LATERAL-TORSIONAL BUCKLING BY ENERGY METHOD

The determination of the critical lateral-torsional buckling loads by longhand classical methods is very complex and tedious, particularly for non-uniform bending, as this will result in a system of differential equations with variable coefficients. In this section, the Rayleigh-Ritz method will be used to determine approximately the critical lateral-torsional buckling loads. In any energy method, it is required to establish expressions for the strain energy stored in the elastic body and the loss of potential energy of the externally applied loads. It is relatively simple to come up with the expression for the strain energy by:

$$U = (1/2) \int_v \sigma^T \varepsilon dv$$

where σ^T = transpose of the stress vector, ε = strain vector, and v = volume of the body. Although the loss of the potential energy of the applied loads is simple in concept as being the negative product of the generalized force and the corresponding deformation during buckling, the expression for the corresponding deformation usually requires considerable geometric analyses.

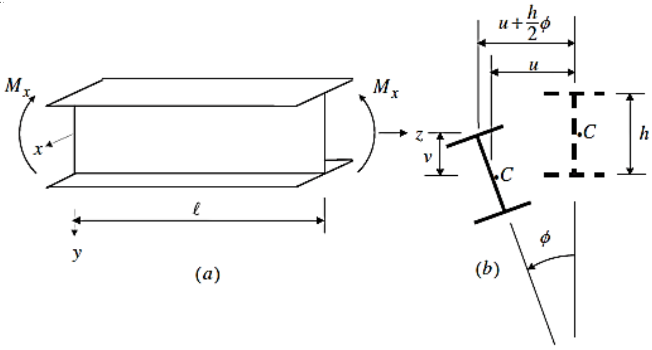
1. Uniform Bending

Consider a prismatic, simply supported doubly symmetric (for simplicity) I-beam subjected to a uniform bending moment M_x as shown in Figure below. The strain energy stored in the beam during buckling consists of two parts: the energy associated with bending about the y-axis and the ener

$$U = \frac{1}{2}\int_0^\ell EI_y(u'')^2 dz + \frac{1}{2}\int_0^\ell GK_T(\phi')^2 dz + \frac{1}{2}\int_0^\ell EI_w(\phi'')^2 dz$$

For a beam subjected to pure bending, the loss of potential energy V is equal to the negative product of the ap s and the corresponding angles due to buckling. Hence

$$V = -2M_x\theta$$



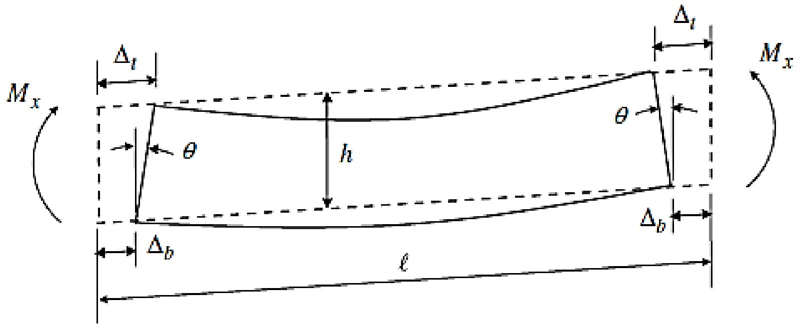
where θ is the angle of rotation about the x-axis at each end of the beam as shown in Figure. By the definition of the simple support, neither twisting of the beam nor lateral deformations of the flanges is allowed at the support. Hence, the top flange deflects more than the bottom flange, as illustrated in Figure below. Thus, the angle θ is:

$$\Delta_t = \frac{1}{4}\int_0^{\ell} (u_t')^2 dz$$

$$\theta = \frac{\Delta_t - \Delta_b}{h}$$

where h is the depth of the cross section.

$$\Delta_b = \frac{1}{4}\int_0^{\ell} (u_b')^2 dz$$



where u_t and u_b are the lateral displacements of the top and bottom of the web, respectively,

$$u_t = u + \frac{h}{2}\phi \qquad u_b = u - \frac{h}{2}\phi$$

$$\Delta_t = \frac{1}{4}\int_0^{\ell} \left(u' + \frac{h}{2}\phi'\right)^2 dz \qquad \Delta_b = \frac{1}{4}\int_0^{\ell} \left(u' - \frac{h}{2}\phi'\right)^2 dz$$

