CHAPTER 3
DESIGN OF CONCRETE STRUCTURES AND FUNDAMENTAL ASSUMPTIONS

3.1 INTRODUCTION

- Unknowns which determined in design process:
  Design is the determination of the ***general shape*** and ***specific dimensions***.
- Goals that should be performed in the design of structure:
  - The ***function for which it was created,***
  - S***afety to withstand the influences that will act on it*** throughout its useful life.
- Primary influences act on the structure:
  The influences are primarily:
  - The ***loads*** and ***other forces*** to which it will be subjected,
  - Other ***detrimental agents***, such as ***temperature fluctuations*** and ***foundation settlements***.
- How architect and the engineer work together to select the concept and system:
  - In the case of a ***building***, ***an architect may present an overall concept*** and with ***the engineer develop a structural system***.
  - For ***bridges*** and ***industrial facilities***, the ***engineer is often directly involved in selecting both the concept and the structural system***.
- General Sequence Adopted in Design of Concrete Structures
  Regardless of the application, the design of concrete structures follows the same general sequence.
  - First, an ***initial structural system is defined***, the ***initial member sizes are selected***, and ***a mathematical model of the structure is generated***.
  - Second, ***gravity and lateral loads are determined*** based on the selected system, member sizes, and external loads. Building loads typically are defined in (ASCE/SEI 7–10), as discussed in ***Chapter 1***.
  - Third, the ***loads are applied*** to the ***structural model and the load effects calculated for each member***. This step may be done on a ***preliminary basis*** or by ***using computer-modeling software***.
    This step is ***more complex for buildings*** in ***Seismic Design Categories D though F*** where the ***seismic analysis requires close coordination of the structural framing system and the earthquake load***s (discussed in Chapter 20).
  - Fourth, maximum load effects at ***critical member sections are identified*** and ***each critical section is designed for moment***, ***axial load***, ***shear***, and ***torsion*** as needed.
  - Fourth step may ***become iterative***, For example:
    - If the member initially selected is ***too small***, ***its size must be increased***, ***load effects recalculated for the larger member***, and ***the members redesigned***.
    - If ***the initial member is too large***, a ***smaller section is selected***; however, loads are ***may not be recalculated, as gravity effects are most often conservative***.
  - Fifth, each member is ***checked for serviceability***.
  - Sixth, ***the reinforcement for each member is detailed***, that is, the number and size of reinforcing bars are selected for the critical sections to provide the required strength.
  - Seventh, ***connections are designed*** to ensure that the building performs as intended.
  - Finally, the ***design information is incorporated in the construction documents***.

  This process is illustrated in ***Figure 3.1-1***. In addition to ***the design methodology***, ***Figure 3.1-1*** indicates the chapters in the textbook and in the ACI Code, (ACI318M, 2014), where the topics are covered.

- ACI code versus textbooks:
    - The ACI Code ***is written based on the assumption that the user understands concrete structural behavior and the design process, whereas this text builds that understanding***.
    - Textbooks are organized so that ***the fundamental theory is presented first***, followed by ***the Code interpretation of the theory***. Thus, ***the text remains relevant even as Code provisions are updated***.

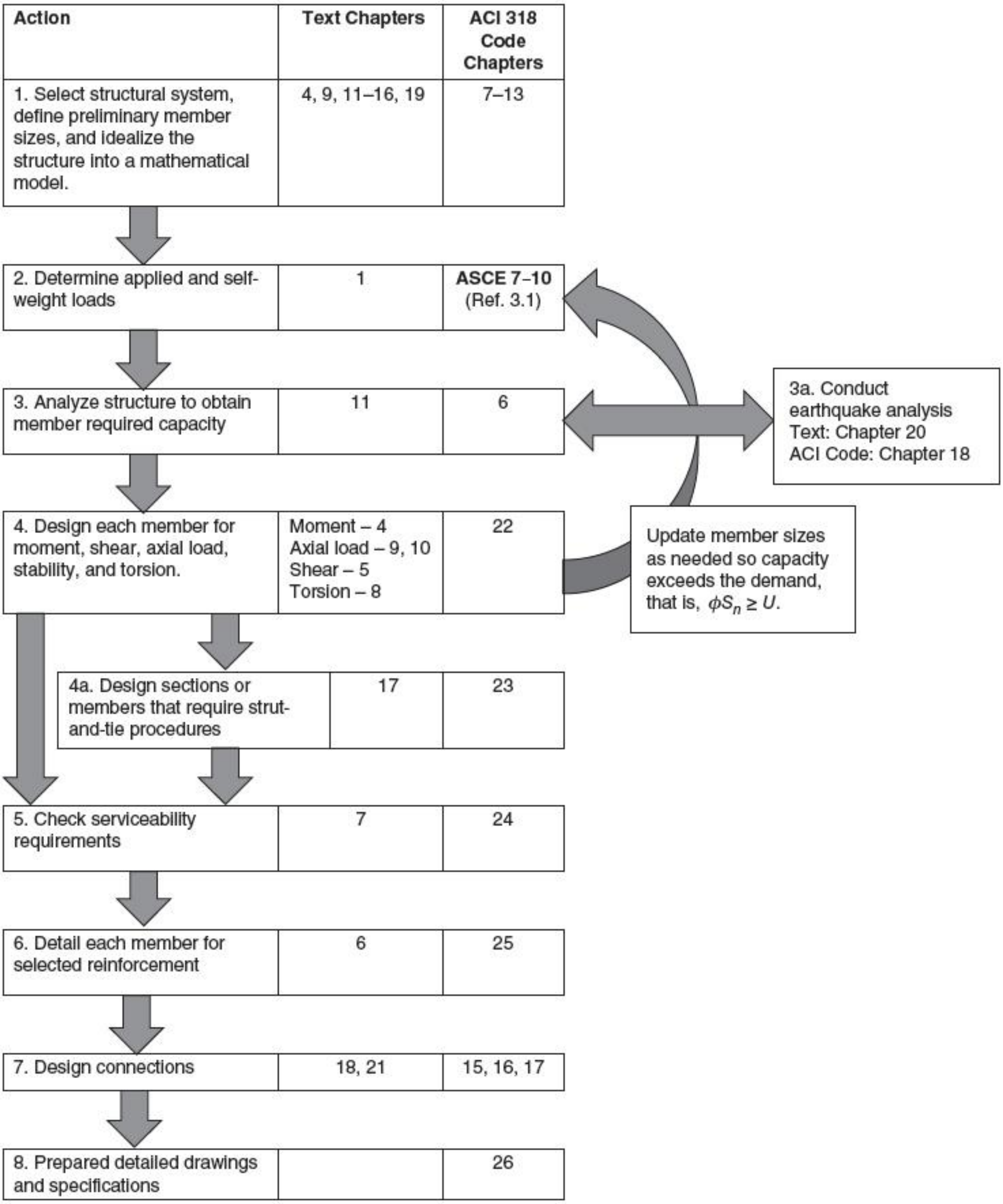

Figure 3.1-1: Design development sequence.

3.2 MEMBERS AND SECTIONS

- The term "***membe***r":
  The term member refers to ***an individual portion of the structure***, such as a ***beam***, ***column***, ***slab***, or ***footing***.
- The term "***section***":
  - Moment, axial load, and shear are distributed along the member, and the member is designed at discrete locations, i.e. discrete sections.
  - The engineer identifies the maximum value of these loads and designs the member at these discrete locations so that the strength at the section exceeds these values. It is not necessary to design every section of a member.
  - For examples, the simply supported and cantilever beams indicated in ***Figure 3.2-1***, have infinite sections but shear forces and bending moments are determined at finite number, discrete number, of sections and then the beams are designed based on critical sections.
    - For the simply supported beam of ***Figure 3.2-1a***, the critical section for flexure is at beam mid-span while the critical section for shear is at support region.
    - For the cantilever beam of ***Figure 3.2-1b***, the critical section for flexure and shear is located at support region.

(a) A simply sumpported beam. (b) A cantilever beam.

Figure 3.2-1: Critical sections for simply supported and cantilever beams.

- Requirements beyond the critical section:
  - The requirement of:

    $$\phi S_n \geq U \qquad \textbf{Eq. 3.2-1}$$

    implies that ***reinforcement for maximum loads can be carried beyond the critical section to ensure that the strength requirements are satisfied for the entire member***.
  - In addition to strength, the reinforcement is designed to provide ***overall structural integrity*** and to ensure that it is anchored to the concrete.

3.3 THEORY, CODES, AND PRACTICE

- The design of concrete structures ***requires an understanding*** the interaction of:
    - Structural theory,
    - The role of building codes,
    - Experience in the practice of structural design itself.
- An example of how practice experience can affect the structural code and theory of structure:
    - A structural failure, a ***practice experience***, may lead to a code revision, i.e. practice experience may alter the design code.
    - The failure may also lead to research, which in turn provides a new theoretical model, i.e. practice experience may alter the structural theory.
    - Changes in practice may also be made to preclude similar failures, even without a code change.
- Insight to the interplay of each of these elements is essential for the engineer to design safe, serviceable, and economical structures.

3.3.1 Theory

- Structural theory includes ***mathematical***, ***physical***, or ***empirical models*** of the behavior of structures.
- For example, in beam theory, equation of ***Eq. 3.3-1*** contains mathematical model of $\kappa \approx y''$, physical models of equilibrium equations and Hook's law, and it may contains an empirical model $E_c \approx 4700\sqrt{f_c'}$.

$$EIy'' = M(x) \quad \textbf{Eq. 3.3-1}$$

- Mathematical and Physical Models:
    - These models have ***evolved over decades*** of ***research*** and ***practice***.
    - They are used to ***predict the nominal strength of members***.
    - The most robust theories derive from statics, equilibrium, and mechanics of materials.
    - Examples include:
        - Equations for the strength of a concrete section for bending, M_n, (Chapter 4),
        - Bending plus axial load, i.e. M_n and P_n, (Chapters 9 and 10).
- Empirical Models:
  Empirical models consists from the following basic steps:
    - Observations:
      In other cases, an ***empirical understanding of structural behavior***, derived from ***experimental observation***, is combined with theory to develop the prediction of member strength.
    - Fitting:
      In this case, ***equations are then fitted to the experimental data to predict the strength***.
    - Adjusting:
      If the experimental strength of a section is highly variable, then the predicted equations are adjusted for use in design to predict a lower bound of the section capacity.

  This approach is used, for example, to calculate the shear strength of a section, V_n, (***Chapter 5***) and anchorage capacity (***Chapter 21***).

3.3.2 Codes

- Building codes provide minimum requirements for the ***life safety*** and ***serviceability*** for structures.
- Code limits the theory:
    - In their simplest application, codes present the theory needed to ensure that ***sectional and member strengths are provided and define the limits on that theory***.
    - For example, ***a structure could be constructed using a large unreinforced concrete beam that relies solely on the tensile strength of the concrete***. Such ***a structure would be brittle***, and an unanticipated load would lead to sudden collapse. Codes ***prohibit such designs***.

- In a similar manner, ***codes prescribe the maximum and minimum amount of reinforcement allowed in a member***.
- Codes also address ***serviceability considerations***, such as ***deflection*** and ***crack control***.

- Codes impose restrictions out the scope of theory:
  Codes may also ***contain restrictions resulting from failures in practice that were not predicted by the theory*** upon which the code is based.
- Code provisions for structural integrity:
  - Codes require reinforcement to ***limit progressive or disproportional collapse***.
  - Disproportional collapse occurs when ***the failure of a single member leads to the failure of multiple adjacent members***.
  - The failure of a single apartment wall in the ***Ronan Point apartment complex*** in 1968 led to the ***failure of several other units,*** see ***Figure 3.3-1*** and ***Figure 3.3-2***.
  - In response to this collapse, ***codes added requirements for integrity reinforcement based on a rational assessment of the failure***.
  - This integrity reinforcement is a ***prescriptive provision***, that is, ***the requirements are detailed in the code and must be incorporated in the structure <u>without associated detailed calculations</u>***.


Figure 3.3-1: Ronan Point apartment complex.

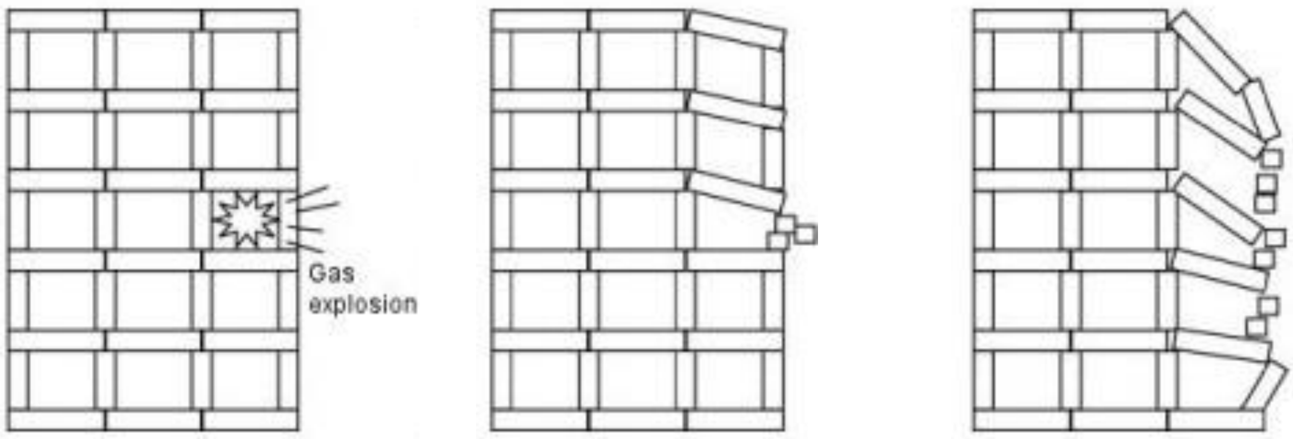

Figure 3.3-2: Chain reaction collapse of a building frame.

- Language of the code:
  - Codes are written in ***terse language***, based on the assumption that the user is a ***competent engineer***.
  - A ***commentar***y accompanies most codes and assists in understanding, ***provides references or background***, and ***offers rationale for the provisions***.
- Codes provide minimum requirements only:
  Because codes provide the minimum requirements for safety and serviceability, ***the engineer is allowed to exceed these requirements***.

3.3.3 Practice

- Structural engineering practice encompasses both the ***art*** and the ***technical practice*** of structural design.
- Throughout history, many extraordinary structures, such as the ***mosques*** and ***cathedrals***, have been ***designed and constructed without the benefit of modern theory and codes***.
- While ***theory and codes provide the mechanics for establishing the strength and serviceability of structures, neither provides the aesthetic, economic, or functional guidance needed for member selection***.
- Questions such as "***Should a beam be slender or stout within the code limits***?" or "***How should the concrete mixture be adjusted for corrosive environments***?" need to be answered by the engineer. To respond, the engineer relies on ***judgment***, ***personal experience***, and the ***broader experience of the profession to adapt the design to meet the overall project requirements***.
- Inclusion of ***long-standing design guidelines for the selection of member sizes is a good example of how that broader experience of the profession is used***.

