u_t')^2 dz$$

$$\theta = \frac{\Delta_t - \Delta_b}{h}$$

where h is the depth of the cross section.

$$\Delta_b = \frac{1}{4}\int_0^{\ell} (u_b')^2 dz$$



where u_t and u_b are the lateral displacements of the top and bottom of the web, respectively,

$$u_t = u + \frac{h}{2}\phi \qquad u_b = u - \frac{h}{2}\phi$$

$$\Delta_t = \frac{1}{4}\int_0^{\ell}\left(u' + \frac{h}{2}\phi'\right)^2 dz \qquad \Delta_b = \frac{1}{4}\int_0^{\ell}\left(u' - \frac{h}{2}\phi'\right)^2 dz$$

$$\theta = \frac{1}{2}\int_0^{\ell}(u')(\phi')dz \qquad V = -M_x\int_0^{\ell}(u')(\phi')dz$$

$$\Pi = U + V$$

$$= \frac{1}{2}\int_0^{\ell} EI_y(u'')^2 dz + \frac{1}{2}\int_0^{\ell} GK_T(\phi')^2 dz + \frac{1}{2}\int_0^{\ell} EI_w(\phi'')^2 dz$$

$$- M_x\int_0^{\ell}(u')(\phi')dz$$

It is now necessary to assume proper buckled shapes u and ϕ. Sine functions are selected for both u and ϕ for the lowest buckling mode as

$$u = A \sin \frac{\pi z}{\ell} \qquad \phi = B \sin \frac{\pi z}{\ell}$$

Since M_x and M_y are defined to be positive when they produce positive curvature, $M_x = EI_x v''$ and $M_y = EI_y u''$. $M_y = \phi\, M_x$. Thus

$$\phi = \frac{EI_y}{M_x} u'' \qquad A = -B \frac{\ell^2}{\pi^2} \frac{M_x}{EI_y}$$

The assumed function for u can now be written $u = -\frac{B\ell^2}{\pi^2} \frac{M_x}{EI_y} \sin \frac{\pi z}{\ell}$

$$\begin{aligned} \Pi &= U + V \\ &= \frac{1}{2} \frac{B^2 M_x^2}{EI_y} \int_0^\ell \sin^2 \frac{\pi z}{\ell} dz + \frac{1}{2} GK_T B^2 \frac{\pi^2}{\ell^2} \int_0^\ell \cos^2 \frac{\pi z}{\ell} dz \\ &\quad + \frac{1}{2} EI_w B^2 \frac{\pi^4}{\ell^4} \int_0^\ell \sin^2 \frac{\pi z}{\ell} dz - \frac{M_x^2 B^2}{EI_y} \int_0^\ell \cos^2 \frac{\pi z}{\ell} dz \end{aligned}$$

Since

$$\int_0^\ell \sin^2 \frac{\pi z}{\ell} dz = \int_0^\ell \cos^2 \frac{\pi z}{\ell} dz = \frac{\ell}{2}$$

$$\Pi = U + V = \frac{1}{4} \left(\frac{GK_T B^2 \pi^2}{\ell} + \frac{EI_w B^2 \pi^4}{\ell^3} - \frac{M_x^2 B^2 \ell}{EI_y} \right)$$

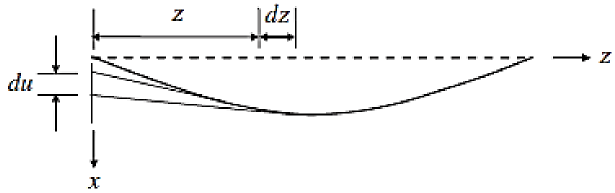
$$\frac{d\Pi}{dB} = \frac{d(U+V)}{dB} = \frac{B}{2} \left(\frac{GK_T \pi^2}{\ell} + \frac{EI_w \pi^4}{\ell^3} - \frac{M_x^2 \ell}{EI_y} \right) = 0$$

$$\frac{GK_T \pi^2}{\ell} + \frac{EI_w \pi^4}{\ell^3} - \frac{M_x^2 \ell}{EI_y} = 0$$

$$M_{x\,cr} = \pm \frac{\pi}{\ell} \sqrt{EI_y (GK_T + \pi^2 EI_w / \ell^2)}$$

2. One Concentrated Load at Mid-span

Consider an element *dz* of the beam at a distance *z* from the left support as shown in Figure. Due to lateral bending, there is a small vertical translation du at the support between the tangents drawn to the elastic curve at the two end points of the element. The value of the translation is, according to the moment-area theorem, given by:



$$du = \frac{M_y}{EI_y} z\,dz$$

For small deformations, the increment in the vertical displacements *dv* corresponding to *du* is:

$$dv = \phi\,du = \frac{M_y}{EI_y}\phi z\,dz$$

Thus the vertical displacement v_0 at the shear center at mid-span is

$$v_0 = \int_0^{\ell/2} dv = \int_0^{\ell/2} \frac{M_y}{EI_y}\phi z\,dz$$

$$M_y = M_x\phi = \frac{P}{2}z\phi \qquad\qquad v_0 = \int_0^{\ell/2} \frac{Pz^2\phi^2}{2EI_y}\,dz$$

$$V = -Pv_0 = -\int_0^{\ell/2} \frac{P^2z^2\phi^2}{2EI_y}\,dz$$

If the load *P* is applied at a distance "*a*" above the shear center, an additional lowering of the load must be considered. If ϕ_0 is the twisting angle of the member at mid-span, the additional lowering of the load is:

$$a(1 - \cos\phi_0) \approx \frac{a\phi_0^2}{2}$$

and an additional loss of the potential energy is

$$\Delta V = -\frac{Pa\phi_0^2}{2}$$

$$\Pi = U + V$$
$$= \frac{1}{2}\int_0^{\ell} GK_T(\phi')^2 dz + \frac{1}{2}\int_0^{\ell} EI_w(\phi'')^2 dz - \frac{P^2}{4EI_y}\int_0^{\ell/2} \phi^2 z^2 dz$$

Assume ϕ to be of the form

$$\phi = B \sin\frac{\pi z}{\ell}$$

$$U + V = -\frac{P^2 B^2}{4EI_y}\int_0^{\ell/2} z^2 \sin^2\frac{\pi z}{\ell} dz + \frac{GK_T B^2 \pi^2}{2\ell^2}\int_0^{\ell} \cos^2\frac{\pi z}{\ell} dz + \frac{EI_w B^2 \pi^4}{2\ell^4}\int_0^{\ell} \sin^2\frac{\pi z}{\ell} dz$$

Substituting the definite integrals

$$\int_0^{\ell/2} z^2 \sin^2\frac{\pi z}{\ell} dz = \frac{\ell^3}{48\pi^2}(\pi^2 + 6)$$

$$\int_0^{\ell} \sin^2\frac{\pi z}{\ell} dz = \int_0^{\ell} \cos^2\frac{\pi z}{\ell} dz = \frac{\ell}{2}$$

$$U + V = -\frac{P^2 B^2 \ell^3}{192 EI_y \pi^2}(\pi^2 + 6) + \frac{GK_T B^2 \pi^2}{4\ell} + \frac{EI_w B^2 \pi^4}{4\ell^3}$$

At the critical load, the first variation of $U + V$ with respect to B must vanish. Thus,

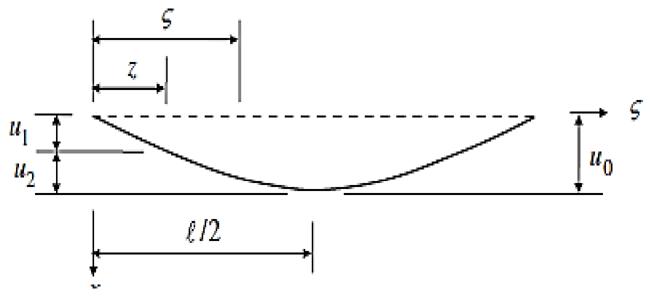
$$\frac{d}{dB}(U + V) = \frac{B}{2}\left[-\frac{P^2 \ell^3}{48 EI_y \pi^2}(\pi^2 + 6) + \frac{GK_T \pi^2}{\ell} + \frac{EI_w \pi^4}{\ell^3}\right] = 0$$

which leads to

$$P_{cr} = \pm\frac{4\pi^2}{\ell^2}\sqrt{\frac{3}{\pi^2 + 6} EI_y \left(GK_T + \frac{EI_w \pi^2}{\ell^2}\right)}$$

3. Uniformly Distributed Load

The procedure described above for the case of a concentrated load at Mid-span can also be used when the I-beam carries a uniformly distributed load. However, the expression for the loss of potential energy of the externally applied load must be determined.



Assume ϕ to be of the form

$$\phi = B\sin\frac{\pi z}{\ell} \qquad v_0 = \int_{\ell/2}^{0}\frac{M_y}{EI_y}\varsigma\,\phi\,d\varsigma$$

$$v_0 = u_0\phi,\ v_1 = u_1\phi,\ \text{and } v_2 = u_2\phi$$

where u_0 and u_1 are the lateral displacements of the beam at mid-span and at a distance z from the support, respectively, and u_2 is equal to u_0 subtracted by u_1 as shown in Figure.