$$\theta = \frac{1}{2}\int_0^{\ell} (u')(\phi')dz \qquad V = -M_x\int_0^{\ell} (u')(\phi')dz$$

$$\Pi = U + V$$

$$= \frac{1}{2}\int_0^{\ell} EI_y(u'')^2dz + \frac{1}{2}\int_0^{\ell} GK_T(\phi')^2dz + \frac{1}{2}\int_0^{\ell} EI_w(\phi'')^2dz$$

$$- M_x\int_0^{\ell} (u')(\phi')dz$$

It is now necessary to assume proper buckled shapes u and ϕ. Sine functions are selected for both u and ϕ for the lowest buckling mode as

$$u = A\sin\frac{\pi z}{\ell} \qquad \phi = B\sin\frac{\pi z}{\ell}$$

Since M_x and M_y are defined to be positive when they produce positive curvature, $M_x = EI_x v''$ and $M_y = EI_y u''$. $M_y = \phi M_x$. Thus

$$\phi = \frac{EI_y}{M_x}u'' \qquad A = -B\frac{\ell^2}{\pi^2}\frac{M_x}{EI_y}$$

The assumed function for u can now be written $u = -\frac{B\ell^2}{\pi^2}\frac{M_x}{EI_y}\sin\frac{\pi z}{\ell}$

$$\begin{aligned}\Pi &= U + V \\ &= \frac{1}{2}\frac{B^2M_x^2}{EI_y}\int_0^\ell \sin^2\frac{\pi z}{\ell}dz + \frac{1}{2}GK_TB^2\frac{\pi^2}{\ell^2}\int_0^\ell \cos^2\frac{\pi z}{\ell}dz \\ &\quad + \frac{1}{2}EI_wB^2\frac{\pi^4}{\ell^4}\int_0^\ell \sin^2\frac{\pi z}{\ell}dz - \frac{M_x^2B^2}{EI_y}\int_0^\ell \cos^2\frac{\pi z}{\ell}dz\end{aligned}$$

Since

$$\int_0^\ell \sin^2\frac{\pi z}{\ell}dz = \int_0^\ell \cos^2\frac{\pi z}{\ell}dz = \frac{\ell}{2}$$

$$\Pi = U + V = \frac{1}{4}\left(\frac{GK_TB^2\pi^2}{\ell} + \frac{EI_wB^2\pi^4}{\ell^3} - \frac{M_x^2B^2\ell}{EI_y}\right)$$

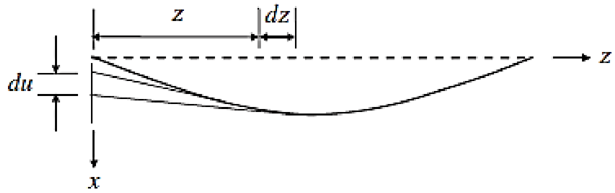
$$\frac{d\Pi}{dB} = \frac{d(U+V)}{dB} = \frac{B}{2}\left(\frac{GK_T\pi^2}{\ell} + \frac{EI_w\pi^4}{\ell^3} - \frac{M_x^2\ell}{EI_y}\right) = 0$$

$$\frac{GK_T\pi^2}{\ell} + \frac{EI_w\pi^4}{\ell^3} - \frac{M_x^2\ell}{EI_y} = 0$$

$$M_{x\,cr} = \pm\frac{\pi}{\ell}\sqrt{EI_y(GK_T + \pi^2EI_w/\ell^2)}$$

2. One Concentrated Load at Mid-span

Consider an element dz of the beam at a distance z from the left support as shown in Figure. Due to lateral bending, there is a small vertical translation du at the support between the tangents drawn to the elastic curve at the two end points of the element. The value of the translation is, according to the moment-area theorem, given by:



$$du = \frac{M_y}{EI_y} z dz$$

For small deformations, the increment in the vertical displacements dv corresponding to du is:

$$dv = \phi du = \frac{M_y}{EI_y} \phi z dz$$

Thus the vertical displacement v_0 at the shear center at mid-span is

$$v_0 = \int_0^{\ell/2} dv = \int_0^{\ell/2} \frac{M_y}{EI_y} \phi z dz$$

$$M_y = M_x \phi = \frac{P}{2} z \phi \qquad v_0 = \int_0^{\ell/2} \frac{P z^2 \phi^2}{2EI_y} dz$$