3.4 BEHAVIOR OF MEMBERS SUBJECT TO AXIAL LOADS

3.4.1 Big Picture for Behavior of Reinforced Concrete Mechanics through Analysis of Axially Loaded Members

- Many of the ***fundamentals of the behavior of reinforced concrete***, through ***the full range of loading from zero to ultimate***, can be ***illustrated*** clearly in ***the context of members subject to simple axial compression or tension***.
- The basic concepts illustrated here will be recognized in later chapters in the analysis and design of beams, slabs, eccentrically loaded columns, and other members subject to more complex loadings.

3.4.2 Axial Compression

- In members that sustain chiefly or exclusively axial compression loads, such as building ***columns***, ***it is economical to make the concrete carry most of the load***.
- Why steel reinforcement are used in an axially loaded member:
  Still, some steel reinforcement is always provided for various reasons.
  - Very few members are ***subjected to truly axial load***; steel is essential for resisting any bending that may exist.
  - If part of the total load is carried by steel with its much greater strength, the ***cross-sectional dimensions of the member can be reduced***—the more so, the larger the amount of reinforcement.
- Two chief column forms:
  - The two chief forms of reinforced concrete columns are shown in Figure 3.4-1.
- The square column:
  - The four longitudinal bars serve as main reinforcement.
  - They are held in place by transverse small-diameter steel ties that:
    - prevent displacement of the main bars ***during construction operations,***
    - counteract any tendency of ***the compression loaded bars to buckle out of the concrete by bursting the thin outer cover***.
- The round column:
  - There are eight main reinforcing bars.
  - These are surrounded by a closely spaced spiral that ***serves the same purpose as the more widely spaced ties*** but also ***acts to confine the concrete within i***t, thereby ***increasing its resistance to axial compression***.
- The discussion that follows applies to ***<u>tied columns</u>***.

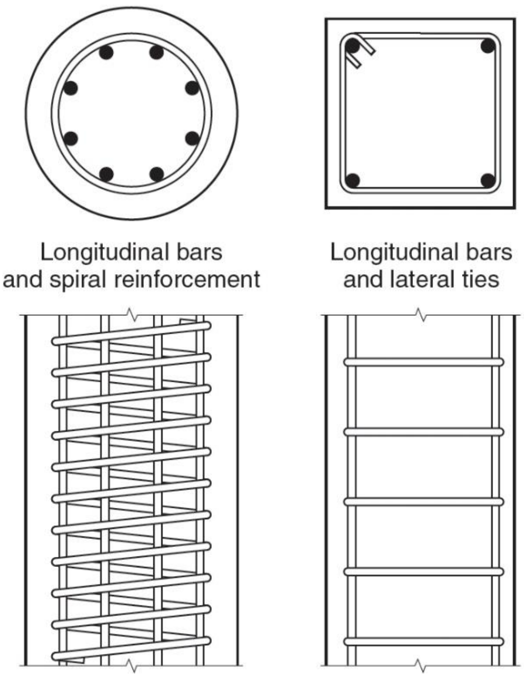

Figure 3.4-1: Reinforced concrete columns.

3.4.2.1 Application of Fundamental Assumption for Analysis of an Axially Loaded Member

- Fundamental assumptions for reinforced concrete behavior of Section 1.8 of Chapter 1 can be adopted to formulate the behavior of axially load member.
- Compatibility and Kinematic Assumption:
  - When axial load is applied, the compression deformation is the same over the entire cross section:
    $\Delta = constant$
  - Hence the strain would be constant for the entire section:
    $\epsilon = \frac{\Delta}{L} = constant$
  - In view of the bonding between concrete and steel, is the same in the two materials:
    $\epsilon_c = \epsilon_{st} = consatnt$
- Stresses-strain Diagrams:
  - Figure 3.4-2 shows two representative stress-strain curves, one for a concrete with compressive strength $f_c' = 28\ MPa$ and the other for a steel with yield stress $f_y = 420\ MPa$.
  - The curves for the two materials are drawn on the same graph using different vertical stress scales.
  - Different Loading Rates for Concrete:
    - ***Curve b*** has the shape that would be obtained in a concrete ***cylinder test***.
    - The rate of loading in most structures is considerably slower than that in a cylinder test, and this affects the shape of the curve.
    - ***Curve c***, therefore, is drawn as being ***characteristic of the performance of concrete under slow loading***.
    - Under these conditions, tests have shown that the maximum reliable compressive strength of reinforced concrete is about $0.85 f_c'$, as shown.

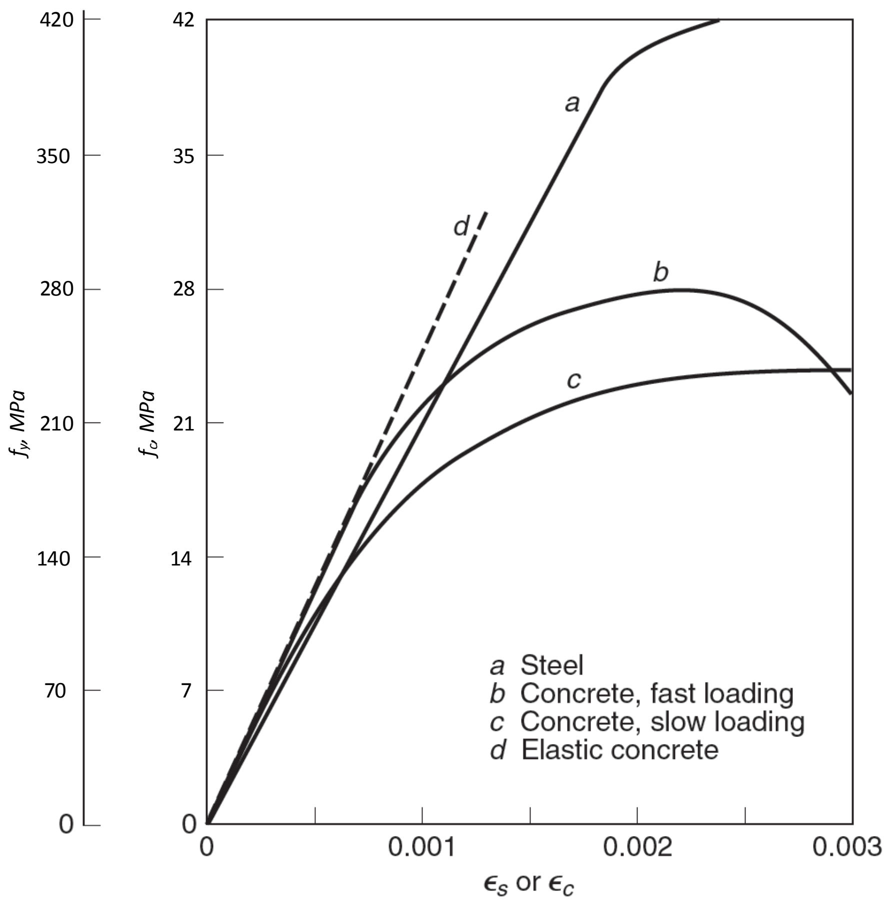

Figure 3.4-2: Concrete and steel stress strain curves.

3.4.2.2 Elastic Behavior

- At low stresses, up to about $f_c'/2$, the concrete is seen to ***behave nearly elastically***, that is, ***stresses and strains are quite closely proportional***; the straight line d represents this range of behavior with little error for both rates of loading.
- For the given concrete, the range extends to a strain of about 0.0005. The steel, on the other hand, is seen to be elastic nearly to its yield point of $420\ MPa$, or to the much greater strain of about 0.002.
- Stress Distribution:
  Because the compression strain in the concrete, at any given load, is equal to the compression strain in the steel,
  $$\epsilon_c = \frac{f_c}{E_c} = \epsilon_s = \frac{f_s}{E_s}$$
  from which the relation between the steel stress f_s and the concrete stress f_c is obtained as:
  $$f_s = \frac{E_s}{E_c} f_c = n f_c \qquad \textbf{Eq. 3.4-1}$$
  where $n = E_s/E_c$ is known as the modular ratio.
- Equilibrium Conditions:
  Let
  A_c = net area of concrete, that is, gross area minus area occupied by reinforcing bars
  A_g = gross area
  A_{st} = total area of reinforcing bars
  P = axial load
  Then from stress distribution and equilibrium condition, namely $\Sigma F_y = 0$, one can obtain:
  $$P = f_c A_c + f_s A_{st} = f_c A_c + n f_c A_{st}$$
  or
  $$P = f_c (A_c + n A_{st}) \qquad \textbf{Eq. 3.4-2}$$
- Concept of the transformed area:
  - The term $A_c + nA_{st}$ can be interpreted as the area of a ***fictitious concrete cross section***, the ***transformed area***, which when subjected to the particular concrete stress f_c results in the same axial load P as the actual section composed of both steel and concrete.
  - This transformed concrete area is seen to consist of the actual concrete area plus n times the area of the reinforcement. It can be visualized as shown in ***Figure 3.4-3***. That is, in ***Figure 3.4-3b*** the three bars along each of the two faces are thought of as being removed and replaced, at the same distance from the axis of the section, with added areas of fictitious concrete of total amount nA_{st}.
  - Alternatively, as shown in ***Figure 3.4-3c*** , one can think of the area of the steel bars as replaced with concrete, in which case one has to add to the gross concrete area A_g so obtained only $(n - 1)A_{st}$ to obtain the same total transformed area.
  - Therefore, alternatively,
    $$P = f_c\big(A_g + (n-1)A_{st}\big) \qquad \textbf{Eq. 3.4-3}$$

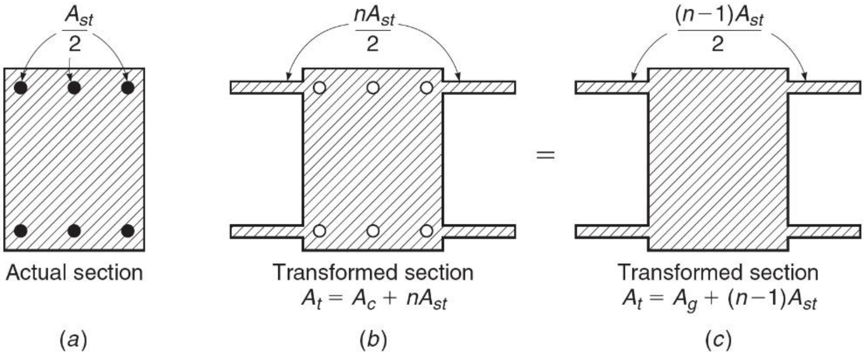

Figure 3.4-3: Transformed section in axial compression.

Example 3.4-1

A column made of the materials defined in ***Figure 3.4-2*** has a cross section of $400 \times 500mm$ and is reinforced by six No. 29 bars, disposed as shown in ***Figure 3.4-3***. Determine the axial load that will stress the concrete to $8.0MPa$. The modular ratio n may be assumed equal to 8, in view of ***the scatter inherent in*** E_c***, it is customary and satisfactory to round off the value of n to the nearest integer and never justified to use more than two significant figures***.

Solution

From Eq. 3.4-3,

$$P = f_c\left(A_g + (n-1)A_{st}\right) \Rightarrow P = 8.0 \times \left((400 \times 500) + (8-1) \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)$$

Solve,

$P = 1821935\ N$

Of this total load, the concrete is seen to carry:

$$P_c = f_c A_c = 8.0 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right) = 1568295\ N$$

and the steel

$$P_s = f_s A_{st} = n f_c A_{st} = 8 \times 8 \times \left(6 \times \frac{\pi \times 29^2}{4}\right) = 253640\ N$$

The percent of load that are supporting by steel would be:

$$Ratio\ of\ P_s = \frac{P_s}{P} \times 100 = \frac{253640}{1821935} \times 100 = 13.9\ \%$$

3.4.2.3 Inelastic Range

- Inspection of ***Figure 3.4-2***, reproduce in below for convenience, shows that the elastic relationships that have been used so far ***cannot be applied beyond a strain of about 0.0005 for the given concrete***.
- To obtain information on the behavior of the member at larger strains and, correspondingly, at larger loads, it is therefore necessary to make direct use of the information in ***Figure 3.4-2***.

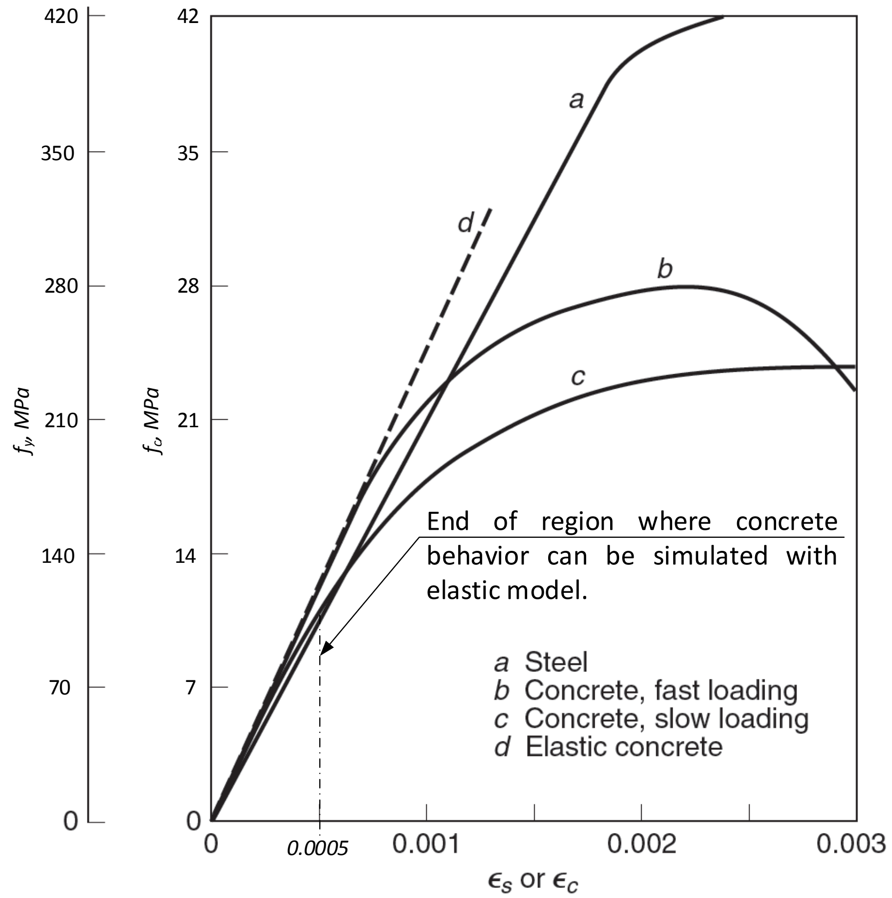

Figure 3.4-2: Concrete and steel stress strain curves. Reproduced for convenience.

Example 3.4-2

Determine the magnitude of the axial load that will produce a strain or unit shortening $\epsilon_c = \epsilon_s = 0.001$ in the column of ***Example 3.4-1***.

