كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الرابعة

أستاذ المادة : ا.م.د.ماجد محمد عبد

اسم المادة باللغة العربية :مقاسات

اسم المادة باللغة الإنكليزية : **MODULES**

اسم الحاضرة الأولى باللغة العربية: المقاس المسطح

اسم المحاضرة الأولى باللغة الإنكليزية: flat Module

محتوى المحاضرة الثامنة

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(3) implies (2)

Let $0 \neq x \in N$ and consider a maximal ideal m containing Ann(x). But then Ax ' A/Ann(x) and then $Ax \otimes M = A/Ann(x) \otimes M = M/Ann(x)M$ which is nonzero because $Ann(x)M \subset mM \neq M$. Now, we have an R-linear map which is injective $0 \to Ax \to N$ and by flatness of M we get that $Ax \otimes M$ injects into $N \times M$, so the latter is nonzero as well.

(2) implies (1)

Let f be an A-linear map between two modules $E \to F$. We claim that $Ker(f) \otimes M = Ker(f \otimes 1)$ and $Im(f) \otimes M = Im(f \otimes 1)$. Indeed, $Ker(f) \to E \to F$ and $E \to F \to F/Im(f)$ are exact, so they remain exact after tensoring with M. Consider a sequence of A-modules $N0 \to N \to N00$ such that $N0 \otimes M$ $f\otimes 1 \to N \otimes M$ $g\otimes 1 \to N00 \otimes M$ is exact. So $(g \circ f) \otimes 1 = 0$ hence $g \circ f = 0$. In conclusion $Im(f) \subset$

Ker(g). Consider now H = Ker(g)/Im(f). Then by flatness H⊗M = (Ker(g)⊗M)/(Im(f)⊗ M) = 0. Therefore H = 0.

Corollary 1.5. Let f : (A, m) → (B, n) a local homomorphism of rings (that is f is a ring homomorphism and f(m) ⊂ n). Then B is A-flat if and only if B is A-faithfully flat.

Proof. Since f(m) ⊂ n we get that mB ⊂ n ≠B.

Proposition 1.6.

(1) Let A be a ring and M an A-flat module. Let N1, N2 be two submodules of M. Then (N1 ∩ Ns) ⊗ M = (N1 ⊗ M) ∩ (N2 ⊗ M), where the objects are regarded as submodules of N ⊗A M.

(2) Therefore, if A → B is flat then for any ideals I,J of A, we have (I ∩ J)B = IB ∩ JB. If J is finitely generated, then (I : J)B = (IB : JB).

(3) If f : A → B is faithfully flat, then for any A-module M the natural map M → M ⊗A B is injective. In particular f is injective. In particular, for any ideal I ⊂ A, IB ∩ A = I.

Proof. For (1), consider the exact sequence of A-modules 0 → N1 ∩ N2 → N → N/N1 ⊕ N/N2, and tensor with M. The resulting exact sequence gives the statement. For (2), let N = A, N1 = I,N2 = J, and M = B. For the second part, let J = (a1,...,ak). But then I : J = ∩k i=1(I : Aai). Fix i, and let 0 → (I : Aai) → A → ·ai A/I which is exact. Since B is A-flat we get that the sequence stays exact after tensoring with B. This gives us 0 → (I : Aai)B → B → ·ai B/IB. Therefore, (I : Aai)B = (IB : Bai) by computing the kernels in two ways. Therefore, (I : J)B = (∩k i=1(I : Aai))B which equals (∩k i=1(I : Aai)B) by the first part of (2). But this last term equals ∩k i=1(IB : Bai) = IB : JB. Finally, let m ∈ M such that m ⊗ 1 = 0 in M ⊗A B. We need m = 0, so let us assume that m 6= 0. But then 0 6= Am ⊂ M and therefore, since B is A-faithfully flat, we get that 0 ≠Am ⊗A B in M ⊗A B. On the hand m ⊗ 1 = 0 so Am ⊗A B = 0 as well. Contradiction. The final statement is obtained by letting M = B.

Lemma 1.7. Let i : E → F be an injective A-linear map. Let M be an A-module and consider u ∈ ker(1M ⊗ i) ⊂ E ⊗A M, where 1M ⊗ i : E ⊗A M → F ⊗A M. Then there exists N finitely generated submodule of M and v ∈ ker(1N ⊗ i) such that v maps to u under the canonical map E ⊗ N → E ⊗ M.

Proposition 1.8. A module M is flat over A if all its finitely generated submodules are flat over A.

Proof. This is a straightforward application of the Lemma. If there exists an R-linear injection $i : E \to F$ and for any element $u \in Ker(i \otimes_A 1_M)$, we can find a finitely generated submodule N of M and $v \in ker(i \otimes_A 1_N)$ such that v maps onto u under the canonical map. But N is flat so $v = 0$ which gives $u = 0$.

Proposition 1.9. Let A be a domain. Then every flat A-module is torsion free. The converse holds, if A is a PID. Proof. Let a 6= 0 in A. Then multiplication by a is injective on A (