UNIVERSITY OF ANBAR

كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الرابعة

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اسم المادة بالغة العربية :مقاسات

اسم المادة باللغة الإنكليزية : **MODULES**

اسم الحاضرة الأولى باللغة العربية: الجمع المباشر

اسم المحاضرة الأولى باللغة الإنكليزية **Direct Sum and direct summand**

محتوى المحاضرة الثامنة

Direct Sum and direct summand

Definition.

Let R be a ring and $\{M_i \mid i \in I\}$ be an arbitrary (possibly infinite) of a nonempty family of R-modules. $\Pi_{i\in I} M_i$ is the direct product of the abelian groups M_i, and $\oplus_{i\in I} M_i$ the direct sum of the of the abelian groups M_i, where $\Pi_{i\in I} M_i = \{f: I \to \cup_{i\in I} M_i \mid f(i) \in M_i, \text{ for all } i \in I\}$.

Define a binary operation "+" on the direct product (of modules) $\prod_{i\in I} M_i$ as follows: for each f, g ∈ $\prod_{i\in I} M_i$ (that is, f,g : I → $\cup_{i\in I} M_i$ and f(i), g(i) ∈ M_i for each i), then f+g : I → $\cup_{i\in I} M_i$ is the function given by i → f(i)+g(i).

i.e (f+g)(i) = f(i)+g(i) for each i ∈I.

Since each M_i is a module, f(i)+g(i)∈M_i for every i, whence f+g ∈ $\prod_{i\in I} M_i$. So ($\prod_{i\in I} M_i$, +) is an abelian group. Now, if r ∈ R and f ∈ $\prod_{i\in I} M_i$, then rf : I → $\cup_{i\in I} M_i$ as (rf)(i) = r(f(i)).

1. $\prod_{i\in I} M_i$ is an R-module with the action of R given by r(f(i)) = (rf(i)) (i.e define α: R x ∈ $\prod_{i\in I} M_i$ → ∈ $\prod_{i\in I} M_i$ by α(r,f) = rf).

2. $\bigoplus_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. (H.W.).

Remark

$\prod_{i\in I} M_i$ is called the (external) direct product of the family of R-modules {M_i | i ∈I} and $\bigoplus_{i\in I} M_i$ is (external) direct sum. If the index set is finite, say i = {1 ,2, …, n}, then the direct product and direct sum coincide and will be written $M_1 \oplus M_2 \oplus \ldots \oplus M_n$.

Definition. ((internal) direct sum) Let R be a ring and N, K submodules of an R-module M such that:

1. M = N + K.

2. N ∩ K = 0. Then N and K is said to be direct summand of M and M = N⊕ K internal direct sum of N and K.

Definition

The external direct sum of the modules M_i, i ∈ I, denoted by $\bigoplus_{i\in I} M_i$, consists of all families (a_i, i ∈ I) with a_i ∈ M_i, such that a_i = 0 for all but finitely many i. Addition and scalar multiplication are defined exactly as for the direct product, so that the external direct sum coincides with the direct product when the index set I is finite.

The R-module M is the internal direct sum of the submodules Mi if each $x \in M$ can be expressed uniquely as $x_{i1} + \cdots + x_{in}$ where $0 \neq x_{ik} \in M_{ik}$, k = 1,...,n. (The positive integer n and the elements x_{ik} depend on x. In any expression of this type, the indices i_k are assumed distinct.) Just as with groups, the internal and external direct sums are isomorphic. To see this without a lot of formalism, let the element $x_{ik} \in M_{ik}$ correspond to the family that has x_{ik} in position i_k and zeros elsewhere. We will follow standard practice and refer to the "direct sum" without the qualifying adjective. Again, as with groups, the next result may help us to recognize when a module can be expressed as a direct sum**.**

Proposition

The module M is the direct sum of submodules M_i if and only if both of the following conditions are satisfied:

(1) $M = \sum_i M_i$, that is, each $x \in M$ is a finite sum of the form $x_{i1} + \cdots + x_{in}$, where $x_{ik} \in M_{ik}$;

(2) For each i, $M_i \cap \sum_{j \neq i} M_j = 0$. (Note that in condition (1), we do not assume that the representation is unique. Observe also that another way of expressing (2) is that if $x_{i1} + \cdots + x_{in} = 0$, with $x_{ik} \in M_{ik}$, then $x_{ik} = 0$ for all k.).

Proof.

A basic property of the direct sum $M = \bigoplus_{i \in I} M_i$ is that homomorphisms $f_i : M_i \to N$ can be "lifted" to M. In other words, there is a unique homomorphism $f : M \to N$ such that for each i, f = $_{fi}$ on Mi. Explicitly, $f(x_{i1} + \cdots + x_{ir}) = f_{i1}(x_{i1}) + \cdots + f_{ir}(x_{ir})$