Solution

At this strain, the steel is seen to be still elastic, so that the steel stress:

$$f_s = \epsilon_s E_s = 0.001 \times 200000 = 200\ MPa$$

The concrete is in the inelastic range, so that its stress cannot be directly calculated, but it can be read from the stress-strain curve for the given value of strain. Considering load rate, there are two possible solution as indicated in below:

<u>**Fast Loading Rate:**</u>

With referring to ***Figure 3.4-4***, the concrete stress would be:

$$f_c \approx 22\ MPa$$

$$P = f_c A_c + f_s A_s = \frac{\left(22 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(200 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000}$$

$$= 5105\ kN$$

$$Ratio\ of\ P_s = \frac{P_s}{P} \times 100 = \frac{\left(200 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{5105 \times 10^3} \times 100 = 15.5\ \%$$

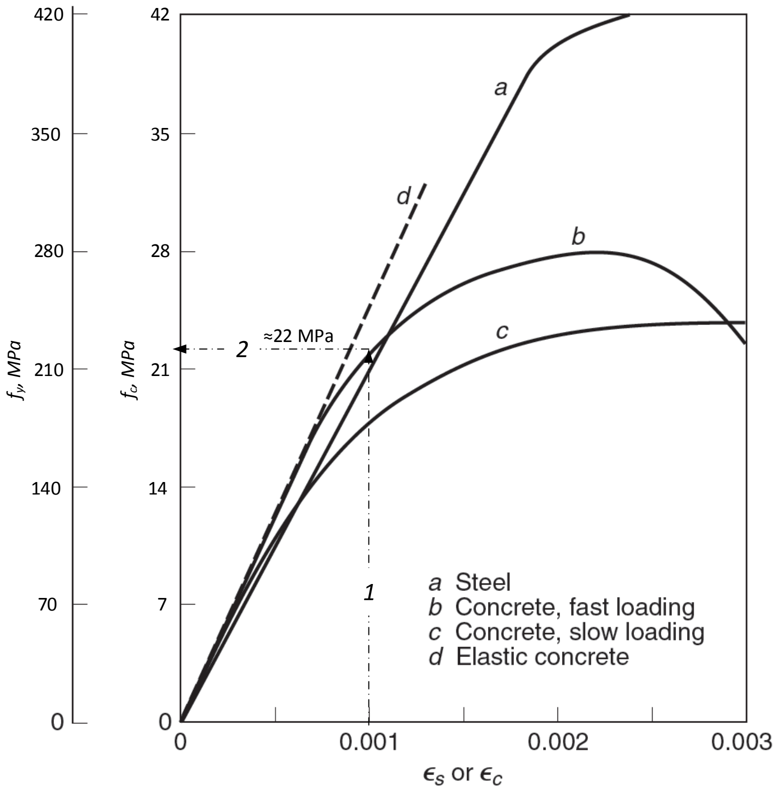

Figure 3.4-4: Concrete stress for the column of Example 3.4-2 when load rate is fast.

Slow Loading Rate:

With referring to ***Figure 3.4-5***, the concrete stress would be:

$f_c \approx 17\ MPa$

$$P = f_c A_c + f_s A_s = \frac{\left(17 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(200 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000}$$

$$= 4125\ kN$$

$$Ratio\ of\ P_s = \frac{P_s}{P} \times 100 = \frac{\left(200 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{4125 \times 10^3} \times 100 = 19.2\ \%$$

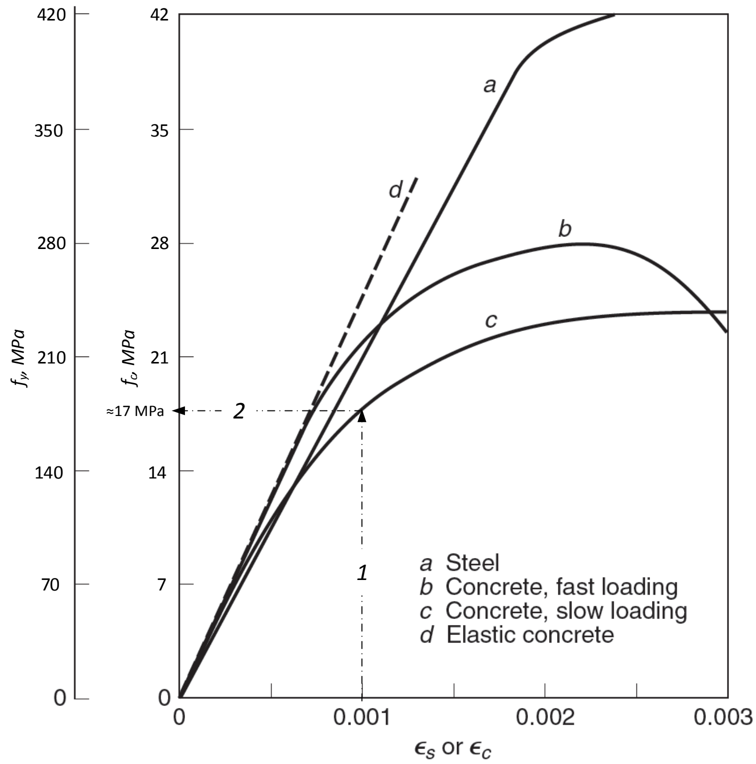

Figure 3.4-5: Concrete stress for the column of Example 3.4-2 when load rate is slow.

Comparison of the results for fast and slow loading shows the following:

- Owing to creep of concrete, ***a given shortening of the column is produced by a smaller load***, $P_{Slow} = 4125$ kN, ***when slowly applied or sustained over some length of time than when quickly applied***.
- More important, the farther the stress is beyond the proportional limit of the concrete, and the more slowly the load is applied or the longer it is sustained, the smaller the share of the total load carried by the concrete and the larger the share carried by the steel.
- In the sample column, the steel was seen to carry ***13.9 percent of the load in the elastic range***, ***15.5 percent for a strain of 0.001 under fast loading***, and ***19.2 percent at the same strain under slow or sustained loading***.

3.4.2.4 Strength

- Importance of Strength:
  The strength is one quantity of chief interest to the structural designer.
- Definition of Strength:
  The strength is the maximum load that the structure or member will carry.
- Parameters to determine the strength:
  Information on ***stresses***, ***strains***, and similar quantities serves chiefly as ***a tool for determining carrying capacity***.
- Performance of the Column:
  The performance of the column discussed so far indicates two things:
  - Large stresses and strains companion to the maximum load:
    - The range of large stresses and strains that precede attainment of the maximum load and subsequent failure.
    - Hence, ***elastic relationships cannot be used***.
  - Different behaviors for different loading rates:
    - The member ***behaves differently under fast and under slow or sustained loading***.
    - It shows ***less resistance to the slow load than to the faster load***.
- Loading rates in usual constructions:
  - Slow rate in general:
    In usual construction, many types of loads, such as the ***weight of the structure*** and any ***permanent equipment housed*** therein, are ***<u>sustained</u>***, and ***others are applied at <u>slow rates</u>***.
  - Suitable concrete strength:
    For this reason, to calculate a reliable magnitude of compressive strength, ***curve c*** of ***Figure 3.4-2*** must be used as far as the concrete is concerned.
- Strain for maximum tensile strength of steel:
  The steel reaches its tensile strength (peak of the curve) at strains on the order of 0.08 (see ***Figure 3.4-6***).

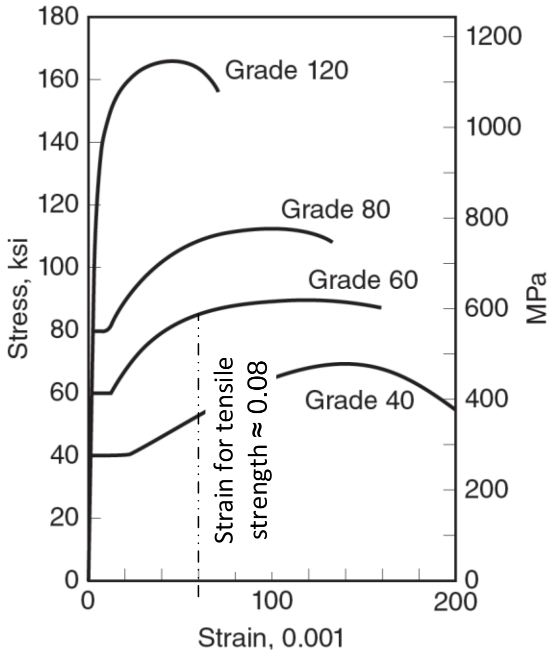

Figure 3.4-6: Strain for tensile strength (peak of the curve).

- Crushing strain for concrete:
  Concrete fails by ***crushing at the much smaller strain of about 0.003*** and, as seen from ***Figure 3.4-2*** (***curve c***).
  $\epsilon_u = 0.003$
- Strains for maximum stresses of concrete:
  Concrete reaches its ***maximum stress in the strain range of 0.002 to 0.003***, see ***Figure 3.4-2*** (***curve c***).

- Yielding strains of steel, ϵ_y:
  - If the small knee prior to yielding of the steel is disregarded, that is, if the steel is assumed to be ***sharp-yielding***, ***Figure 3.4-7***, the strain at which it yields is:

$$\epsilon_y = \frac{f_y}{E_s}$$ **Eq. 3.4-4**

  - For Grade 60 steel:

$$\epsilon_y = \frac{420}{200000} \approx 0.002$$

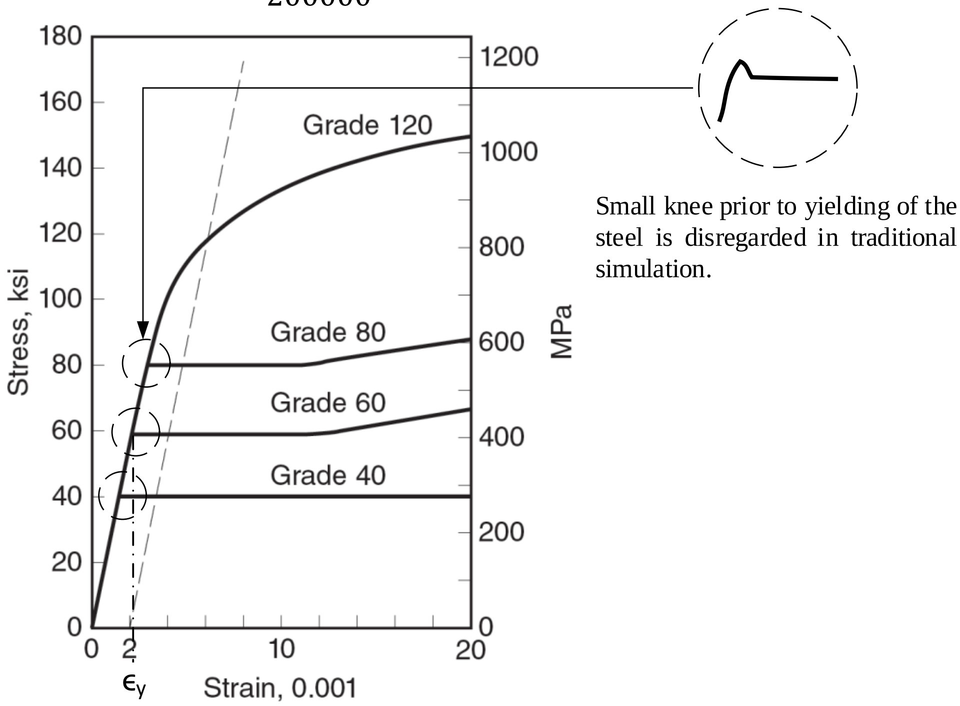

Figure 3.4-7: Idealized stress-strain diagram of steel and the corresponding yield strain.

- Compatibility conditions and section nature:
  Because the strains in steel and concrete are equal in axial compression, ***the rebars for an axially compressed column are yielded at a strain of*** ϵ_y ***before concrete crushing at strain of*** ϵ_u.
- Nominal compressive capacity, P_n:
  Based on previous discussion, the nominal, theoretical, strength of an axially compressed column is:

$$P_n = 0.85 f_c' A_c + f_y A_{st}$$ **Eq. 3.4-5**

  The factor 0.85 is adopted to calibrate concrete compressive strength at slow loading rate of usual construction to that of fast loading rate for cylindrical test, f_c'.

Example 3.4-3

Determine the nominal compressive strength for the column of ***Example 3.4-1***.

Solution

Based on Eq. 3.4-5, the nominal compressive strength of a column is:

$$P_n = 0.85 f_c' A_c + f_y A_{st} =$$

$$= \frac{\left(0.85 \times 28 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(420 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000}$$

$$= 6330\ kN \blacksquare$$

$$Ratio\ of\ P_s = \frac{P_s}{P} \times 100 = \frac{\left(420 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{6330 \times 10^3} \times 100 = 26.3\ \%$$

3.4.2.5 Summary

- In the ***elastic range***, ***the steel carries a relatively small portion of the total load of an axially compressed member***.
- As ***member strength is approached***, there occurs ***a redistribution of the relative shares of the load resisted by concrete and steel, the latter taking an increasing amount***.
- The nominal capacity, at which the member is on the point of failure, consists of the contribution of the steel when it is stressed to the yield point plus that of the concrete when its stress has attained a value of $0.85 f_c'$, as reflected in ***Eq. 3.4-5***.

3.4.3 Axial Tension

- Concrete is not suitable for tension members in general:
    - The tension strength of concrete is only a small fraction of its compressive strength.
    - It follows that reinforced concrete is not well suited for use in tension members because the concrete will contribute little, if anything, to their strength.
- Members where concrete is subjected to direct tension:
    - Still, there are situations in which reinforced concrete is stressed in tension, chiefly in ***tie-rods in structures such as arches and trusses***, see ***Figure 3.4-8***, or in uplift piles, see ***Figure 3.4-9***.


Figure 3.4-8: Concrete trusses with members under tensions.

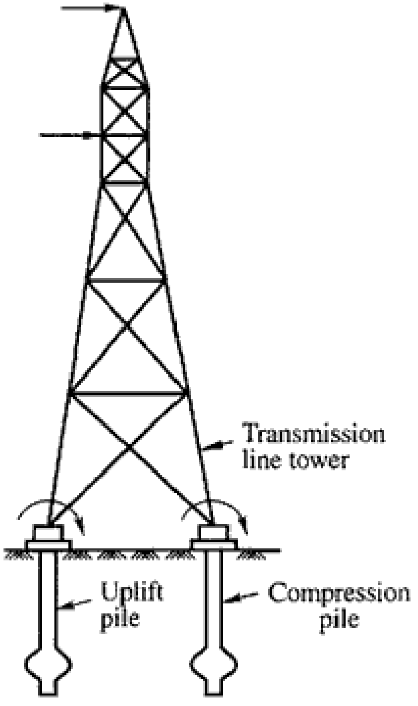

Figure 3.4-9: Tensile piles.

