كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : **Mathematical Analysis**

اسم الحاضرة الثامنة باللغة العربية: المتتابعات الكوشية

اسم المحاضرة الثامنة باللغة الإنكليزية: **Caushy sequences**

Example: find a sub Seq. of the following seq.

1. $< x_n > = < \sqrt{n} >$

   Solution:

   $< \sqrt{n} > = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$

   Let $< m > = < 2n >$ **increasing Seq. in N, the Sequence is**

   $$< Xm > = < \sqrt{2n} > = \sqrt{2}, \sqrt{4}, \sqrt{6}, \ldots$$

   Let $< m > = < n + 3 >$ **increasing seq in N, the sub seq is**

   $$< m > = < \sqrt{n+3} > = \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$$

Theorem: Let $< x_n >$ **be a convergent Seq and** $\lim_{n\to\infty} X_n = x$ **then the sub seq** $< X_{nm} >$ **also conv. To** x, **where** $n \to \infty$

Proof:

Since $x_n \to x, \forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x) < \epsilon, \forall n > k$

Choose $nr > k$, **then** $\forall m > r \to nm > nr > k$

$\Longrightarrow d(x_{nm}, x) < \epsilon, \forall nm > k$

$\Longrightarrow < x_{nm} > \to x.$

Definition: Let (X, d) **be a metrices space and** $< x_n >$ **be a seq. in** X **we say that**

$< x_n >$ **is a principle. (Caushy) seq. if** $\forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x_m) < \epsilon, \forall n, m > k.$

Example: prove that $< \frac{1}{n} >$ **is Caushy seq in R?**

Solution: $\forall\ \epsilon > 0$, **to find** $k \in N$ **s.t** $d(x_n, x_m) < \epsilon, \forall n, m > k, \forall n, m > k.$

Let $m > n \to d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

Since $\epsilon > 0$ (by Arch. Prop) $\rightarrow\ \exists k \in N$ s.t

$k\epsilon > 2 \rightarrow \frac{2}{k} < \epsilon$

$\forall\ n > k\ , d(x_n, x_m) = |x_n - x_m| < \frac{2}{n} < \epsilon, \forall n, m > k \rightarrow < X_n >$ is Caushy seq.

Theorem: I metric space (X, d), every Convergent seq. is Caushy.

Remark: The Converse of the above theorem. Is not true by the following example.

Example: Let $X = IR^{++}$ positive numbers $d(x, y) = |x - y|, \forall x, y \in R^{++}, \forall n > k$.

$< x_n > = < \frac{1}{n} >$ is Caushy seq.

But $\frac{1}{n} \rightarrow 0 \notin R^{++}$

$\therefore < \frac{1}{n} >$ is not Conv

Theorem: In metric Space (x, d) every Caushy seq. is bounded.

Example 3.11: Let $< x_n > = (-1)^n$ be a seq.

$< x_n >$ is bounded seq, but not Caushy Seq

Since $d(-1, 1) = 1 < \epsilon, \forall\ \epsilon > 0$

If $\epsilon = \frac{1}{2} \rightarrow 2 < \frac{1}{2} \rightarrow C!$

Theorem: For any real number $r, \exists$ rational Caushy Seq $< x_n >$ Conv to r.

Definition: Let (X, d) be a metric space we say that X is Compete. If every Cauchy Seq. In X coverage to a point in X.

i.e.: X is complete. If $\forall < X_n >$ Cauchy Seq. $\rightarrow \exists \overline{x} \in X \; s.t \; X_n \rightarrow X$.

Theorem: Cantor's theorem for Nested sets.

Proof:

Let (X, d) be a Complete matric Space and $< E_n >$ be a seq of closed bounded Subset of X such that $E_1 \supset E_2 \supset \cdots E_n \supset E_{n+1} \; \forall n$ and the Sequence of Positive numbers $< daim \, E_n > \rightarrow 0$, then $\cap \, E_n =$ Singleton point

Remark: The condition of closed sets of Cantor's theorem is necessary.

Example: Let $E_n = \left(0 \, , \frac{1}{n}\right)$ be the open intervals, $E_{n+1} \subset E_n$, and $daim(E_n) = \frac{1}{n} \rightarrow 0, \forall n$ E_n is bounded and not closed. Prove that $\cap \, E_n = \emptyset$

Proof:

Suppose $\cap \, E_n \neq \emptyset \rightarrow \; \exists r \in E_n \; s.t$

$r \in \left(0 \, , \frac{1}{n}\right), \forall n$

Since $r > 0$, by Arch.pvop ,$\exists k \in N \; s.t$

$kr > 1 \rightarrow \frac{1}{k} < r \rightarrow C!$

➔ $\cap \, E_n = \emptyset$

Corollary: Let $< \pm n >$ be aseq of closed intervals, $I_n = [a_n, b_n]$ such that

1. $I_n \supset I_{n+1}$
2. $\lim_{n \rightarrow \infty} |I_n| = 0$, then $\cap \, I_n =$singleton Point

Theorem: R^n **is Complete metric Space,** $n \geq 1$

i.e.: (Every Cauchy sequence in R^n **is Convergent)**

Theorem: **Let** $< X_n >, < Y_n >$ **and** $< Z_n >$ **real Sequence s.t** $\forall n, X_n \leq Y_n \leq Z_n$ **and**

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} Z_n = a \text{ then } \lim_{n\to\infty} Y_n = a$$

Theorem: **let** $< X_n >$ **be a real sequence such that** $< X_n >$ **Converge to 0 and**

$$X_n \geq 0, p > 0 \text{ then } < X_n^p > \text{ converges to } 0$$

Proof:

$< X_n^p > = x_1^p, x_2^p, x_3^p, \ldots$

Since $< X_n > \to 0 \to \forall \epsilon > 0, \exists k \in N\ s.t$

$|X_{n-0}| = |X_n| < \epsilon^p, \forall n > k$ **and**

$|X_n . X_n \ldots . X_n| = |X_n||X_n| \ldots . . |X_n| = |X_n|^p < \left(\epsilon^{\frac{1}{p}}\right)^p, \forall n > k$

$< X_n^p > \to 0$.