$$V = -P v_0 = -\int_0^{\ell/2} \frac{P^2 z^2 \phi^2}{2EI_y} dz$$

If the load P is applied at a distance "a" above the shear center, an additional lowering of the load must be considered. If ϕ_0 is the twisting angle of the member at mid-span, the additional lowering of the load is:

$$a(1 - \cos \phi_0) \approx \frac{a\phi_0^2}{2}$$

and an additional loss of the potential energy is

$$\Delta V = -\frac{Pa\phi_0^2}{2}$$

$$\Pi = U + V$$
$$= \frac{1}{2}\int_0^{\ell} GK_T(\phi')^2 dz + \frac{1}{2}\int_0^{\ell} EI_w(\phi'')^2 dz - \frac{P^2}{4EI_y}\int_0^{\ell/2} \phi^2 z^2 dz$$

Assume ϕ to be of the form

$$\phi = B \sin\frac{\pi z}{\ell}$$

$$U + V = -\frac{P^2B^2}{4EI_y}\int_0^{\ell/2} z^2 \sin^2\frac{\pi z}{\ell} dz + \frac{GK_T B^2 \pi^2}{2\ell^2}\int_0^{\ell} \cos^2\frac{\pi z}{\ell} dz + \frac{EI_w B^2 \pi^4}{2\ell^4}\int_0^{\ell} \sin^2\frac{\pi z}{\ell} dz$$

Substituting the definite integrals

$$\int_0^{\ell/2} z^2 \sin^2\frac{\pi z}{\ell} dz = \frac{\ell^3}{48\pi^2}(\pi^2 + 6)$$

$$\int_0^{\ell} \sin^2\frac{\pi z}{\ell} dz = \int_0^{\ell} \cos^2\frac{\pi z}{\ell} dz = \frac{\ell}{2}$$

$$U + V = -\frac{P^2B^2\ell^3}{192EI_y\pi^2}(\pi^2 + 6) + \frac{GK_T B^2\pi^2}{4\ell} + \frac{EI_w B^2\pi^4}{4\ell^3}$$

At the critical load, the first variation of $U + V$ with respect to B must vanish. Thus,

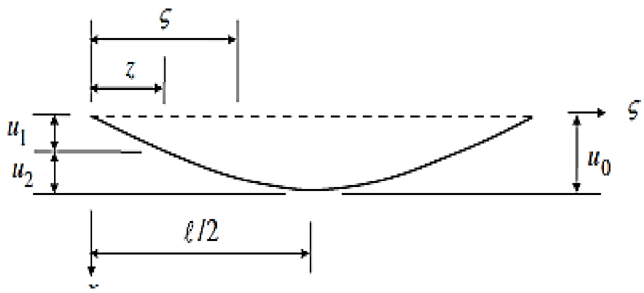
$$\frac{d}{dB}(U + V) = \frac{B}{2}\left[-\frac{P^2\ell^3}{48EI_y\pi^2}(\pi^2 + 6) + \frac{GK_T\pi^2}{\ell} + \frac{EI_w\pi^4}{\ell^3}\right] = 0$$

which leads to

$$P_{cr} = \pm\frac{4\pi^2}{\ell^2}\sqrt{\frac{3}{\pi^2 + 6}EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

3. Uniformly Distributed Load

The procedure described above for the case of a concentrated load at Mid-span can also be used when the I-beam carries a uniformly distributed load. However, the expression for the loss of potential energy of the externally applied load must be determined.



Assume ϕ to be of the form

$$\phi = B \sin \frac{\pi z}{\ell} \qquad v_0 = \int_{\ell/2}^{0} \frac{M_y}{EI_y} \varsigma \, \phi \, d\varsigma$$

$$v_0 = u_0 \phi, \; v_1 = u_1 \phi, \text{ and } v_2 = u_2 \phi$$

where u_0 and u_1 are the lateral displacements of the beam at mid-span and at a distance z from the support, respectively, and u_2 is equal to u_0 subtracted by u_1 as shown in Figure.