- Reinforcement for concrete tensile members:
  Such members consist of one or more bars embedded in concrete in a symmetric arrangement ***similar to compression members*** (see ***Figure 3.4-1***).
- Elastic behavior under small tensile forces:
    - When the tension force in the member is ***small enough for the stress in the concrete to be considerably below its tensile strength***, ***both steel and concrete behave elastically***.
    - In this situation, ***all the expressions derived for elastic behavior in compressio***n in ***Section 3.4.2.2*** are ***identically valid for tension***. In particular, ***Eq. 3.4-2*** becomes:

$$P = f_{ct}(A_c + nA_{st}) \quad \textbf{Eq. 3.4-6}$$

where f_{ct} is the tensile stress in the concrete.

- Elastic Cracked Section:
    - When the ***load is further increased***, however, ***the concrete reaches its tensile strength at a stress and strain on the order of one-tenth of what it could sustain in compression***. At this stage, ***the concrete cracks across the entire cross section***.
    - When this happens, it ceases to resist any part of the applied tension force, since, evidently, ***no force can be transmitted across the air gap in the crack***.
    - At any load larger than that which caused the concrete to crack, ***the steel is called upon to resist the entire tension force***.
    - Correspondingly, at this stage:

$$P = f_s A_{st} \qquad \textbf{Eq. 3.4-7}$$

- Tensile Strength (It is determined based on f_y instead of f_u):
    - With further increased load, the tensile stress f_s in the steel reaches the yield point f_y.
    - When this occurs, the tension members cease to exhibit small, elastic deformations but instead ***stretch a sizable and permanent amount at substantially constant load***. ***This does not impair the strength of the member***.
    - Its ***elongation***, however, ***becomes so large*** (***approximately 1 percent or more of its length***) as to render it useless.
    - Therefore, ***the maximum useful strength*** P_{nt} ***of a tension member is the force that will just cause the steel stress <u>to reach the yield point</u>***. That is,

$$P_{nt} = f_y A_{st} \qquad \textbf{Eq. 3.4-8}$$

- Tensile Strength under Service Conditions:
    - To provide adequate safety, the force permitted in a tension member under normal service loads should be limited to about:

$$P_{Tension\ under\ service\ conditions} = \frac{1}{2} P_{nt}$$

    - Because the concrete has cracked at loads considerably smaller than this, ***concrete does not contribute to the carrying capacity of the member in service***.
    - It does serve, however, as ***fire*** and ***corrosion protection*** and ***often improves the appearance of the structure***.
- Tension in Watertight Structures:
    - There are situations, though, in which ***reinforced concrete is used in axial tension under conditions in which the occurrence of tension cracks must be prevented***.
    - A case in point is a circular tank, see ***Figure 3.4-10***, ***to provide watertightness, the hoop tension caused by the fluid pressure must be prevented from causing the concrete to crack***.
    - In this case, Eq. 3.4-2 can be used to determine ***a safe value for the axial tension force P*** by using, for the concrete tension stress f_{ct}, an appropriate fraction of the tensile strength of the concrete, that is, of the stress that would cause the concrete to crack.

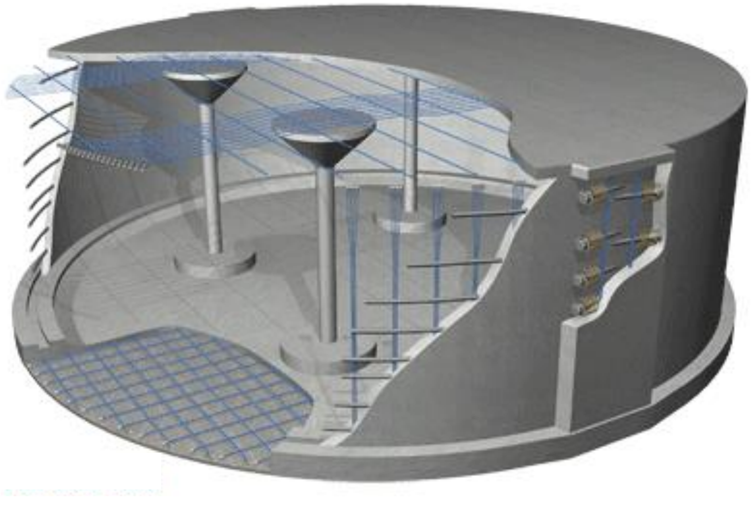

Figure 3.4-10: Circular tank under tensile hoop stresses.

3.5 ADDITIONAL EXAMPLE

Additional Example 3.5-1

A 400 × 500 mm column is made of the same concrete and reinforced with the same six No. 29 bars as the column in ***Example 3.4-1***, except that a steel with yield strength $f_y = 280\ MPa$ is used. The stress-strain diagram of this reinforcing steel is shown in ***Figure 3.5-1*** for $f_y = 280\ MPa$. For this column determine:

- The axial load that will stress the concrete to 8 MPa.
- The load at which the steel starts yielding.
- The maximum load.
- The share of the total load carried by the reinforcement at these three stages of loading.

Compare results with those calculated in the examples for $f_y = 420\ MPa$, keeping in mind, in regard to relative economy, that the price per pound for reinforcing steels with 280 and 420 MPa yield points is about the same.

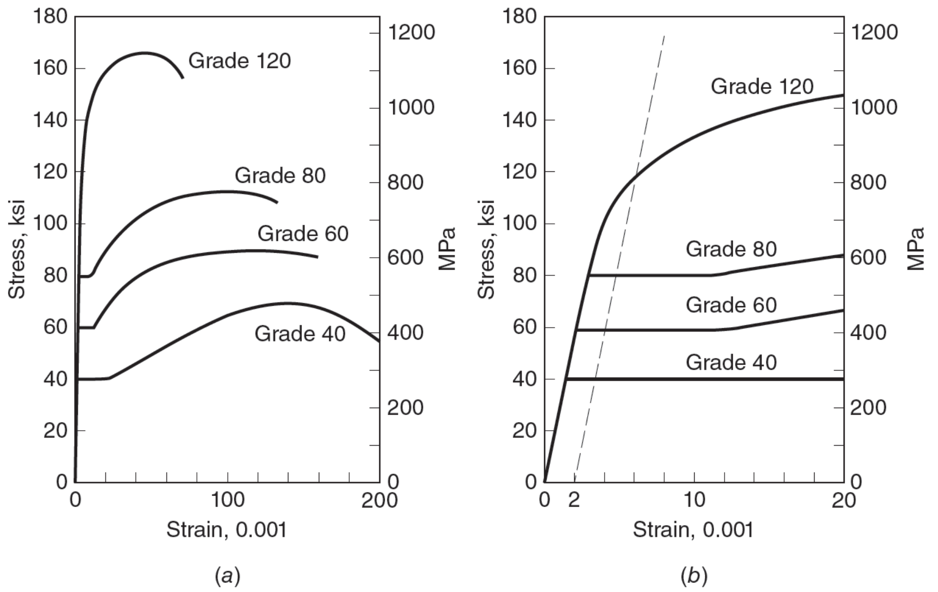

Figure 3.5-1: Typical stress-strain curves for reinforcing bars.

Solution

- The axial load that will stress the concrete to 8 MPa:
  With referring to concrete stress-strain diagram of ***Figure 3.4-2***, concrete behaves elastically when subjected to compressive stress of 8 MPa. Hence the compressive axial force, P, can be determined from ***Eq. 3.4-2***:
  $P = f_c(A_c + nA_{st})$
  As steel elastic modulus, E_s, has a constant value of $200000\ MPa$ irrespective of steel yield stresses, therefore, the modular ratio, n, is equal to 8 as for ***Example 3.4-1***. Substitute into ***Eq. 3.4-2*** to obtain:
  $$P_{@\ stress\ of\ 8\ MPa} = \frac{8.0 \times \left((400 \times 500) + (8-1) \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000} = 1821\ kN\ ■$$
  $$P_{s\ ratio\ @\ elastic\ range} = \frac{8 \times 8 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)}{1821 \times 1000} \times 100 = 13.9\ \%$$
  As steel yield stress has no effect of elastic behavior, these values are same as those of ***Example 3.4-1***.
- The load at which the steel starts yielding:
  The steel yield at strain of:
  $$\epsilon_y = \frac{f_y}{E_s} = \frac{280}{200000} = 0.0014$$
  From concrete stress-strain curve of Figure 3.4-2, reproduce in below for convenience:
  $f_{c\ slow} = 21\ MPa$

$$P_{for\ steel\ yeild} = f_c A_c + f_y A_s$$
$$= \frac{\left(21 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(280 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000}$$
$$= 5226\ kN\ \blacksquare$$
$$P_{s\ ratio\ @\ yeild} = \frac{\left(280 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{5226 \times 1000} \times 100 = 21.2\ \%$$

These values are smaller than those of steel with grade of 420 MPa. Therefore, at yield range, the steel with grade of 420 MPa has more contribution than grade of 280 MPa.

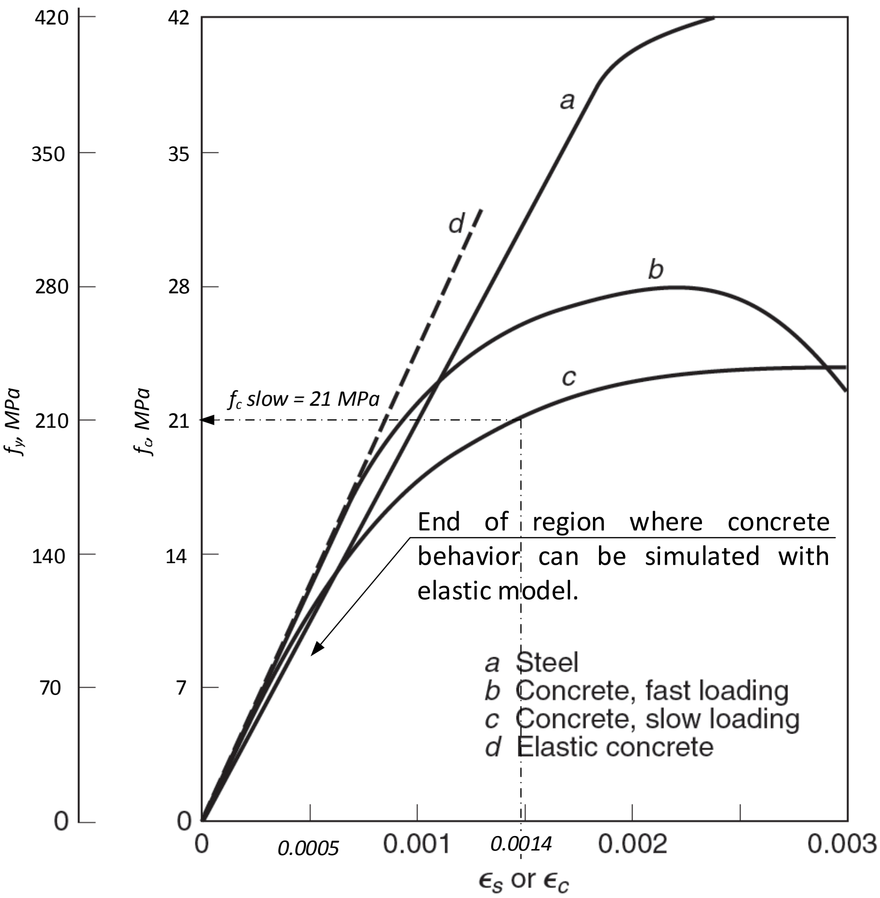

Figure 3.4-2: Concrete and steel stress strain curves. Reproduced for convenience.

- The maximum load:

  The maximum load, nominal strength P_n, can be determined from Eq. 3.4-5:

$$P_n = 0.85 f_c' A_c + f_y A_{st} =$$
$$= \frac{\left(0.85 \times 28 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(280 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{1000}$$
$$= 5775\ MPa$$
$$P_{s\ ratio\ @\ ultimate\ range} = \frac{\left(280 \times \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)}{5775 \times 1000} \times 100 = 19.2\ \%$$

These values are smaller than those of steel with grade of 420 MPa. Therefore, at ultimate, the steel with grade of 420 MPa has more contribution than grade of 280 MPa.

Additional Example 3.5-2

The area of steel, expressed as a percentage of gross concrete area, for the column of ***Additional Example 3.5-1*** is lower than would often be used in practice. Recalculate the comparisons of ***Additional Example 3.5-1***, using f_y of $280\,MPa$ as before, but for a 400 × 500 mm column reinforced with eight No. 36 bars. Compare your results with those of ***Additional Example 3.5-1***.

Solution

- The axial load that will stress the concrete to 8 MPa:

$$P_{@\,stress\,of\,8\,MPa} = \frac{8.0 \times \left((400 \times 500) + (8-1) \times \left(8 \times \frac{\pi \times 38^2}{4}\right)\right)}{1000} = 2108\,kN\ \blacksquare$$

$$P_{s\,ratio\,@\,elastic\,range} = \frac{8 \times 8 \times \left(8 \times \frac{\pi \times 36^2}{4}\right)}{2108 \times 1000} \times 100 = 24.7\,\%$$

- The load at which the steel starts yielding:

$$P_{for\,steel\,yeild} = f_c A_c + f_y A_s = \frac{\left(21 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(280 \times \left(8 \times \frac{\pi \times 36^2}{4}\right)\right)}{1000}$$
$$= 6397\,kN\ \blacksquare$$

$$P_{s\,ratio\,@\,yeild} = \frac{\left(280 \times \left(8 \times \frac{\pi \times 36^2}{4}\right)\right)}{6397 \times 1000} \times 100 = 35.6\,\%$$

- The maximum load:

$$P_n = 0.85 f_c' A_c + f_y A_{st} =$$
$$= \frac{\left(0.85 \times 28 \times \left(400 \times 500 - \left(6 \times \frac{\pi \times 29^2}{4}\right)\right)\right) + \left(280 \times \left(8 \times \frac{\pi \times 36^2}{4}\right)\right)}{1000}$$
$$= 6946\,MPa$$

$$P_{s\,ratio\,@\,ultimate\,range} = \frac{\left(280 \times \left(8 \times \frac{\pi \times 36^2}{4}\right)\right)}{6946 \times 1000} \times 100 = 32.8\,\%$$

- Comments:
  Using larger amount of reinforcement, leads to a larger steel contribution in elastic, yield, and strength ranges.

Additional Example 3.5-3

A square concrete column with dimensions 550 × 550 mm is reinforced with a total of eight No. 32 bars arranged uniformly around the column perimeter. Material strengths are $f_y = 420\,MPa$ and $f_c' = 28\,MPa$, with stress-strain curves as given by curves a and c of ***Figure 3.4-2***. Calculate the percentages of total load carried by the concrete and by the steel as load is gradually increased from 0 to failure, which is assumed to occur when the concrete strain reaches a limit value of 0.0030. Determine the loads at strain increments of 0.0005 up to the failure strain, and graph your results, plotting load percentages vs. strain. The modular ratio may be assumed at n = 8 for these materials.

