كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة باللغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : **Mathematical Analysis**

اسم الحاضرة الثانية باللغة العربية: مبرهنات حول الاعداد الحقيقية

اسم المحاضرة الثانية باللغة الإنكليزية: **Theorems of real numbers**

Proposition:
If $\emptyset \neq S \subset R$ and $\sup(S) = M$, then $\forall p < M\ \ \exists x \in S$ s.t $p < x \leq M$
i.e.: if $\sup(S) = M$ then $\forall \epsilon > 0$, $\exists x \in S$ s.t $M - \epsilon < x \leq M$
proof:
Suppose that $\sup(S) = M$ then $\forall x \in S,\ x \leq M$
T.P $\forall x \in S$, $p < x$?
Let $x \leq p$, $\forall x \in S$
$\rightarrow$ p is upper bounded for S, but by hypothesis $p < M = \sup(S)$
………. C!
$\therefore\ \exists\, x \in S \ni p < x \leq M$.

Theorem: The set N of natural numbers is unbounded above in R
Proof:
Suppose N is bounded above.
By completeness axiom
N has a supreme M
Let $\sup(N) = M$
From proposition above $\exists n \in N$ s.t $M - 1 < n < M$.
Then $M - 1 < n \rightarrow M < n + 1$,
But $n + 1 \in N$
And $n + 1 > M = \sup(N) \rightarrow C!$
Therefore, N is unbounded above

Theorem: Archimedan property
If $x \in R^{++}$ then for any $y \in R$, there exists $n \in N$ s.t $n > y$

Detention: let F a field, F is called Archimedean filed, if for any $x \in F$, $\exists n\ \in N$ s.t $n > x$
i.e.: N is abounded above in F
Example:
1. R is Archimedean field

2. **Q is Archimedean field**
3. $s = \{a + b\sqrt{2} : a, b \in Q\}$ **is Archimedean field**

Theorem: Denseness property

Between any two distinct reals, there exists infinitely many rationales and irrationals

Detention: (irrational numbers Q')

Let Q' be a complement of Q in the real number R.

i.e.: $Q' = R - Q$, **we called is set of irrational numbers**

remark: $R = Q \cup Q'$

Theorem: prove that $\sqrt{2}$ **is irrational number**

i.e.: There are no rational numbers whose square is 2

i.e.: $\nexists\ x \in Q \ni x^2 = 2$

proof:

suppose $\sqrt{2}$ **is rational number i.e.** $\sqrt{2} = \frac{m}{n}$

So $2 = \frac{m^2}{n^2}$, **then** $m^2 = 2n^2$

Case 1:

m and n are odd.

Since m is odd $\rightarrow m^2$ **is odd**

Since n is odd $\rightarrow n^2$ **is odd**

But $2n^2$ **is even** $\rightarrow\ m^2 = 2n^2 \rightarrow C!$

Case 2:

m is even and n is odd, then $m = 2p$

and $m^2 = 4p^2$, $\rightarrow 4p^2 = 2n^2 \rightarrow 2p^2 = n^2 \rightarrow C!$

Case 3:

m is odd and n is even, then, since m is odd

$\rightarrow m^2$ **is odd, and** $2n^2$ **is even** $\rightarrow m^2 = 2n^2 \rightarrow C!$

$\therefore\ \sqrt{2}$ **is irrational number**

Theorem: Q is not Complete field

Theorem: for every real $x > 0$ **and every integer** $n > 0$ **there is one and only one positive real y such that** $y^n = x$

i.e.: $\forall x > 0, \forall n \in N, \exists!, y \in R^+ \ s.t \ y = \sqrt[n]{x}$

Theorem: if $\frac{m}{n}$ and $\frac{p}{q}$ are rationales and $q \neq 0$ then $\frac{m}{n} + \sqrt{2}\frac{p}{q}$ is irrational number

***Proof*:** Suppose $\frac{m}{n} + \sqrt{2}\ \frac{p}{q}$ is rational

Then there is $r, s \in Z, s \neq 0 \ s.t \ \frac{m}{n} + \sqrt{2}\frac{p}{q} = \frac{r}{s}$

So $\sqrt{2}\frac{p}{q} = \frac{r}{s} - \frac{m}{n} \rightarrow \sqrt{2} = \frac{p}{q}\left(\frac{rn-sm}{sn}\right) \in Q$

So $2 = \left(\frac{q(nr-sm)}{psn}\right)^2 \rightarrow !$ with theorem: $\nexists\ x \in Q \ni x^2 = 2$

Theorem: Between any two distinct rationales there is an irrational number.

Example:

1. Prove $x^2 \geq 0, \forall x \in R$
2. Let a, b be tow real s.t $a \leq b + \epsilon \ \forall \epsilon > 0$ then $a \leq b$

   Proof (2):

   Suppose $a > b$. Then $a + a > b + a$

   $\frac{2a}{2} > \frac{b+a}{2}$

   $a > \frac{b+a}{2}$ ………….(1)

   Take $\epsilon = \frac{a-b}{2} > 0$ (Since $> b$, then $a - b > 0 \rightarrow \frac{a-b}{2} > 0$)

   $a \leq b + \epsilon \rightarrow a \leq b + \frac{a-b}{2} = \frac{2b+a-b}{2} = \frac{a+b}{2} < a$

   From (1) ……………. C!

   $a \leq b$

Example 1.3:

1. Q is order field $(A_1 \rightarrow A_{14})$
2. C is field but not order

   since: if $x = 1 \rightarrow x = \sqrt{1} \rightarrow x^2 = -1 < 0 \rightarrow C!$

   since: $(x^2 \geq 0, \forall x \in R)$