Substituting the expression for the moment M_y and the rotation ϕ, the vertical displacement of the beam at midspan takes the following form:

$$v_0 = \frac{-1}{EI_y} \int_{\ell/2}^{0} \frac{w_y}{2}(\ell\varsigma - \varsigma^2)\varsigma \left(B \sin \frac{\pi\varsigma}{\ell} \right)^2 d\varsigma = \frac{w_y B^2}{2EI_y} \int_0^{\ell/2} (\ell\varsigma^2 - \varsigma^3) \sin^2 \frac{\pi\varsigma}{\ell} d\varsigma$$

$$v_0 = \frac{w_y B^2 \ell^4}{768 \pi^4 EI_y}(5\pi^4 + 12\pi^2 + 144)$$

$$v_2 = \int_{\ell/2}^{z} \frac{M_y}{EI_y}(\varsigma - z)\phi d\varsigma = \frac{w_y B^2}{2EI_y} \int_{\ell/2}^{z} (\varsigma^2 - \ell\varsigma)(\varsigma - z)(\sin \frac{\pi\varsigma}{\ell})^2 d\varsigma$$

$$v_2 = \frac{w_y B^2}{768 \pi^4 EI_y}$$

$$\times \left[\begin{aligned} & 5\pi^4\ell^4 - 48\pi^2\ell^3 z - 16\ell^3\pi^4 z + 12\ell^4\pi^2 - 96\ell^2\pi^2 z^2 \cos^2 \frac{\pi z}{\ell} \\ & + 48\ell^2\pi^2 z^2 + 144\ell^4 \cos^2 \frac{\pi z}{\ell} - 96\ell^4\pi \cos \frac{\pi z}{\ell} \sin \frac{\pi z}{\ell} \\ & + 96\ell^3\pi^2 z \cos^2 \frac{\pi z}{\ell} - 16\pi^4 z^4 + 32\ell\pi^4 z^3 + 192\pi\ell^3 z \cos \frac{\pi z}{\ell} \sin \frac{\pi z}{\ell} \end{aligned} \right]$$

$$V = -2w_y \int_0^{\ell/2} (v_0 - v_2)dz = -\frac{w_y^2 B^2 \ell^5}{240\pi^4 EI_y}(\pi^4 + 45)$$

$$\frac{EI_y}{2}\int_0^{\ell} (u'')^2 dz = \frac{EI_y}{2}\int_0^{\ell} (-\frac{M_y}{EI_y})^2 dz = \frac{1}{2EI_y}\int_0^{\ell} (-M_x\phi)^2 dz$$

$$= \frac{w_y^2 B^2}{8EI_y}\int_0^{\ell} (\ell z - z^2)^2 \sin^2 \frac{\pi z}{\ell} dz$$

$$= \frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45)$$

$$U = \frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45) + \frac{GK_T B^2 \pi^2}{4\ell} + \frac{EI_w B^2 \pi^4}{4\ell^3}$$

and

$$U + V = -\frac{w_y^2 B^2 \ell^5}{480\pi^4 EI_y}(\pi^4 + 45) + \frac{GK_T B^2 \pi^2}{4\ell} + \frac{EI_w B^2 \pi^4}{4\ell^3}$$

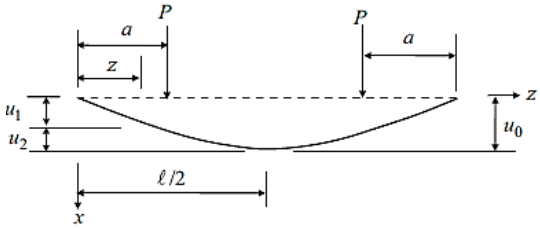
At the critical load, the first variation of $U + V$ with respect to B must vanish. Thus,

$$\frac{d}{dB}(U + V) = \frac{B}{2}\left[-\frac{w_y^2 \ell^5}{120EI_y \pi^4}(\pi^4 + 45) + \frac{GK_T \pi^2}{\ell} + \frac{EI_w \pi^4}{\ell^3}\right] = 0$$

which leads to

$$(w_y \ell)_{cr} = \pm\frac{2\pi^3}{\ell^2}\sqrt{\frac{30}{\pi^4 + 45} EI_y \left(GK_T + \frac{EI_w \pi^2}{\ell^2}\right)}$$

H.W. Show that the critical load in case of two concentrated applied



$$P_{cr} = \sqrt{EI_y(GK_T + \pi^2 EI_w/\ell^2)} \Bigg/ \sqrt{\ell\left(-\frac{4a^3}{3\pi^2} + \frac{a^2\ell}{\pi^2} - \frac{a\ell^2}{\pi^4}\cos\frac{2\pi a}{\ell} + \frac{\ell^3}{2\pi^5}\sin\frac{2\pi a}{\ell}\right)}$$