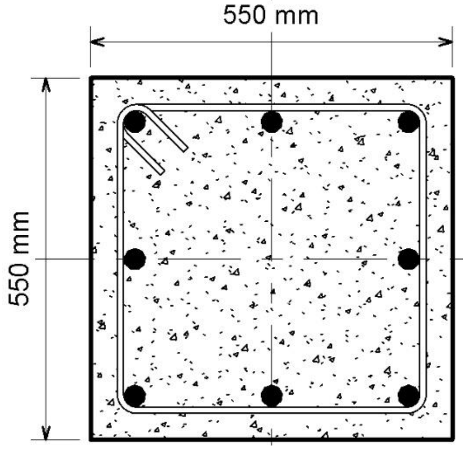

Figure 3.5-2: Cross section for column of Additional Example 3.5-3.

Solution

<u>At $\epsilon = 0$:</u>

$P_c = P_s = 0$

<u>At $\epsilon = 0.0005$:</u>

From ***Figure 3.4-2***, reproduce in below,

$f_c(@\, \epsilon\ of\ 0.0005) = 10.5\ MPa$

From ***Article 3.4.2.3*** and ***Figure 3.4-2***, it is found that concrete behaves almost in linear up to a strain ϵ of 0.0005, then its behavior can be determined from Eq. 3.4-3:

$$P_{@\, \epsilon\ of\ 0.0005} = f_c\left(A_g + (n-1)A_{st}\right) = \frac{10.5 \times \left(550^2 + (8-1) \times 8 \times \frac{(\pi \times 32^2)}{4}\right)}{1000} = 3649\ kN$$

$$Ratio_{of\ P_c @\, \epsilon\ of\ 0.0005} = \frac{f_c A_c}{P} = \frac{10.5 \times \left(550^2 - 8 \times \frac{(\pi \times 32^2)}{4}\right)}{3649 \times 1000} \times 100 \approx 85\%$$

$$Ratio_{of\ P_s @\, \epsilon\ of\ 0.0005} = \frac{n f_c A_s}{P} = \frac{8 \times 10.5 \times 8 \times \frac{(\pi \times 32^2)}{4}}{3649 \times 1000} \times 100 \approx 15\%$$

<u>At $\epsilon = 0.001$:</u>

From ***Figure 3.4-2***, reproduce in below,

$f_c(@\, \epsilon\ of\ 0.001) = 17\ MPa, f_s(@\, \epsilon\ of\ 0.001) = \epsilon E_s = 0.001 \times 200000 = 200\ MPa$

As the strain is within the inelastic range, hence correspond forces can be determined from following relation:

$$P_{@\, \epsilon\ of\ 0.001} = f_c A_c + f_s A_s = \frac{17 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right) + 200 \times \left(8 \times \frac{\pi \times 32^2}{4}\right)}{1000} = 6320\ kN$$

$$Ratio_{of\ P_c @\, \epsilon\ of\ 0.001} = \frac{f_c A_c}{P} = \frac{17 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right)}{6320 \times 1000} \times 100 \approx 80\ \%$$

$$Ratio_{of\ P_s @\, \epsilon\ of\ 0.001} = \frac{f_s A_s}{P} = \frac{200 \times 8 \times \frac{(\pi \times 32^2)}{4}}{6320 \times 1000} \times 100 \approx 20\ \%$$

<u>At $\epsilon = 0.0015$:</u>

From ***Figure 3.4-2***, reproduce in below,

$f_c = 21\ MPa, f_s = \epsilon E_s = 0.0015 \times 200000 = 300\ MPa$

$$P_{@\, \epsilon\ of\ 0.0015} = f_c A_c + f_s A_s = \frac{21 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right) + 300 \times \left(8 \times \frac{\pi \times 32^2}{4}\right)}{1000} = 8148\ kN$$

$$Ratio_{of\ P_c @\, \epsilon\ of\ 0.0015} = \frac{f_c A_c}{P} = \frac{21 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right)}{8148 \times 1000} \times 100 \approx 76\ \%$$

$$Ratio_{of\ P_s @\, \epsilon\ of\ 0.0015} = \frac{f_s A_s}{P} = \frac{300 \times 8 \times \frac{(\pi \times 32^2)}{4}}{8148 \times 1000} \times 100 \approx 24\ \%$$

<u>At $\epsilon = 0.002$:</u>

From ***Figure 3.4-2,*** reproduce in below,

$f_c = 23\ MPa, f_s = \epsilon E_s = 0.002 \times 200000 = 400\ MPa$

$$P_{@\, \epsilon\ of\ 0.002} = f_c A_c + f_s A_s = \frac{23 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right) + 400 \times \left(8 \times \frac{\pi \times 32^2}{4}\right)}{1000} = 9383\ kN$$

$$Ratio_{of\ P_c @\, \epsilon\ of\ 0.002} = \frac{f_c A_c}{P} = \frac{23 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right)}{9383 \times 1000} \times 100 \approx 73\ \%$$

$$Ratio_{of\ P_s\ @\ \epsilon\ of\ 0.002} = \frac{f_s A_s}{P} = \frac{400 \times 8 \times \frac{(\pi \times 32^2)}{4}}{9383 \times 1000} \times 100 \approx 27\ \%$$

<u>At $\epsilon = 0.0025$:</u>

From ***Figure 3.4-2,*** reproduce in below,

$f_c = 23.5\ MPa, f_s = f_y = 420\ MPa$

$$P_{@\ \epsilon\ of\ 0.0025} = f_c A_c + f_s A_s = \frac{23.5 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right) + 420 \times \left(8 \times \frac{\pi \times 32^2}{4}\right)}{1000}$$
$$= 9660\ kN$$

$$Ratio_{of\ P_c @\ \epsilon\ of\ 0.0025} = \frac{f_c A_c}{P} = \frac{23.5 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right)}{9660 \times 1000} \times 100 \approx 72\ \%$$

$$Ratio_{of\ P_s\ @\ \epsilon\ of\ 0.0025} = \frac{f_s A_s}{P} = \frac{420 \times 8 \times \frac{(\pi \times 32^2)}{4}}{9660 \times 1000} \times 100 \approx 28\ \%$$

<u>At $\epsilon = 0.003$:</u>

From ***Figure 3.4-2,*** reproduce in below,

$f_c = 24\ MPa, f_s = f_y = 420\ MPa$

$$P_{@\ \epsilon\ of\ 0.003} = f_c A_c + f_s A_s = \frac{24 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right) + 420 \times \left(8 \times \frac{\pi \times 32^2}{4}\right)}{1000} = 9808\ kN$$

$$Ratio_{of\ P_c @\ \epsilon\ of\ 0.003} = \frac{f_c A_c}{P} = \frac{24 \times \left(550^2 - \left(8 \times \frac{\pi \times 32^2}{4}\right)\right)}{9808 \times 1000} \times 100 \approx 72\ \%$$

$$Ratio_{of\ P_s\ @\ \epsilon\ of\ 0.003} = \frac{f_s A_s}{P} = \frac{420 \times 8 \times \frac{(\pi \times 32^2)}{4}}{9808 \times 1000} \times 100 \approx 28\ \%$$

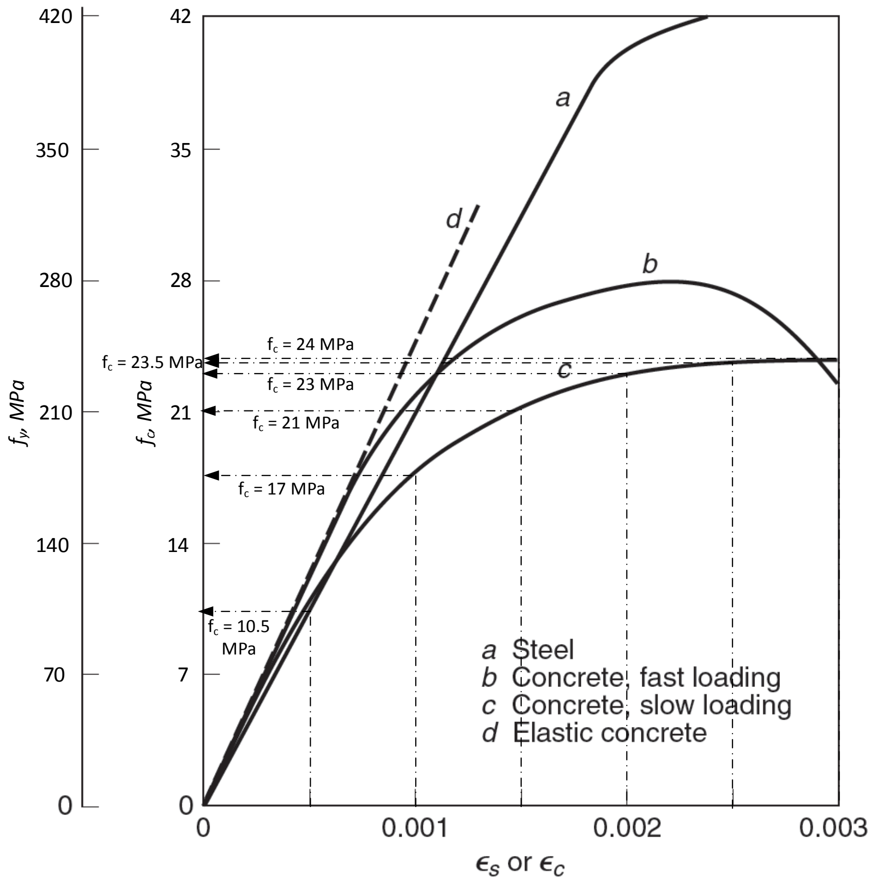

Figure 3.4-2: Concrete and steel stress strain curves. Reproduced for convenience.

Above ratios are summarized in ***Table 3.5-1*** and ***Figure 3.5-3*** below.

Table 3.5-1: Force ratios versus different strains in column of Additional Example 3.5-3.

Strain	Ratio of Pc	Ratio of Ps
0	0	0
0.00005	85	15
0.0001	80	20
0.00015	76	24
0.0002	73	27
0.00025	72	28
0.0003	72	28

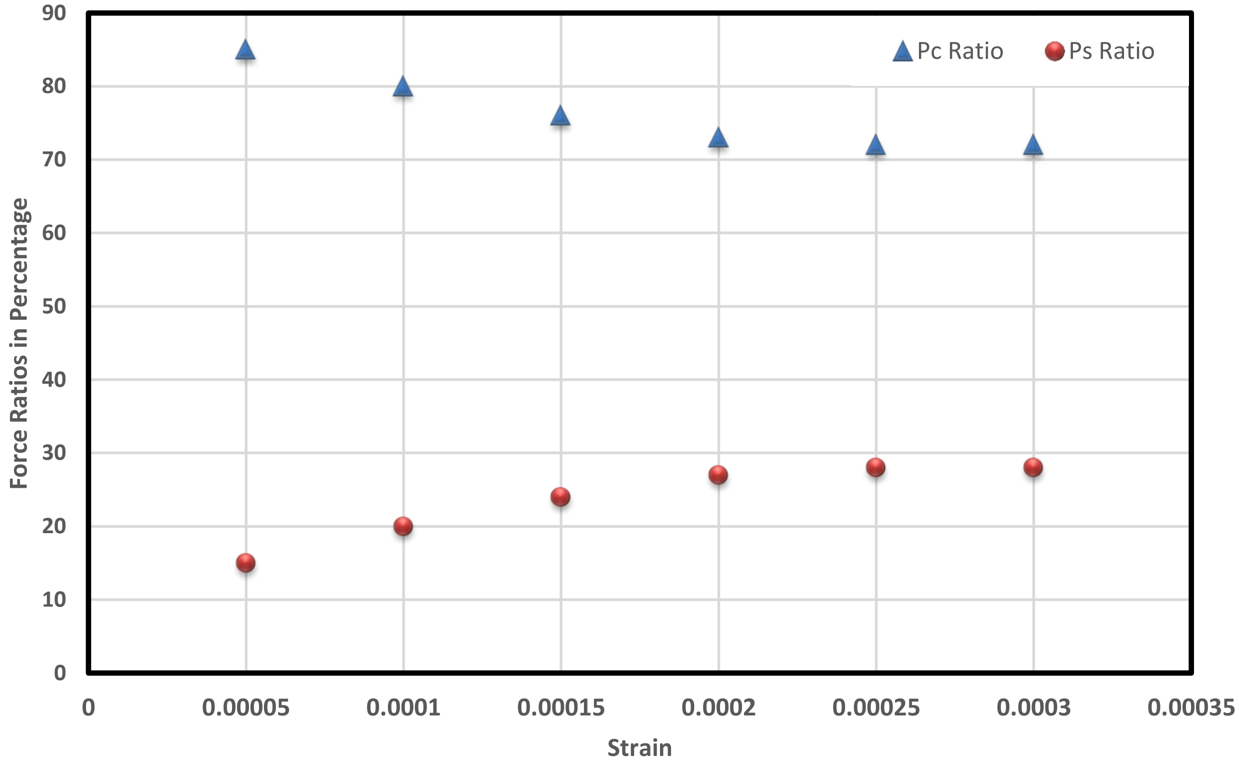

Figure 3.5-3: Force ratios versus different strains in column of Additional Example 3.5-3.

REFERENCES

ACI318M. (2014). Building Code Requirements for Structural Concrete (ACI 318M-14) and Commentary (ACI 318RM-14). Farmington Hills: American Concrete Institute.

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CONTENTS

CHAPTER 4
FLEXURE ANALYSIS AND DESIGN OF BEAMS

4.1 BENDING OF HOMOGENOUS BEAMS

- For the homogenous beams (i.e., the beams made from single homogenous material like steel or wood) and in the elastic range, the bending stresses can be computed based on the following relation:

  $f = \frac{M.y}{I}$

  where

  f is the bending stress at distance y from the neutral axis.

  M is the bending moment at the section,

  I is the moment of inertia of cross section about neutral axis.

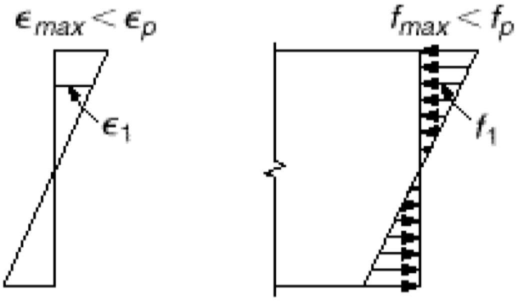


Figure 4.1-1: Stress distribution according to conventional flexural formula.

- For this homogenous elastic beam, the neutral axis passes through the center of gravity of the section.
- In general, the conventional flexure formula, $M.c/I$, is not applicable for RC beams as it has been derived for homogenous materials with linear elastic behavior.
- In Article 4.2 below a more fundamental flexural formula has been derived to take into account the nonlinear and composite nature of RC beams.

4.2 CONCRETE BEAM BEHAVIOR

4.2.1 Behavior of Plain Concrete

Plain concrete beams are inefficient flexure members because the tension strength in bending is a small fraction of the compression strength. Then we will focus on the analysis of reinforced concrete beams only.

4.2.2 Reinforce Concrete Beam Behavior

4.2.2.1 Suitability of Conventional Bending Formula for Analysis of RC Beams

As the reinforced concrete beam

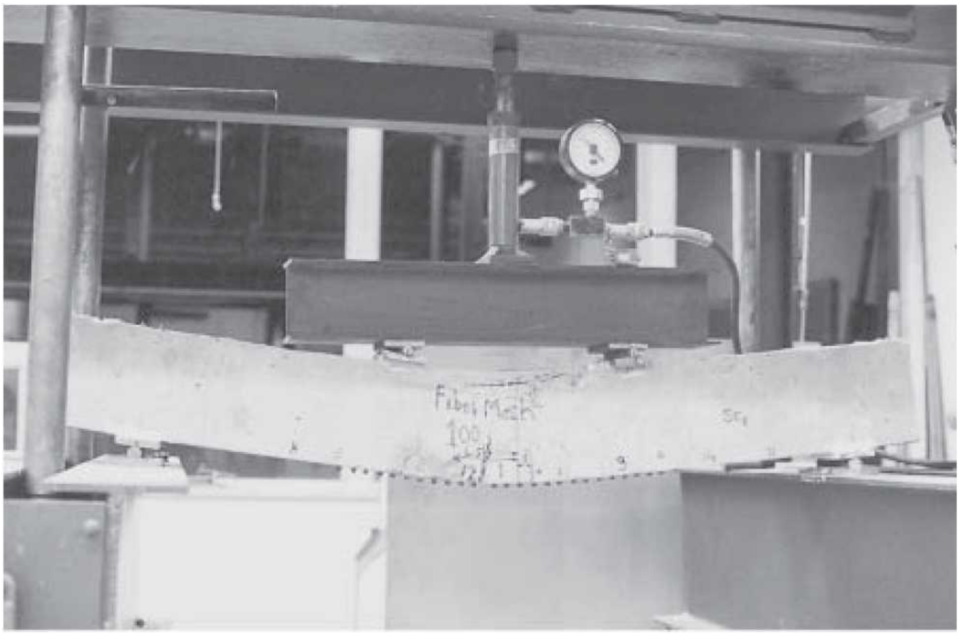
- Is made from two materials
- Cracks in the concrete,
- Behaves no-linear in concrete and steel

above conventional bending relation of $(M.c)/I$ for the homogenous beam cannot be applied to analysis of RC beams.

4.2.2.2 More Rigorous Relations

4.2.2.2.1 Model Beam and Experimental Works

- Experiment works pertained to flexural behavior of RC beams are usually conducted through model beam in below:



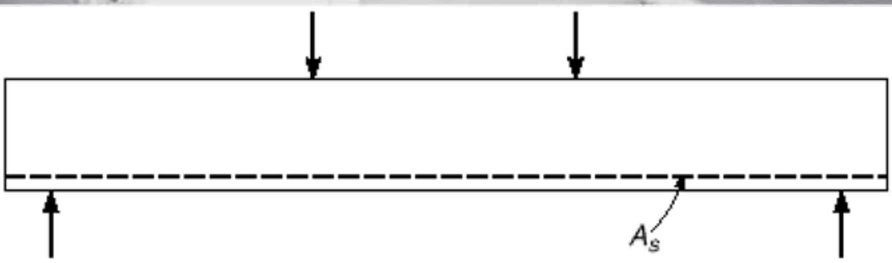


Figure 4.2-1: Model beam for experimental works of RC beams behavior in flexure.

- A beam loaded at third points mainly due to the fact that the mid region is under ***pure bending***, then ***the analysis can exclude the effect of shear stresses and focusing on flexure stresses only***.
- When the load on above beam is gradually increased from zero to the magnitude that will cause the beam to fail ***following three different stages of behavior can be clearly distinguished***.

4.2.2.2.2 Stresses Elastic and Section Uncracked

- At low loads, as long as the maximum tensile stress in the concrete is smaller than the tensile strength of concrete, the entire concrete is effective in resisting stress, in compression on one side and in tension on the other side of the neutral axis.
- At this stage, all stresses in the concrete are of small magnitude and are proportional to strains (i.e. the stresses are varied linearly with the depth). The distribution of strains and stresses in concrete and steel over the depth of the section is as shown in Figure 4.2-2 below.

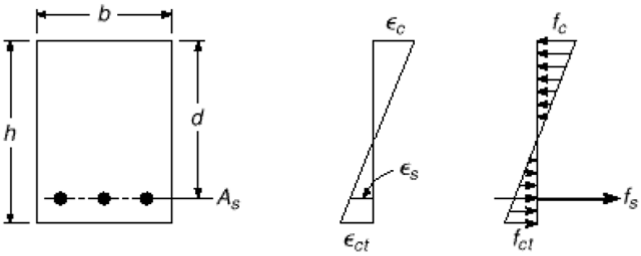


Figure 4.2-2: Strain and stress distribution during elastic uncracked stage.

- Then the only difference from the homogenous beam is in the presence of the steel reinforcement.
- It can be shown (see any text on strength of materials) that one can take account of the presence of the steel reinforcement by replacing the actual steel-and-concrete cross section with a *fictitious section* thought of as consisting of concrete only. In this "***Transformed Section***," the actual area of the reinforcement is replaced with an equivalent concrete area equal to $(n-1)A_s$, located at the level of the steel, as shown in Figure 4.2-3 below:

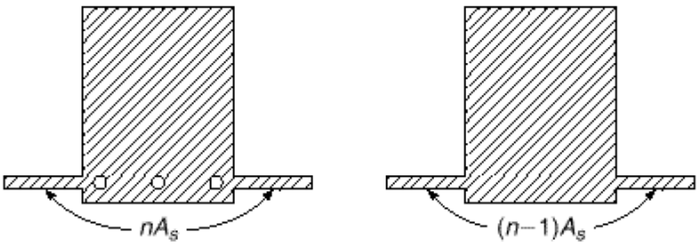


Figure 4.2-3: Transformed section for elastic uncracked RC beam.

where

$$n = \frac{Es}{Ec}$$

is the modular ratio.

- Once the transformed section has been obtained, the usual methods

$$f = \frac{M.c}{I}$$

of analysis of elastic homogeneous beams apply.

- **Computing of E_s and E_c:**

As discussed in Chapter 2, according to (ACI318M, 2014), article 19.2.2, modulus of elasticity, E_c, for concrete can be estimated based on following correlation:

  - For values of w_c between 1440 and 2560 kg/m^3

    $E_c = w_c^{1.5} 0.043\sqrt{f_c'}\ (in\ MPa)$

  - For normalweight concrete

    $E_c = 4700\sqrt{f_c'}\ (in\ MPa)$

According to the (ACI318M, 2014) (**20.2.2.2**), modulus of elasticity, E_s, for nonprestressed bars and wires shall be permitted to be taken as:

$E_{Steel} = E_s = 200\ 000\text{MPa}$

- This stage ends when tensile stress in concrete reaching a limit state. As discussed in Chapter 2, concrete tensile strength can be predicated based on
  - Direct Tensile Strength f_t'.
  - Split-Cylinder Strength f_{ct}.
  - Modulus of Rupture f_r

Example 4.2-1

For the beam shown in Figure 4.2-4 below, find the maximum magnitude of the load "P" such that the section stays in the uncracked elastic state.

Given:

- fc′ = 25 MPa
- fy = 400 MPa
- Neglect the beam selfweight.

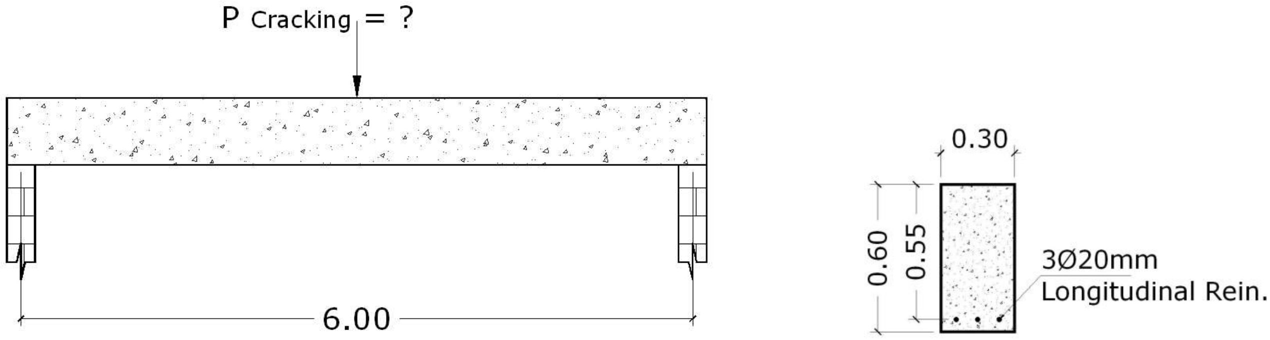
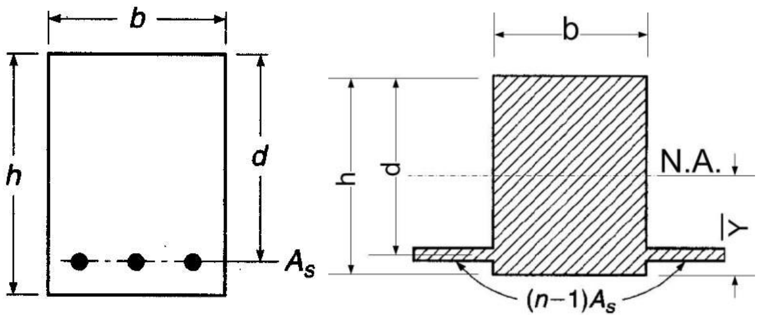


Figure 4.2-4: Beam of elastic uncracked section for Example 4.2-1.

Solution

- As the flexure formula is derived for the ***homogenous section***, then the steel must be transformed for the equivalent concrete to obtain a homogenous section that formed from a single material:



$\because E_S = 200\ 000\ MPa, and\ E_C = 4700\sqrt{f_c'} = 4700\sqrt{25} = 23\ 500\ MPa, \therefore n \approx 8.5$

$A_S = \left(\pi \frac{20^2}{4}\right) \times 3 = 942\ mm^2$

$\therefore (n-1)A_s = 7\ 065\ mm^2$

$\sum M_{of\ Area\ about\ lower\ face} = \bar{y}.A$

$\bar{y}.(300 \times 600 + 7\ 065) = (300 \times 600) \times 300 + (7\ 065) \times 50$

$\Rightarrow\ \bar{y} = 290mm\ < 300\ ok.$

- Compute the moment of inertia for the transformed section:

$I_{N.A.} = \left[\left(300 \times \frac{600^3}{12}\right) + (300 \times 600 \times 10^2)\right] + 7\ 065\ \times (290 - 50)^2$

$I_{N.A.} = 5.82 \times 10^9\ mm^4$

- Use the flexure formula to compute the cracking moment:

$M_{Crack} = \frac{f_r \times I_{N.A.}}{c}$

$f_r = 0.62\sqrt{f_c'} = 0.62\sqrt{25} = 3.1\ MPa$

$M_{Crack} = \frac{3.1 \frac{N}{mm^2} \times (5.82 \times 10^9 mm^4)}{290mm}\ 62.2 \times 10^6\ N.mm = 62.2\ kN.m$

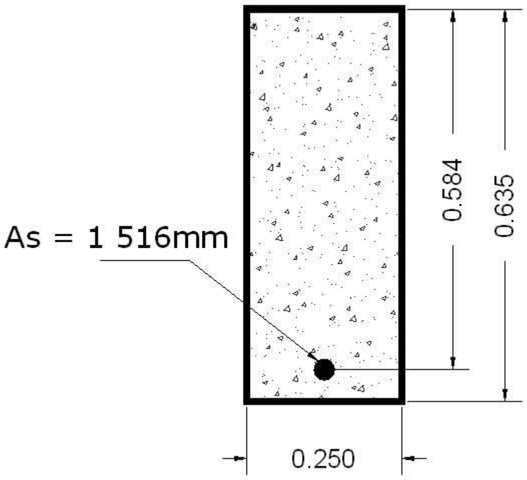
- Compute the P_{Crack}:

$M_{Crack} = \frac{P_{Crack} \times L}{4} \Rightarrow P_{Crack} = \frac{62.2\ kN.m\ \times 4}{6m} = 41.5\ kN$■

4.2.2.2.3 Home Work for Article 4.2.2.2.2: Analysis of Uncracked Elastic Section

Problem 4.2-1

A rectangular beam with dimensions of b= 250mm, h = 635mm, and d = 584mm. The f_c' = 28 MPa, f_y = 400 MPa, and E_{Steel} = 200,00 MPa. Check the state of section and determine the stresses caused by a bending moment of M = 61 kN.m.



Hint:

Start your solution with assumption that the section under a moment of 61 kN.m stills within the 1st Stage. This assumption should be checked later.

Answers:

$f_{c\,Ten.} = 3.04\ MPa < f_r$

Then, the section is uncracked elastic section.

$f_{c\,Comp.} = 3.37\ MPa$

$f_{Steel} = 20.2\ MPa$

4.2.2.2.4 Second Stage: Elastic Cracked Section

- When the load is further increased, the tension strength of the concrete is reached.
- Tension cracks develop and propagate quickly upward to or closed to level of the neutral plane, which in turn shift upward with progressive cracking.
- In ***well-designed beams the width of these cracks is so small*** (***hairline cracks***) that they are not harmful from the view point of the either corrosion protection or appearance ***(i.e. in current design philosophy, the design is based on permitting of hairline cracks***).
- In cracked section, the concrete does not transmit any tension stresses. Hence the steel is called upon to resist the entire tension. If the concrete stress do not exceeded approximately $f_c'/2$ and the steel stress has not reached the yield point, stresses and strains continue to be closely proportional. Then the distribution of strains and stresses are as shown below.

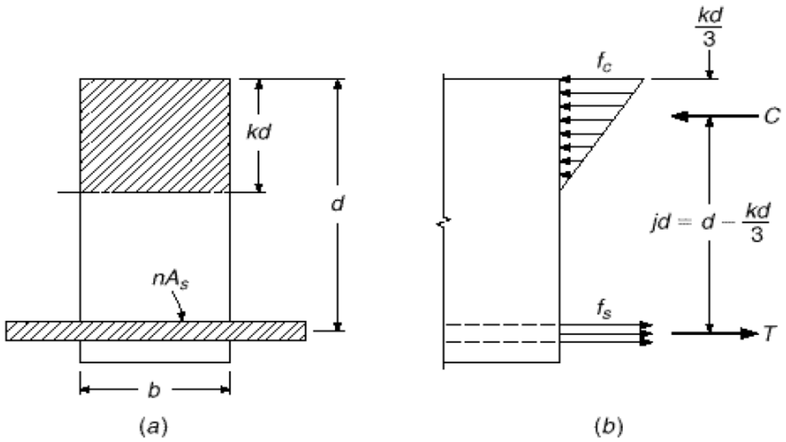


Figure 4.2-5: Strain and stress distribution during elastic cracked stage.

