UNIVERSITY OF ANBAR

كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الرابعة

أستاذ المادة : ا.م.د.ماجد محمد عبد

اسم المادة باللغة العربية :مقاسات

اسم المادة باللغة الإنكليزية : **MODULES**

اسم الحاضرة الأولى باللغة العربية: الزمرة والحلقة والمقاس

اسم المحاضرة الأولى باللغة الإنكليزية :**Submodule:**

محتوى المحاضرة الثانية

Definition 3.3 (Submodules, sums, and quotients). Let M be an R-module.

(a) A submodule of M is a non-empty subset N _ M satisfying m+n 2 N and am 2 N for all m;n 2 N and a 2 R. We write this as N _ M. Of course, N is then an R-module itself, with the same addition and scalar multiplication as in M.

(b) For any subset S _ M the set

hSi := fa1m1+_ _ _+anmn : n 2 N;a1; : : : ;an 2 R;m1; : : : ;mn 2 Sg _ M

of all R-linear combinations of elements of S is the smallest submodule of M that contains S. It is called the submodule generated by S. If S = fm1; : : : ;mng is finite, we write hSi = hfm1; : : : ;mngi also as hm1; : : : ;mn i. The module M is called finitely generated if M = hSi for a finite set S _ M.

(c) For submodules N1; : : : ;Nn _ M their sum

N1+_ _ _+Nn = fm1+_ _ _+mn : mi 2 Ni for all i = 1; : : : ;ng

is obviously a submodule of M again. If moreover every element m 2 N1 +_ _ _+Nn has a unique representation as m = m1 +_ _ _+mn with mi 2 Ni for all i, we call N1 +_ _ _+Nn a direct sum and write it also as N1__ _ __Nn.

(d) If N _ M is a submodule, the set

M=N := fx : x 2 Mg with x := x+N

of equivalence classes modulo N is again a module [G2, Proposition 15.15], the so-called quotient module of M modulo N.

Example 3.4.

(a) Let R be a ring. If we consider R itself as an R-module, a submodule of R is by definition the same as an ideal I of R. Moreover, the quotient ring R=I is then by Definition 3.3 (d) an R-module again.

Note that this is the first case where modules and vector spaces behave in a slightly different way: if K is a field then the K-vector space K has no non-trivial subspaces.

(b) The polynomial ring K[x1; : : : ;xn] over a field K is finitely generated as a K-algebra (by x1; : : : ;xn), but not finitely generated as a K-module, i. e. as a K-vector space (the monomials 1;x1;x2

1; : : : are linearly independent). So if we use the term “finitely generated” we always have to make sure to specify whether we mean “finitely generated as an algebra” or “finitely generated as a module”, as these are two different concepts.

Exercise 3.5. Let N be a submodule of a module M over a ring R. Show:

(a) If N and M=N are finitely generated, then so is M.

(b) If M is finitely generated, then so is M=N.

(c) If M is finitely generated, N need not be finitely generated.

Definition 3.6 (Morphisms). Let M and N be R-modules.

(a) A morphism of R-modules (or R-module homomorphism, or R-linear map) from M to N is a map j : M !N such that

j(m+n) = j(m)+j(n) and j(am) = aj(m)

for all m;n 2 M and a 2 R. The set of all such morphisms from M to N will be denoted HomR(M;N) or just Hom(M;N); it is an R-module again with pointwise addition and scalar multiplication.

(b) A morphism j : M ! N of R-modules is called an isomorphism if it is bijective. In this case, the inverse map j□1 : N !M is a morphism of R-modules again [G2, Lemma 13.25 (a)]. We call M and N isomorphic (written M _=
N) if there is an isomorphism between
them.

Example 3.7.

(a) For any ideal I in a ring R, the quotient map j : R ! R=I; a 7! a is a surjective R-module homomorphism.

(b) Let M and N be Abelian groups, considered as Z-modules as in Example 3.2 (d). Then a Z-module homomorphism j : M !N is the same as a homomorphism of Abelian groups, since j(m+n) = j(m)+j(n) already implies j(am) = aj(m) for all a 2 Z.

(c) For any R-module M we have HomR(R;M)_=
M: the maps
M !HomR(R;M); m 7! (R!M;a 7! am) and HomR(R;M)!M; j 7! j(1)
are obviously R-module homomorphisms and inverse to each other. On the other hand, the module HomR(M;R) is in general not isomorphic to M: for the Z-module Z2 we have HomZ(Z2;Z) = 0 by (b), as there are no non-trivial group homomorphisms from Z2 to Z.

(d) If N1; : : : ;Nn are submodules of an R-module M such that their sum N1 __ _ __Nn is direct, the morphism
N1__ _ __Nn !N1__ _ __Nn; (m1; : : : ;mn) 7! m1+_ _ _+mn
is bijective, and hence an isomorphism. One therefore often uses the notation N1 __ _ __Nn for N1 __ _ __Nn also in the cases where N1; : : : ;Nn are R-modules that are not necessarily submodules of a given ambient module M.

Example 3.8 (Modules over polynomial rings). Let R be a ring. Then an R[x]-module M is the same as an R-module M together with an R-module homomorphism j : M !M:

“)” Let M be an R[x]-module. Of course, M is then also an R-module. Moreover, multiplication with x has to be R-linear, so j : M !M;m 7! x _m is an R-module homomorphism.

“(” If M is an R-module and j : M !M an R-module homomorphism we can give M the structure of an R[x]-module by setting x _m := j(m), or more precisely by defining scalar multiplication where ji denotes the i-fold composition of j with itself, and j0 := idM.

Remark 3.9 (Images and kernels of morphisms). Let j :M!N be a homomorphism of R-modules.

(a) For any submodule M0 _ M the image j(M0) is a submodule of N [G2, Lemma 13.21 (a)].

In particular, j(M) is a submodule of N, called the image of j.

(b) For any submodule N0 _ N the inverse image j□1(N0) is a submodule of M [G2, Lemma 13.21 (b)]. In particular, j□1(0) is a submodule of M, called the kernel of j.

Proposition 3.10 (Isomorphism theorems).

(a) For any morphism j : M !N of R-modules there is an isomorphism

M=kerj !imj; m 7! j(m):

(b) For R-modules N0 _ N _ M we have

(M=N0)=(N=N0) _=

M=N:

(c) For two submodules N;N0 of an R-module M we have

(N +N0)=N0 _=

N=(N \N0):

Proof. The proofs of (a) and (b) are the same as in [G2, Proposition 15.22] and Exercise 1.22, respectively. For (c) note that N !(N +N0)=N0; m 7! m is a surjective R-module homomorphism with kernel N \N0, so the statement follows from (a). _

Exercise 3.11. Let N be a proper submodule of an R-module M. Show that the following statements are equivalent: