كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : Mathematical Analysis

اسم الحاضرة الخامسة باللغة العربية: خواص المجاميع المغلقه والفضاءات المتراصة

اسم المحاضرة الخامسة باللغة الإنكليزية: Properties of closed sets compact spaces

Remark: the infinite union of closed sets is not necessary closed set

Example: let $S_n = \left\{ \left[\frac{-n}{n+1}, \frac{n}{n+1}\right] : n \in N \right\}$, S_n is closed interval, Is $\bigcup_{n=1}^{\infty} S_n$ is closed?

Solution:

If $n = 1 \rightarrow S_1 = \left[\frac{-1}{2}, , \frac{1}{2}\right]$

If $n = 2 \rightarrow S_2 = \left[\frac{-2}{3}, \frac{2}{3}\right]$

. .

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When $n \rightarrow \infty \Longrightarrow \lim_{n\rightarrow\infty} \frac{\pm n}{n+1} = \lim_{n\rightarrow\infty} \frac{\pm\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \pm 1$

$\therefore \bigcup_{n=1}^{\infty} s_n = (-1, 1)$ open set

Theorem: The infinite intersection of closed set S is closed?

Definition: let X be a metric space and $S \subseteq X, p \in X$, p is called an accumulation point of S if every open set contain p , contains another point q s.t $p \neq q$, $q \in S$.

i.e.: p is a cc. point of S if $\forall U$, U is open set $p \in U$, then $U - P \cap S \neq \phi$

Remark: Since every open set is Union balls. So, we can define acc. Point as following:

P is acc. Point of S, if $\forall r > 0$ $B(p, r) - \{p\} \cap S \neq \phi$

- S' is the closure of all acc. Point of S (Derived set)
- $\overline{S}$ is the closure of S and $\overline{S} = S \cup S'$

* **P is not acc. Point, if $\exists U$, U is open and $p \in U$ S.t $U - \{p\} \cap S = \phi$. (i.e. $\exists r > 0$, $B(r, p) - \{ p \} \cap S = \phi$**

Example: let $S = \{1, 5\}$, find S' and $\overline{S}$

Solution: To find S' there are some cases $x = 1$, $x = 5$, $x < 1$, $x > 5$, $1 < x < 5$

If $x = 1 \rightarrow$ x is not acc. Point since , $\exists\ r > 0$

$B(x, r) - \{x\} \cap S = \emptyset$, when $r = 1$

$B(1, 1) - \{1\} \cap \{1, 5\} = (0, 2) - \{1\} \cap [1, 5\} = \emptyset$

If $x = 5 \rightarrow$ x is not acc. Point, since $\exists r > 0$, $B(x, r) - \{x\} \cap S = \emptyset$, when $r = 1$

$\rightarrow B(5, 1) - \{5\} \cap \{1, 5\} = (4, 6) - \{5\} \cap \{1, 5\} = \emptyset$

If $x < 1 \rightarrow$ x are not acc. Point since $x \in (x - 1, 1)$ & $(x - 1, 1) \cap S = \emptyset$

If $x > 5 \rightarrow x$ are not acc. Point, since $x \in (5, x + 1)$& $(5, x + 1) \cap S = \emptyset$

If $1 < x < 5$ are not acc. Point since, $x \in (1, 5)$ and $(1, 5) \cap S = \emptyset$

So, S has no a acc. Point then $S' = \emptyset$ and $\overline{S} = S \cup S' = S \cup \emptyset = S$.

Let $s = \left\{1 , \frac{1}{2} , \frac{1}{3} , \dots \right\} = \left\{\frac{1}{n} , n = 1, 2, 3, \dots . \right\}$ show that $S' = \{0\}$

If $S = (a, b)$, find S'

Solution:

If $x = a \rightarrow x$ is acc. Point since $\forall r > 0$, $a \in B(0, r) = (a - r, a + r)$ and $B(a, r) - \{a\} \cap S \neq \emptyset$

If $x = b \rightarrow x$ is acc. Point, since $\forall r > 0$, $b \in B(b, r)$

$B(b, r) = (b - r, b + r)$ and $B(b, r) - \{b\} \cap (a, b) \neq \emptyset$

If $a < x < b \rightarrow x$ are acc. Point since $\forall\ r > 0$,

$x \in B(x, r) = (x - r, x + r)$ and $B(x, r) - \{x\} \cap S \neq \emptyset$

That is $(x - r, x + r) - \{x\} \cap (a, b) \neq \emptyset$

If $x < a \rightarrow x$ are not acc. Point since $x \in (x - 1, a)$ & $(x - 1, a) \cap S = \emptyset$

If $x > b \rightarrow x$ are not acc. Point, since $x \in (b, x + 1)$ and $(b, x + 1) \cap (a, b) = \emptyset \therefore S' = [a, b] \rightarrow \overline{S} = S \cup S' = [a, b]$

Definition: A sub set A of a metric space X is said to be dense if $\overline{A} = X$

Example: prove that $\overline{Q} = R$ (i.e., Q dense set in R)

Solution: If $x \in R$, then x is acc. Point in Q.

Since any open interval Contain x Contains infinitely rational and irrationals Then $Q' = R$

So $\overline{Q} = Q \cup Q' = Q \cup R = R$

Definition: a metric space is called separable if it has a countable dense subset.

Example: R separable since Q countable and $Q \subseteq R$, with Q dense in R

Theorem: let X be a metric space, $S \subseteq X$ then

1- S is closed iff $S' \subset X$

2- $\overline{S}$ is closed set

3- $\overline{S} = S$ iff S closed set

4- $\overline{S}$ is smallest closed set contains S.

Compact Space

Definition: let (X, d) be a metric space, $\emptyset \neq S \subseteq X$, if the set $\{U_\lambda : U_\lambda \text{ open set}, \lambda \in \wedge\}$ is a family of open subsets of X such that $S \subseteq \bigcup_{\lambda \in \wedge} U_\lambda$, then the family $\{U_\lambda\}$ is called open cover for S in X.

- If the family $\{U_\lambda\}$ is finite and $S \subseteq \bigcup_{\lambda \in \wedge} U_\lambda$ then $\{U_\lambda\}$ is called finite cover.
- Let $\{U_\lambda\}$ and $\{U_\alpha\}$ be to open cover for S and $U_\lambda \in \{U_\alpha\}\ \forall \lambda$, then $\{U_\lambda\}$ is called subcover for $\{U_\alpha\}$

Def: let A be a subset of a metric space (X, d), A is called compact set if every open cover for A in X has a finite subcover.

Example: Any finite subset B of matric space (X, d) is compact set

Example: R is not compact

Example: Any open interval A=(a,b) is not compact

Example: Any closed interval A=[a,b] is Compact.

Proof:

Since we can restrict any open cover for A to finite subcover such as :

Let $\epsilon > 0, B = \{(a-\epsilon, a+\epsilon, (a,b), (b-\epsilon, b+\epsilon)\}$

Theorem: ((Bolzano weir strass theorem))

In compact space X, every infinite subset S of X has at least one accumulation point.

Theorem: In compact metric space, every closed subset is compact.

Proof: X be a compact metric space, and A be a closed subset of X, then A^c is open. T.P A is compact.

Let $B = \{U_\lambda : U_\lambda \text{ is open set in } X, \forall \lambda \in \wedge\}$ be open cover for A.

Then $A \subseteq \bigcup_{\lambda\in\wedge} U_\lambda$

Sine $X = A \cup A^c \subseteq (\bigcup_{\lambda\in\wedge} U_\lambda) \cup A^c$,

But A^c is open set then $\bigcup_{\lambda\in\wedge} U_\lambda \cup A^c$ is open cover for X, since X is compact set , then there exists a finite member $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ such that

$$X = A^c \cup \left(\bigcup_{i=1}^{n} U_{\lambda i}\right)$$

Since that $X = A^c \cup (\bigcup_{i=1}^{n} U_{\lambda i})$ .Since $\cap A^c = \emptyset$, then $A \subseteq \bigcup_{i=1}^{n} U_{\lambda i}$

$\Rightarrow$ B has a finite subcover $\{U_{\lambda 1}, U_{\lambda 2}, \dots\dots, U_{\lambda n}\}$. For A, $\Rightarrow$ A is compact.

Theorem: Let (X,d) be a metric space, $A \subseteq X$, If A is compact, Then A is closed

Theorem: Let (X,d) be a metric space, $A \subseteq X$, If A is compact, Then A is bounded

Remark: In metric space

Compact $\rightarrow$ Closed + bounded

$\not\leftarrow$

Theorem: Let $\{I_n : n = 1,2,3,\dots\}$ be a family of closed interval if $I_{n+1} \subset I_n, \forall n, \text{then } \bigcap_{n=1}^{\infty} I_n = \emptyset$

Theorem: (Hien-Bord Theorem)

Every closed and bounded subset of R^n, $n \geq 1$, is compact.