Substituting the expression for the moment M_y and the rotation ϕ, the vertical displacement of the beam at midspan takes the following form:

$$v_0 = \frac{-1}{EI_y}\int_{\ell/2}^{0}\frac{w_y}{2}(\ell\varsigma - \varsigma^2)\varsigma\left(B\sin\frac{\pi\varsigma}{\ell}\right)^2 d\varsigma = \frac{w_y B^2}{2EI_y}\int_{0}^{\ell/2}(\ell\varsigma^2 - \varsigma^3)\sin^2\frac{\pi\varsigma}{\ell}d\varsigma$$

$$v_0 = \frac{w_y B^2\ell^4}{768\pi^4 EI_y}(5\pi^4 + 12\pi^2 + 144)$$

$$v_2 = \int_{\ell/2}^{z}\frac{M_y}{EI_y}(\varsigma - z)\phi\,d\varsigma = \frac{w_y B^2}{2EI_y}\int_{\ell/2}^{z}(\varsigma^2 - \ell\varsigma)(\varsigma - z)(\sin\frac{\pi\varsigma}{\ell})^2 d\varsigma$$

$$v_2 = \frac{w_y B^2}{768\pi^4 EI_y}$$

$$\times\left[\begin{aligned} &5\pi^4\ell^4 - 48\pi^2\ell^3 z - 16\ell^3\pi^4 z + 12\ell^4\pi^2 - 96\ell^2\pi^2 z^2\cos^2\frac{\pi z}{\ell} \\ &+ 48\ell^2\pi^2 z^2 + 144\ell^4\cos^2\frac{\pi z}{\ell} - 96\ell^4\pi\cos\frac{\pi z}{\ell}\sin\frac{\pi z}{\ell} \\ &+ 96\ell^3\pi^2 z\cos^2\frac{\pi z}{\ell} - 16\pi^4 z^4 + 32\ell\pi^4 z^3 + 192\pi\ell^3 z\cos\frac{\pi z}{\ell}\sin\frac{\pi z}{\ell}\end{aligned}\right]$$

$$V = -2w_y \int_0^{\ell/2} (v_0 - v_2)dz = -\frac{w_y^2 B^2 \ell^5}{240\pi^4 EI_y}(\pi^4 + 45)$$

$$\frac{EI_y}{2}\int_0^{\ell} (u'')^2 dz = \frac{EI_y}{2}\int_0^{\ell} (-\frac{M_y}{EI_y})^2 dz = \frac{1}{2EI_y}\int_0^{\ell} (-M_x\phi)^2 dz$$

$$= \frac{w_y^2 B^2}{8EI_y}\int_0^{\ell} (\ell z - z^2)^2 \sin^2\frac{\pi z}{\ell} dz$$

$$= \frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45)$$

$$U = \frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45) + \frac{GK_T B^2 \pi^2}{4\ell} + \frac{EI_w B^2 \pi^4}{4\ell^3}$$

and

$$U + V = -\frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45) + \frac{GK_T B^2 \pi^2}{4\ell} + \frac{EI_w B^2 \pi^4}{4\ell^3}$$

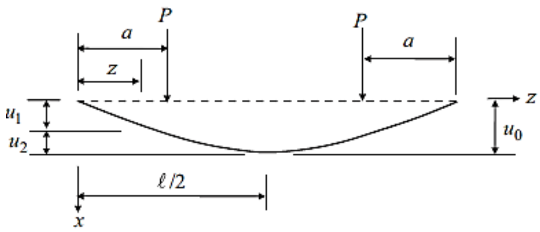
At the critical load, the first variation of $U + V$ with respect to B must vanish. Thus,

$$\frac{d}{dB}(U + V) = \frac{B}{2}\left[-\frac{w_y^2 \ell^5}{120EI_y\pi^4}(\pi^4 + 45) + \frac{GK_T\pi^2}{\ell} + \frac{EI_w\pi^4}{\ell^3}\right] = 0$$

which leads to

$$(w_y\ell)_{cr} = \pm\frac{2\pi^3}{\ell^2}\sqrt{\frac{30}{\pi^4 + 45} EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

H.W. Show that the critical load in case of two concentrated applied



$$P_{cr} = \sqrt{EI_y(GK_T + \pi^2 EI_w/\ell^2)}\Bigg/\sqrt{\ell\left(-\frac{4a^3}{3\pi^2} + \frac{a^2\ell}{\pi^2} - \frac{a\ell^2}{\pi^4}\cos\frac{2\pi a}{\ell} + \frac{\ell^3}{2\pi^5}\sin\frac{2\pi a}{\ell}\right)}$$