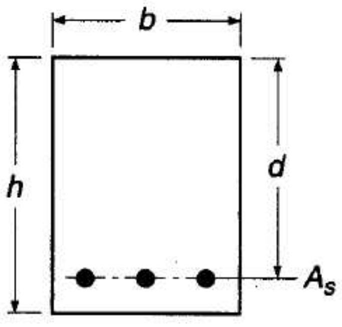
- This situation (elastic cracked section) ***generally occurs in structures under normal service loads (unfactored loads)***.
- The stresses and strains in the elastic cracked section can be computed based on transform the steel to an equivalent concrete, and then use the conventional flexure formula $f = M.y/I$.

Example 4.2-2

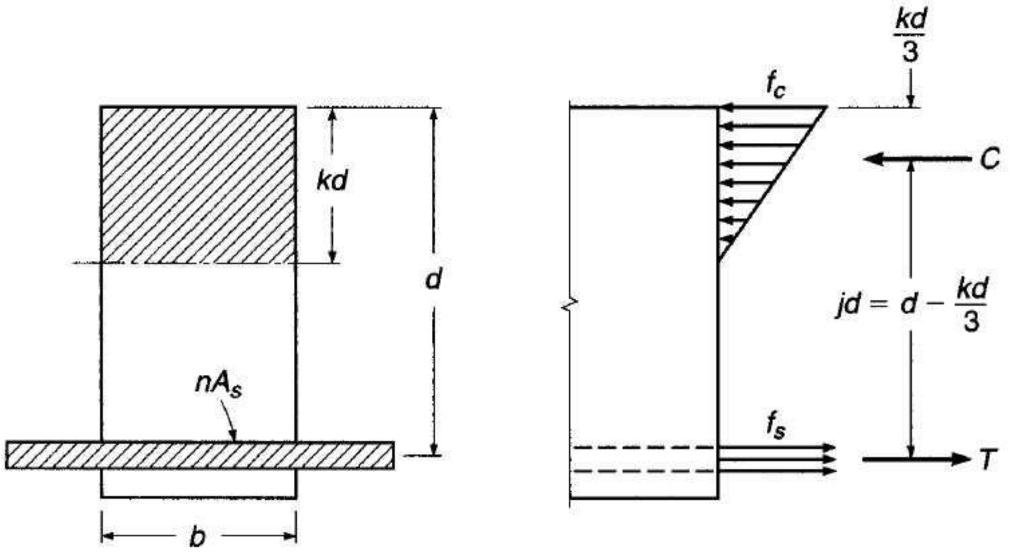
Resolve Example 4.2-1, to compute the maximum magnitude of the load P for the cracked elastic section (assuming that the elastic limit of concrete is equal to $f_c'/2$).

Solution

- As was discussed in the mechanics of materials the application of the conventional flexure formula ($f = M.c/I$) is based on the assumption of homogenous and linear section.
- Linearity of section is assured for concrete and will be assumed for the steel (and should be checked later). While homogeneity of section will be assured through the transformed section concept. That based on transformation of original nonhomogeneous original section shown



- In to the equivalent homogenous section shown below:



Then

$A_S = \left(\pi \frac{20^2}{4}\right) \times 3 = 942\ mm^2$

$\because E_S = 200\ 000\ MPa, and\ E_C = 4700\sqrt{f'_c} = 4700\sqrt{25} = 23\ 500\ MPa,$

$\therefore n \approx 8.5$

$\therefore nA_s = 8\ 007\ mm^2$

- Application of flexure formula:
  As the section is transformed to a homogenous one, then the flexure formula can now be applied:
  - Compute kd
    As the N.A. passes through the section centroid:
    $$\sum_{i=1}^{2} M\ of\ Area\ about\ N.A. = 0\ \Rightarrow (300 \times kd) \times kd/2\ = 8007mm^2 \times d(1-K)$$
    $$k^2 + 0.0979k - 0.0979 = 0$$
    $$k = \frac{-0.0979 \mp \sqrt{(0.0979^2 + 4 \times 0.0979}}{2 \times 1} = 0.267,\ \ kd = 147mm$$
  - Compute IN.A.
    $$I_{N.A.} = \frac{kd^3 \times b}{3} + nAs \times (d - kd)^2$$
    $$I_{N.A.} = \frac{147^3 \times 300}{3} + 8007 \times (550 - 147)^2 = 1.62 \times 10^9 mm^4$$
  - Compute M
    $$\because f_c = \frac{(f'_c)}{2} = \frac{M.c}{I_{N.A.}}$$
    $$\therefore M = \frac{25}{2} \times \frac{1.62 \times 10^9}{147mm} = 137.7\ kN.m$$
    $$\because M = \frac{PL}{4},\qquad \therefore P = M.\frac{4}{L} = \frac{137.7 \times 4}{6} = 91.8\ kN$$
- Check the assumption of $f_s \le f_y$ as assumed:
  $$f_s = \frac{M.c}{I} \times n = \frac{137.7 \times 10^6 \times (550 - 147)}{1.62 \times 10^9} \times 8.5 = 291MPa$$
  $\because f_s = 291\ MPa < 400\ MPa\ \therefore ok.$
  Then the assumption of $f_s \le f_y$ is correct and the solution that based on it is a final solution.
  $\therefore P = 91.8\ kN$ ■

Example 4.2-3

Show that the neutral axis of cracked elastic reinforced concrete section with rectangular shape under flexure stress can be located based on the following relation:

$$k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n$$

where

ρ is reinforcement ratio that defined as follows:

$$\rho = \frac{A_s}{bd}$$

Solution

As the neutral axis for an elastic beam passes through the centroid of its cross sectional area, then:

$$\sum_{i=1}^{2} \text{M of Area about N.A.} = 0 \Rightarrow (b \times kd) \times kd/2 = nA_S \times d(1 - K)$$

$$\left[\left(\frac{bd^2}{2}\right)k^2 + (nA_S d)k - nA_S d = 0\right] \div d$$

$$\left[\left(\frac{bd}{2}\right)k^2 + (nA_S)k - nA_S = 0\right] \div bd$$

$$\left(\frac{1}{2}\right)k^2 + (n\rho)k - n\rho = 0$$

Quadratic formula can be used to solve the above quadratic equation[1]:

$$k = \frac{\left(-n\rho \pm \sqrt{(n\rho)^2 + 4 \times \frac{1}{2} \times n\rho}\right)}{2 \times \frac{1}{2}}$$

As the negative distance has no meaning in our case, then the answer will be in terms of positive root:

$$k = \left(-n\rho \pm \sqrt{(n\rho)^2 + 2n\rho}\right)$$

As the dimension factor k cannot be a negative value, then the actual root for above equation will be:

$$k = \sqrt{(n\rho)^2 + 2n\rho} - n\rho \quad \blacksquare$$

Example 4.2-4

Relocate the neutral axis of Example 4.2-2, based on the general relation that has been derived in Example 3.

Solution

$$n = 8.5$$

$$A_S = 942 \text{ mm}^2$$

$$\rho = \frac{A_s}{bd} = \frac{942 \text{ mm}^2}{550 \times 300 \text{ mm}^2} = 5.71 \times 10^{-3}$$

$$n\rho = 0.0485$$

$$k = \sqrt{(0.0485)^2 + 2 \times 0.0485} - 0.0485 = 0.267 \quad \blacksquare$$

---

[1] Quadratic equation is a equation that has the following general form:

$$ax^2 + bx + c = \quad where\ a \neq 0$$

This equation can be solved based on the *Quadratic Formula*:

$$x = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a}$$

Example 4.2-5 Analysis of Working Stresses in a Reinforced Concrete Beam with General Shape

The simply supported beam shown in Figure 4.2-6 below has the following data: $f_{c\,allowable} = 7\,MPa$, $f_{s\,allowable} = 124\,MPa$, and $n = 12$.

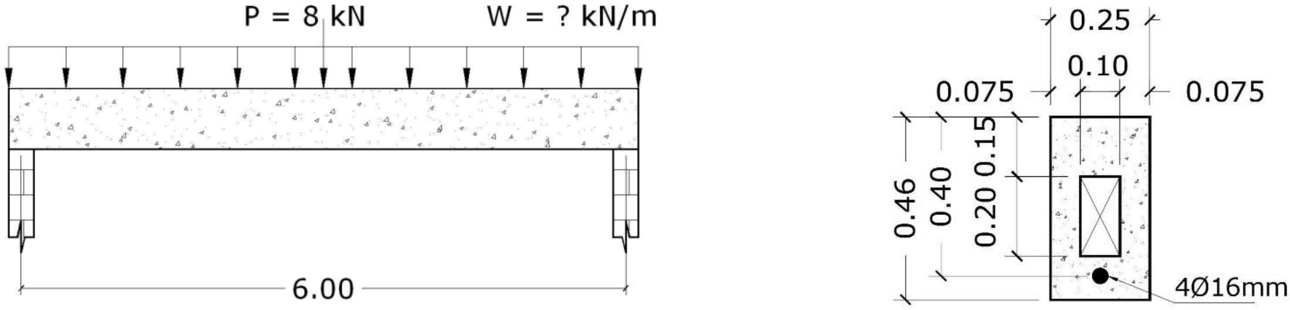


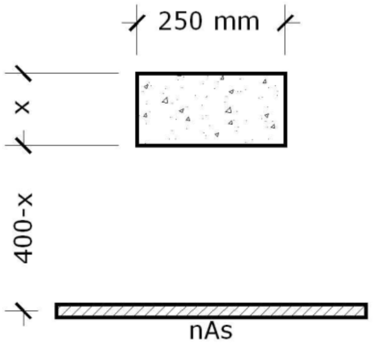
Figure 4.2-6: Simply supported beam for Example 4.2-5.

Compute the following:

- Section allowable bending moment.
- Total uniform load (including beam selfweight) that could the beam carry in addition to a concentrated load of 8 kN at mid-span.

Solution

- Compute the section moment of inertia of transformed section:
  - Assume that the neutral axis to be above the hollow section.



$$A_s = 4 \times \frac{\pi 16^2}{4} = 804\ \text{mm}^2 \quad \Rightarrow nA_s = 12 \times 804\text{mm}^2 = 9\,648\text{mm}^2$$

$$\sum \text{Area Moment about N.A.} = 0$$

$$250\text{mm} \times x \times \frac{x}{2} = 9\,648\text{mm}^2 \times (400 - x)$$

$$x^2 + 77.2x - 30\,874 = 0$$

$$x = \frac{-77.2 \pm \sqrt{77.2^2 + 4 \times 1 \times 30\,874}}{2 \times 1} = 141\text{mm}$$

$$I_{N.A.} = \frac{141^3 \times 250}{3} + 9\,648 \times (400 - 141)^2 = 881 \times 10^6\text{mm}^4$$

  - Compute the allowable bending moment based on concrete allowable stresses:

$$F_{c\,\text{allowable}} = \frac{M_{\text{allowable}} \cdot c_{\text{Top}}}{I_{N.A.}} \Rightarrow M_{allowable} = \frac{7MPa \times 881 \times 10^6 mm^4}{141mm} = 43.7\ kN.m$$

  - Compute the allowable bending moment based on steel allowable stresses:

$$F_{s\,allowable} = n\frac{M_{allowable} \cdot c_{Bottom}}{I_{N.A.}} \Rightarrow M_{allowable} = \frac{124MPa \times 881 \times 10^6 mm^4}{12 \times (400 - 141)mm}$$
$$= 35.1\ kN.m$$

  - Compute the allowable bending moment:

$$M_{\text{allowable}} = \text{Minimum of } 43.7\ \text{kN.m and } 35.1\ \text{kN.m} = 35.1\ \text{kN.m} \blacksquare$$

- Compute the allowable total load:

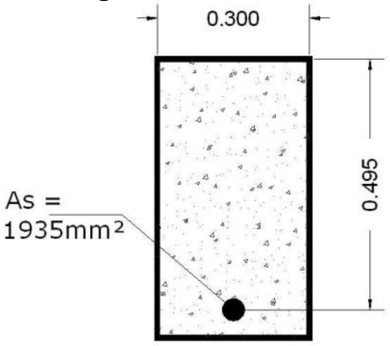
$$M = \frac{WL^2}{8} + \frac{PL}{4} \Rightarrow 35.1\ kN.m = \frac{W\frac{kN}{m} \times 6^2 m^2}{8} + \frac{8kN \times 6m}{4}$$

$$W = 5.13\frac{kN}{m} \blacksquare$$

4.2.2.2.5 Home Work of Article 4.2.2.2.4: Analysis of Working Stresses in Beams with Rectangular Sections

Problem 4.2-2

For the beam shown below if the $E_s = 200\,000\,MPa$, $E_C = 20\,000\,MPa$, $f_c' = 21MPa$, and $f_y = 400\,MPa$, determine the maximum stresses in the steel and concrete if the applied bending moment is 115kN.m.



Answers

$k = 0.396\ kd = 196\,mm\ I_{N.A.} = 2.48 \times 10^9 mm^4$

$f_c = 9.09\,MPa < \frac{f_c'}{2}\ Ok.$ ■ $f_s = 139\,MPa < f_y\ Ok.$ ■

Problem 4.2-3

What is the maximum allowable bending moment for the beam of Problem 1, if the maximum allowable stresses are $f_s = 152MPa$ and $f_C = 8.33\,MPa$.

Answers

$M_{Allowable\ Based\ on\ Steel\ Allowable\ Stresses} = 126\ kN.m$
$M_{Allowable\ Based\ on\ Concrete\ Allowable\ Stresses} = 105\ kN.m$
$M_{Allowable} = Minimum\ (126, 105) = 105\ kN.m$ ■

Problem 4.2-4

What is the maximum allowable bending moment for the beam of Problem 1, if the maximum allowable stresses are $f_s = 132\,MPa$ and $f_C = 9.33\,MPa$.

Answers

$M_{Allowable\ Based\ on\ Steel\ Allowable\ Stresses} = 109\ kN.m$
$M_{Allowable\ Based\ on\ Concrete\ Allowable\ Stresses} = 118\ kN.m$
$M_{Allowable} = Minimum\ (109, 118) = 109\ kN.m$ ■

Problem 4.2-5

A concrete beam shown below has a simple span of 5m. It has $f_c = 9.00MPa$, $f_s = 124MPa$ and n = 10.

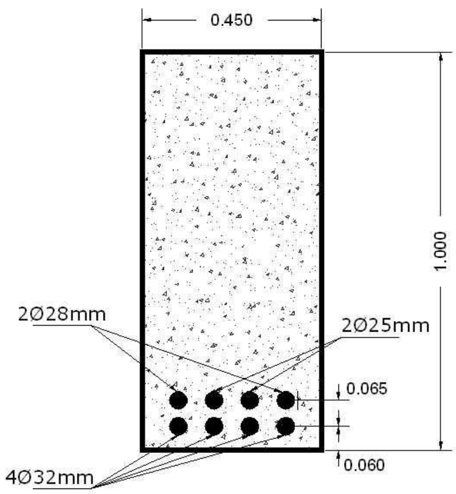
For this beam compute the following:

- Section allowable bending moment.
- Value of concentrated force "P" that the beam could carry at it midspan.