DESIGN SIMPLIFICATION FOR LATERAL-TORSIONAL BUCKLING

The preceding sections determined the critical loading for beams with several different boundary conditions and loading configurations. A simply supported wide flange beam subjected to uniform bending has been shown to be in neutral equilibrium (unstable) when the applied moment reaches the value

$$M_{cr} = \frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

The critical concentrated load applied at mid-span of the same beam has been found by the energy method to be

$$P_{cr} = \frac{4\pi^2}{\ell^2}\sqrt{\frac{3}{\pi^2+6}EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

Likewise, the critical uniformly distributed load on the same beam has been found to be

$$(w_y\ell)_{cr} = \frac{2\pi^3}{\ell^2}\sqrt{\frac{30}{\pi^4+45}EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

$$M_{cr} = \frac{P_{cr}\ell}{4} = 1.36\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

and

$$M_{cr} = \frac{(w_y\ell)_{cr}\ell}{8} = 1.13\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

Examination of these equations reveals that it may be possible to express the critical moment in the form

$$M_{cr} = \alpha\frac{\pi}{\ell}\sqrt{EI_y\left(GK_T + \frac{EI_w\pi^2}{\ell^2}\right)}$$

where the coefficient a is equal to 1.0 for uniform bending, 1.13 for a uniformly distributed load, and 1.36 for a concentrated load at applied at mid-span.

Various lower-bound formulas have been proposed for a, but the most commonly accepted are the following:

$$C_b = 1.75 + 1.05\left(\frac{M_1}{M_2}\right) + 0.3\left(\frac{M_1}{M_2}\right)^2 \leq 2.3$$

Their original equation has been modified slightly to give the following:

$$C_b = \frac{12.5M_{\max}}{2.5M_{\max} + 3M_A + 4M_B + 3M_C} \leq 3.0$$

M_B is the absolute value of the moment at the centerline, M_A and M_C are the absolute values of the quarter point and three quarter-point moments, respectively, and M_{max} is the maximum moment regardless of its location within the brace points. The unbraced length L_p required for compact sections to reach the plastic bending moment M_p is

$$L_p = 1.76r_y\sqrt{\frac{E}{\sigma_y}}$$

where $E =$ elastic modulus, $r_y =$ radius of gyration with respect to the weak axis, and $\sigma_y =$ mill specified minimum yield stress.

The limiting value of the unbraced length for girders of compact sections to buckle in the elastic range is given by L_r. In the presence of residual stress, the maximum elastic critical moment is defined by

$$M_{cr} = S_x(\sigma_y - \sigma_r) = 0.7S_x\sigma_y = \sigma_{cr}S_x$$

where S_x = elastic section modulus about the x-axis, σ_r = residual stress 0.3 σ_y for both rolled and welded shapes.

$$\sigma_{cr} = \frac{C_b\pi^2E}{\left(\frac{L_b}{r_{ts}}\right)^2}\sqrt{\left(\frac{L_b}{r_{ts}}\right)^2\frac{I_yGK_T}{\pi^2E\frac{\sqrt{I_yI_w}}{S_x}S_x^2} + 1}$$

$$= \frac{C_b\pi^2E}{\left(\frac{L_b}{r_{ts}}\right)^2}\sqrt{1 + 0.0779\frac{K_Tc}{S_xh_0}\left(\frac{L_b}{r_{ts}}\right)^2}$$

$$r_{ts}{}^2 = \frac{\sqrt{I_y E I_w}}{S_x}$$

where $I_w = I_y h_0^2/4$ for doubly symmetric I-beams with rectangular flanges and $c = h_0\sqrt{I_w/I_w}/2$ and hence, $c = 1.0$ for a doubly symmetric I-beam.

$$L_r = 1.95 r_{ts} \frac{E}{0.7\sigma_y}\sqrt{\frac{K_T c}{S_x h_0}}\sqrt{1 + \sqrt{1 + 6.767\left(\frac{0.7\sigma_y}{E}\frac{S_x h_0}{K_T c}\right)^2}}$$

when $L_p < L_b \leq L_r$, the nominal flexural strength M_n of compact sections is linearly interpolated between the plastic moment M_p and the elastic critical moment $M_r = 0{:}7 S_x \sigma_y$ as

$$M_n = C_b\left[M_p - (M_p - 0.7\sigma_y S_x)\left(\frac{L_b - L_p}{L_r - L_p}\right)\right] \leq M_p$$

Lateral-distortional buckling is basically a combined mode of lateral-torsional buckling (global buckling) and local buckling, and the derivation of a closed form solution is, therefore, not straightforward.