DESIGN SIMPLIFICATION FOR LATERAL-TORSIONAL BUCKLING

The preceding sections determined the critical loading for beams with several different boundary conditions and loading configurations. A simply supported wide flange beam subjected to uniform bending has been shown to be in neutral equilibrium (unstable) when the applied moment reaches the value

$$M_{cr} = \frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

The critical concentrated load applied at mid-span of the same beam has been found by the energy method to be

$$P_{cr} = \frac{4\pi^2}{\ell^2}\sqrt{\frac{3}{\pi^2+6}EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

Likewise, the critical uniformly distributed load on the same beam has been found to be

$$(w_y\ell)_{cr} = \frac{2\pi^3}{\ell^2}\sqrt{\frac{30}{\pi^4+45}EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

$$M_{cr} = \frac{P_{cr}\ell}{4} = 1.36\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

and

$$M_{cr} = \frac{(w_y\ell)_{cr}\ell}{8} = 1.13\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

Examination of these equations reveals that it may be possible to express the critical moment in the form

$$M_{cr} = \alpha\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

where the coefficient a is equal to 1.0 for uniform bending, 1.13 for a uniformly distributed load, and 1.36 for a concentrated load at applied at mid-span.

Various lower-bound formulas have been proposed for a, but the most commonly accepted are the following:

$$C_b = 1.75 + 1.05\left(\frac{M_1}{M_2}\right) + 0.3\left(\frac{M_1}{M_2}\right)^2 \leq 2.3$$

Their original equation has been modified slightly to give the following:

$$C_b = \frac{12.5M_{\max}}{2.5M_{\max} + 3M_A + 4M_B + 3M_C} \leq 3.0$$

M_B is the absolute value of the moment at the centerline, M_A and M_C are the absolute values of the quarter point and three quarter-point moments, respectively, and M_{max} is the maximum moment regardless of its location within the brace points. The unbraced length L_p required for compact sections to reach the plastic bending moment M_p is

$$L_p = 1.76 r_y \sqrt{\frac{E}{\sigma_y}}$$

where E = elastic modulus, r_y = radius of gyration with respect to the weak axis, and σ_y = mill specified minimum yield stress.

The limiting value of the unbraced length for girders of compact sections to buckle in the elastic range is given by L_r. In the presence of residual stress, the maximum elastic critical moment is defined by

$$M_{cr} = S_x(\sigma_y - \sigma_r) = 0.7S_x\sigma_y = \sigma_{cr}S_x$$

where S_x = elastic section modulus about the x-axis, σ_r = residual stress 0.3 σ_y for both rolled and welded shapes.

$$\sigma_{cr} = \frac{C_b\pi^2 E}{\left(\frac{L_b}{r_{ts}}\right)^2}\sqrt{\left(\frac{L_b}{r_{ts}}\right)^2 \frac{I_y G K_T}{\pi^2 E \frac{\sqrt{I_y I_w}}{S_x} S_x^2} + 1}$$

$$= \frac{C_b\pi^2 E}{\left(\frac{L_b}{r_{ts}}\right)^2}\sqrt{1 + 0.0779\frac{K_T c}{S_x h_0}\left(\frac{L_b}{r_{ts}}\right)^2}$$

$$r_{ts}^{\ 2} = \frac{\sqrt{I_y E I_w}}{S_x}$$

where $I_w = I_y h_0^2/4$ for doubly symmetric I-beams with rectangular flanges and $c = h_0\sqrt{I_w/I_w}/2$ and hence, $c = 1.0$ for a doubly symmetric I-beam.

$$L_r = 1.95 r_{ts} \frac{E}{0.7\sigma_y}\sqrt{\frac{K_T c}{S_x h_0}}\sqrt{1+\sqrt{1+6.767\left(\frac{0.7\sigma_y}{E}\frac{S_x h_0}{K_T c}\right)^2}}$$

when $L_p < L_b \leq L_r$, the nominal flexural strength M_n of compact sections is linearly interpolated between the plastic moment M_p and the elastic critical moment $M_r = 0{:}7 S_x \sigma_y$ as

$$M_n = C_b\left[M_p - (M_p - 0.7\sigma_y S_x)\left(\frac{L_b - L_p}{L_r - L_p}\right)\right] \leq M_p$$

Lateral-distortional buckling is basically a combined mode of lateral-torsional buckling (global buckling) and local buckling, and the derivation of a closed form solution is, therefore, not straightforward.