كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : **Mathematical Analysis**

اسم الحاضرة الثالثة باللغة العربية: الفضاء المتري

اسم المحاضرة الثالثة باللغة الإنكليزية: **Metric Space**

Definition: let X be anon-empty set and $d: X \times X \to R^+$ be a mapping. We say that order (X, d) is metric space if it is satisfying the following:

1. $d(x, y) \geq 0 , \forall x, y \in X$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$
4. $d(x, y) = 0 \leftrightarrow x = y$

Not: d is called metric mapping $d(x, y)$ is a distance between x and y

Remark: A mapping $d: X \times X \to R^+$ is called a pseudo metric for X iff d satisfies (1,2,3) in the above definition and $d(x, x) = 0 , \forall x \in X$

Cauchy - Shwarz inequality

Let $a = (a_1, a_2, \dots, a_n)$ *and* $b = (b_1, b_2, \dots b_n)$ be two tripe of complex number , then:

$$\sum_{i=1}^{n} |a_i . b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} . \left(\sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}}$$

Minkowskis inequality

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{\frac{1}{p}} \quad , p \geq 1$$

Example: if $X = R$ *and* $d(x, y) = |x - y|$, show that (X,d) is a metric space.

Solution:

1. $d(x, y) = |x - y| \geq 0$ by def. of Absolute value

2. $d(x,y) = |x-y| = |-(y-x)| = |y-x| = d(y,x)$
3. $d(x,z) = |x-z| = |x-y+y-z|$
$\leq |x-y| + |y-z|$
$= d(x,y) + d(y,z)$
4. $d(x,y) = 0$ iff $x = y$
$d(x,y) = 0$ iff $|x-y| = 0$
iff $x - y = 0$
iff $x = y$
$\therefore (X,d)$ is a metric space

Discrete metric space

Let $X \neq \emptyset$ and $d: X \times X \to R$ s.t

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

$\forall x, y$, show that (X,d) is metric space

Solution:

1. $d(x,y) \geq 0$, $\forall x, y \in X$ (by def. d)
2. $d(x,y) = d(y,x)$?
if $x = y \to d(x,y) = 0 = d(x,y)$
if $x \neq y \to d(x,y) = 1 = d(y,x)$
3. Let $x, y, z \in X$ T.P $d(x,y) \leq d(x,y) + d(y,z)$?
if $x = z$ then $d(x,z) = 0$
since $d(x,y) \geq 0$ and $d(y,z) \geq 0$ then
$d(x,z) \leq d(x,y) + d(y,z)$
if $x \neq z$ then $d(x,z) = 0$
since $d(x,z) = 1$ and either $x \neq y$ or $x \neq z$, $y = z$
either: $d(x,z) = d(x,y) = d(y,z) = 1$
or: $d(x,z) = d(x,y) = 1$ and $d(y,z) = 0$
then: $d(x,z) \leq d(x,y) + d(y,z)$
$1 \leq 1 + 1$
$1 \leq 1 + 0$

Example: show that (X, d) is pseudo metric space but not metric where

$d: X \times X \to R$, $d(x, y) = |x^2 - y^2|$, $for all\ x, y \in R$.

Solution:

Let $x, y, z, \in R$

1- $d(x, y) = |x^2 - y^2| \geq 0$, by def Abs. Value

2- $d(x, y) = |x^2 - y^2| = |-(y^2 - x^2)| = |y^2 - x^2| = d(y, x)$

3- $d(x, y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2| \leq |x^2 - z^2| + |z^2 - y^2|$

$\leq d(x, z) + d(z, y)$

4- $d(x, x) = |x^2 - x^2| = 0$, $\forall x \in R$

$\therefore (X, d)$ pseudo metric space but not metric space,

since, if $d(x, y) = 0 \to |x^2 - y^2| = 0 \to x^2 - y^2 = 0 \to x^2 = y^2$

$\to x = y$

ex: let $x = 1$, $y = -1$

then $d(x, y) = d(1, -1) = |1^2 - (-1)^2| = 0$, but $1 \neq -1$

Definition: let (X, d) be a metric space $S, T \subseteq X$, $p \in S$ then

1- The distance between p and S is

$d(p, S) = \inf\{d(p, x) : x \in S\}$

2- The distance between S and T is

$d(S, T) = \inf\{d(x, y) : x \in S , y \in T\}$

3- Diameter of S is $d(S) = \sup\{d(x, y) : x, y \in S\}$

4- S is called bounded, if $\exists\ M \in R^{++}$, s.t $d(x, y) \leq M$, $\forall x, y \in S$.

Definition: let (X, d) be a metric space and $S \subseteq X$, S is called open set, if $\forall x \in S$, $\exists r > 0$ s.t $B(x, r) \subset S$

Example: if $S = \emptyset$, then S is open set

If $x \in S \to \exists r > 0\ s.t\ B(x, r) \subset S$

$F \quad \to \quad F \quad or \quad T \quad : \quad T$

If S = X , then S is open set

Solution: Since all balls is contains in X

Any open interval is open set. But the convers is not true

Solution:

Let $\boldsymbol{x \in s \rightarrow x \in (a,b) \subseteq (a,b) = S.}$

So. S is open set

Example: Let $\boldsymbol{S = (-1,1) \cup (2,3)}$

Suppose that $\boldsymbol{x \in s}$**, then** $\boldsymbol{x \in (-1,1) \; or \; x \in (2,3)}$

Then $\boldsymbol{x \in (-1,1) \subset S \; or \; x \in (2,3) \subset S}$

∴ S is open set. But is not open interval

Theorem: every ball in a metric space is open set.

Proof:

$\boldsymbol{\forall y \in B(x,r), \exists w > 0, s.t \; B(y,w) \subset B(x,r)}$?

Let $\boldsymbol{w = r - d(x,y) > 0}$

Let $\boldsymbol{Z \in B(y,w) \longrightarrow d(z,y) < w}$

$\boldsymbol{d(Z,y) \leq d(x,y) + d(y,z)}$

$\boldsymbol{\leq d(x,y + w}$

$\boldsymbol{= d(x,y) + r - d(x,y)}$

$\boldsymbol{= r}$

Then $\boldsymbol{Z \in B(x,r) \longrightarrow B(y,w) \subset B(x,r)}$

This is true for all y in B(x,r)

So B(x,r) is open set

$\boldsymbol{S = \{x\}, x \in R}$ **is not open set. Since there is not open interval in S Containing x and Contained in S**

i.e (($\boldsymbol{\forall r > 0, \exists B(x,r) = (x - r, x + r) \subset S}$ **))**