Notes on Problem 4.2-4

Sometimes, beam width is not sufficient to put the required reinforcement in a single layer, and then the reinforcement is put in two or more layers.

For analysis purposes, theses layers are usually replaced with a single layer that has an area equal to area of all layers and located at centroid of steel layers.



Answers

- Section allowable bending moment.

  $A_{of\ Rebar\ 25mm} = 490\ mm^2$
  $A_{of\ Rebar\ 28mm} = 615\ mm^2$
  $A_{of\ Rebar\ 32mm} = 804\ mm^2$
  $\bar{y}_{Measured\ from\ reinforcement\ center\ to\ beam\ lower\ face} = 86.5\ mm < 92.5\ mm\ Ok.$
  $d = 913\ mm$
  $As = 5\ 426\ mm^2$
  $\rho = 13.2 \times 10^{-3}$
  $n\rho = 0.132$
  $k = 0.398$
  $kd = 364\ mm$
  $nA_s = 54\ 260\ mm^2$
  $I_{N.A.} = 23.6 \times 10^9 mm^4$
  $M_{Allowable\ Based\ on\ Steel\ Allowable\ Stresses} = 533\ kN.m$
  $M_{Allowable\ Based\ on\ Concrete\ Allowable\ Stresses} = 584\ kN.m$
  $M_{Allowable} = 533\ kN.m$ ■
- Value of concentrated force "P" that the beam could carry at it mid-span.

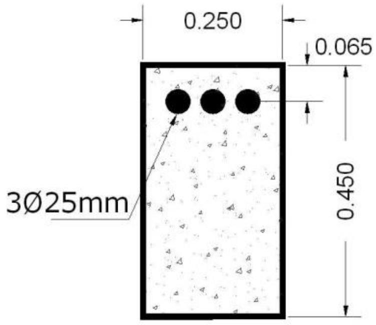
  $W_{Selfweight} = 10.8 \frac{kN}{m}$ $P = 399\ kN$ ■

Problem 4.2-6

The figure shown below is a cross-section of a cantilever beam supporting a uniform load of $2.5 \frac{kN}{m}$ including its own weight and a concentrated load of $30\ kN$ at its free end. It has $f_c = 8.00\ MPa$, $f_s = 124\ MPa$, and $n = 10$.

For this beam compute the following:

- The safe resisting moment of the beam.
- The maximum beam span.



Answers

- The safe resisting moment of the beam:

  $d = 385\ mm$ $As = 1\ 470\ mm^2$ $\rho = 15.1 \times 10^{-3}$ $n\rho = 0.151$
  $k = 0.419$ $kd = 163\ mm$ $nA_s = 14\ 700\ mm^2$ $I_{N.A.} = 1.12 \times 10^9 mm^4$
  $M_{Allowable\ Based\ on\ Steel\ Allowable\ Stresses} = 61.2\ kN.m$
  $M_{Allowable\ Based\ on\ Concrete\ Allowable\ Stresses} = 55.0\ kN.m$
  $M_{Allowable} = Minimum\ (61.2, 55.0) = 55.0\ kN.m$ ■
- The maximum beam span.

  $L = 1.71\ m$ ■

4.2.2.2.6 Third Stage: Flexure Strength

- When the load is still further increased, flexure strength of the beam is reached. Failure can be caused in one of the following two ways:
  - ***SECONDARY COMPRESSION FAILURE (TENSION-CONTROLLED SECTION):***
    - When relatively ***moderate amount of reinforcement are employed, at some value of the load the steel will reach its yield point***.
    - At that stress the reinforcement yields suddenly and stretched a large amount, and the tension cracks in the concrete widen visibly and propagate upwards, with simultaneous significant deflection of the beam. When this happens, the strains in the remaining compression zone of the concrete increase to such degree that crushing of the concrete. Then the stresses history will be as shown in the figures below:

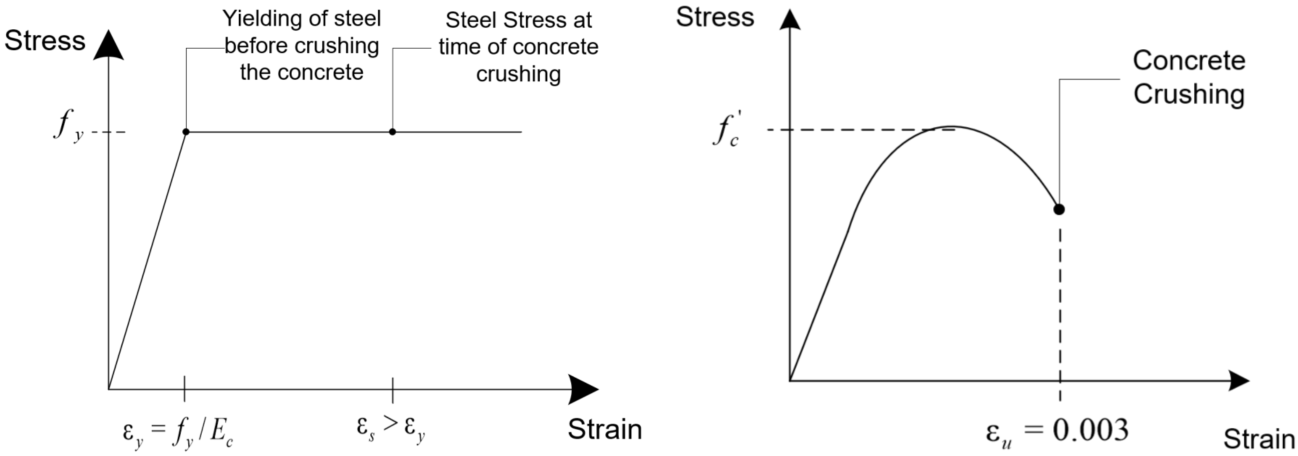


Figure 4.2-7: Stress –strain state for secondary compressive failure.

  - ***COMPRESSION FAILURE (COMPRESSION CONTROLLED SECTION)*:**
    - On the other hand, ***if large amount of reinforcement or normal amount of steel with very high strength are employed, the compression strength of the concrete may be exhausted before the steel start yielding***.
    - Then the stresses history during the compression failure will be as shown below:

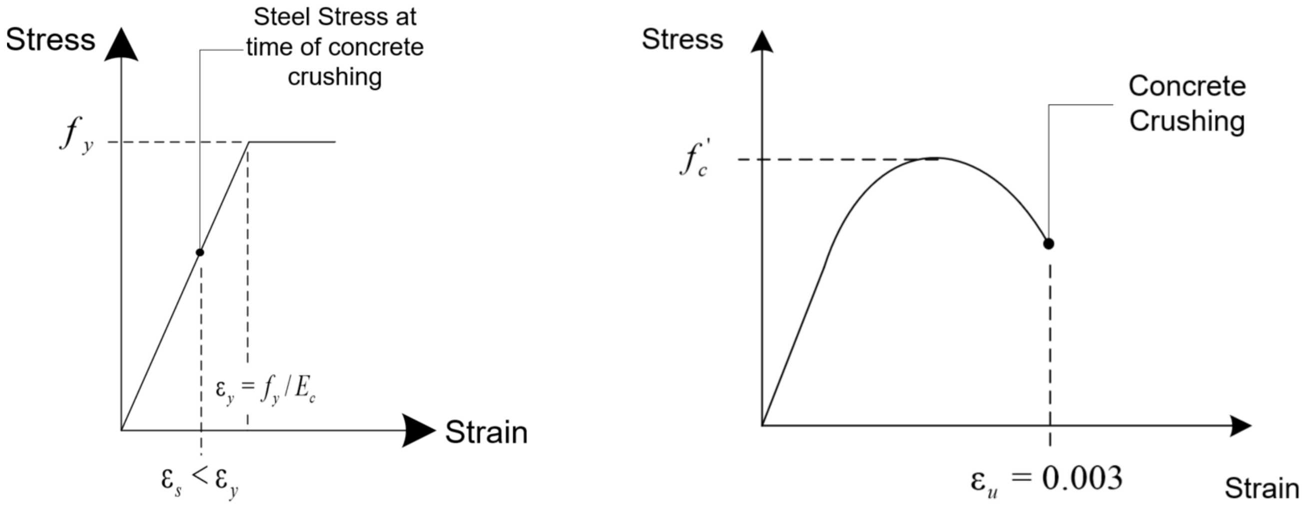


Figure 4.2-8: Stress –strain state for compressive failure.

    - Compression failure is sudden, of an almost explosive nature, and occurs without warning.
    - For this reason, (ACI318M, 2014), ***Article 21.2.1***, required to dimension beams in such a manner that should they be overloaded, failure would be initiated by yielding of the steel rather than by crushing of concrete (*Secondary Compression Failure*).

4.2.2.2.7 Nominal Flexure Strength M_n of a Rectangular Section with Secondary Compression Failure.

- It is clear that at or near ultimate loads, stresses are no longer proportional to the strain, then the conventional flexure formula ($f = \frac{M.c}{I}$) cannot be applied for the analysis and design of the section.
- And the analysis and design of the section must be based on the direct application of the basic principles (compatibility relation, stress-strain relation, and the equilibrium conditions) and as follow:
  - **COMPATIBILITY CONDITIONS**
    Based on
    - The ***kinematic assumption*** of the ***plane section before loading remain plane after loading***. This assumption is adopted by (ACI318M, 2014) in ***article 22.2.1.2***.
    - The ***assumption of the secondary compression failure*** (i.e., the failure starting with the yielding of the steel and the crushing of concrete). According to (ACI318M, 2014), ***article 22.2.2.1***, concrete crushing occurs when maximum strain at the extreme concrete compression fiber reaches a value of $\epsilon_u = 0.003$.

    the strain distribution will be as shown below:

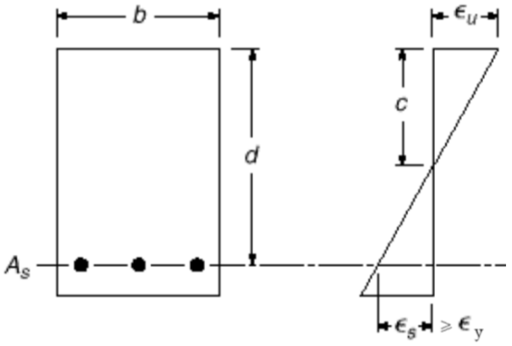


Figure 4.2-9: Strain distribution.

    - The kinematic assumption remains applicable even when materials behave inelastically, (Popov, 1968).
  - **STRESS-STRAIN RELATION:**
    - Based on the actual stress-strain relations of the concrete and reinforcing steel, the stress distribution will be as shown below:

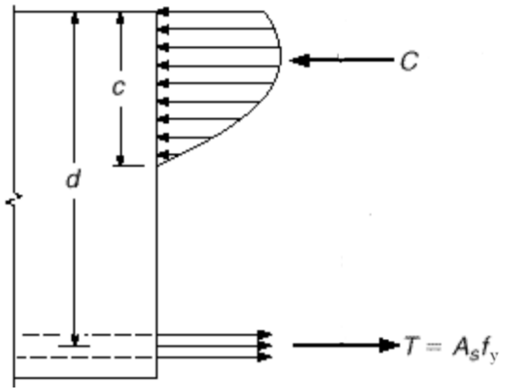


Figure 4.2-10: Stress distribution for a beam with secondary compression failure.

    - It is not really necessary to know the shape of the concrete stress distribution. What is necessary to know is, (ACI318M, 2014) ***article 22.2.2.3***:
      - The total resultant compression force "*C*" in the concrete.
      - Its vertical location.

      Evidently, then, one can think of the actual complex stress distribution as replaced by a fictitious one of some simple geometric shape, provided that this fictitious distribution results in the same total compression force "C" applied at the same location as in the actual member when it is on the point of failure.

- Historically, a number of simplified, fictitious equivalent stress distributions have been proposed by investigators in various countries. The one generally accepted was first proposed by ***C. S. Whitney*** and was subsequently elaborated and checked experimentally by others. The actual stress distribution immediately before failure and the fictitious equivalent distribution are shown in Figure below, (ACI318M, 2014), ***article 22.2.2.4***,

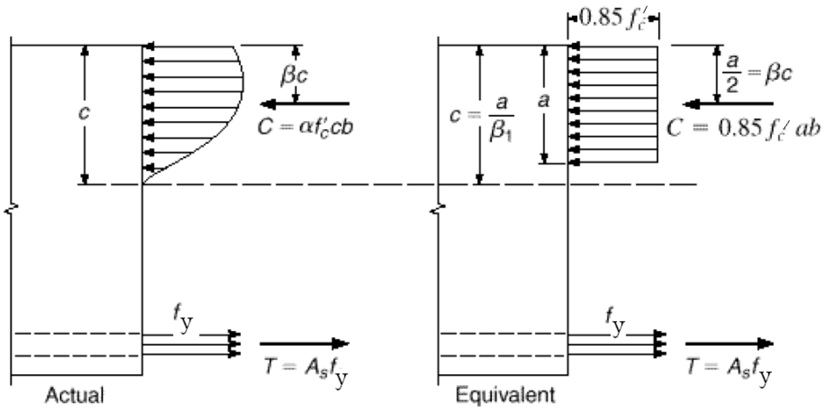


Figure 4.2-11: Whitney simplified equivalent stress distributions.

- According to ACI code (**22.2.2.4.3**) β_1 can be computed based on Table below:

Table 4.2-1: Values of β1 for equivalent rectangular concrete stress distribution, Table 22.2.2.4.3 of (ACI318M, 2014).

f_c', MPa	β_1	
$17 \le f_c' \le 28$	0.85	(a)
$28 < f_c' < 55$	$0.85 - \frac{0.05(f_c' - 28)}{7}$	(b)
$f_c' \ge 55$	0.65	(c)

- **EQUILIBRIUM CONDITIONS:**

  According to (ACI318M, 2014), ***article 22.2.1.1***, equilibrium shall be satisfied at each section:

  $$\because \sum F_x = 0 \implies 0.85 f_c' ba = A_s f_y$$

  $$\therefore a = \frac{A_s f_y}{0.85 f_c' b}$$

  $$\because \sum M_{\text{About Centroid of Compressive Force C}} = 0$$

  $$\therefore M_n = A_s f_y \left(d - \frac{a}{2}\right)$$

  Substitute the value of "a" into above equation:

  $$M_n = A_s f_y \left(d - \frac{1}{2}\,\frac{A_s f_y}{0.85 f_c' b}\right) \quad \text{or } M_n = A_s f_y d \left(1 - \frac{1}{2}\,\frac{A_s f_y}{0.85 f_c' bd}\right)$$

  Let $\rho = \frac{A_s}{bd}$

  $$M_n = \rho f_y bd^2 \left(1 - 0.59\,\frac{\rho f_y}{f_c'}\right) \quad \blacksquare$$