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MODULES : [illegible]

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Injective module: [illegible]

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. Injective modules

Definition

An R-module N is called injective if and only if for any monomorphism f: A→B (where A and B are any two modules) and any homomorphism g: A→N, there exists a homomorphism h: B→N such that h∘f=g:

Remark

Every R-module can be embedded in an injective R-module. That is for every R-module M, there exists an injective R-module N and a monomorphism β:M→N.

Theorem

Let N be an R-module. Then the following statements are equivalent:

1) N is injective module.

2) Every exact sequence of the form 0→N→f→M split where M is any R-module.

3)N is isomorphism to direct summand of an injective R-module M.

Proof.

1⟹ 2

Let 0→N→f→M be an exact sequence and let I_N: N→N be the identity homomorphism on N. Consider the following diagram:

By (1) N is injective. So there exists a homomorphism h: M→N such that $h \circ f = I_N$. Hence h is a left inverse of the homomorphism f. Then the sequence is split.

2⟹3

Let N be an R-module. Then there exists an injective module M over the ring R and let f: N→M be a monomorphism. So 0→N →M is an exact. Hence by (2), this sequence is split. Then Im(f) is a direct summand of M. But Im(f) ≃N. Then N is a direct summand of M and this means N is an isomorphism to direct summand of an injective R-module.

$3 \Longrightarrow 1$

From (3), there exists an injective R-module X such that $X \simeq N \oplus Y$ for some submodule Y of X. Now we need to prove that N is injective module. Let $f: A \to N$ be a monomorphism mapping and $g: A \to N$ be a homomorphism. Consider the following commutative diagram:

$$X = N \oplus Y$$

X is an injective R-module, so there exists a homomorphism $h: B \to X$ such that Since

$$h \circ f = j \circ g \ldots\ldots\ldots\ldots\ldots\ldots \text{ (1)}$$

where $j: N \to X$ is the injective homomorphism. Define $h^{/}: B \to N$ such that

$$h^{/} = \rho \circ h \ldots\ldots\ldots\ldots\ldots\ldots \text{ (2)}$$

So, $h^{/}$ is a homomorphism where $\rho : X \to N$ is the projection homomorphism. Now we must prove that $h^{/} \circ f = g$.

$$h^{/} \circ f = (\rho \circ h) \circ f \qquad \text{by (2)}$$

$$= \rho \circ (h \circ f)$$

$$= \rho \circ (j \circ g) \qquad \text{by (1)}$$

$$= (\rho \circ j) \circ g.$$

$$= I_N \circ g$$

=g

So, N is injective module.

Remarks

1.An R-module J is an injective module if J satisfies one of the equivalent conditions above theorem

2.An R-module m is injective if and only if for every ideal I of R and for every homomorphism f: I→M, there exists a homomorphism g: R→M such that g∘i=f where i: I→R.

Theorem.

If R is an integral domain, then every injective R-module is divisible.

Proof. Let M be an injective R-module. We must prove that M is divisible module. Let x an element belongs to M and let a be a non-zero element belong to R. Suppose that I=(a) the ideal of R generated by a. Define a homomorphism f: I→M such that f(ra)=rx for all r in R. Now to prove f is well define:

$r_1a = r_2a$

$(r_1-r_2)\ a=0$

Since R is an integral domain, a≠0, so $r_1-r_2=0$. Hence $r_1=r_2$ and then $r_1x=r_2x$ and finally $f(r_1a)-f(r_2a)$. It is clear that f is a homomorphism. Since M is injective module, then there exists g: R→M is a homomorphism such that g∘i=f where i: I→R is the inclusion homomorphism. Let y=g (1). So, y in M. To prove that x=ay:

X=1.x=f(1-a) =a.f(1)=a((g◦i)(1))=ag(1)=ay. Then x is a divisible element. Thus, M is a divisible module.

As a result, from above Theorem, we present the following.

Corollary

If R is an integral domain, then every torsion free divisible R-module is injective.

Proof. Let M be a divisible torsion free module over the integral domain R. To prove that M is injective. Let I be a non-trivial ideal of R and let f: I→M be any homomorphism. Suppose that 0≠a belong to I. So, f(a) belong to M. Since M is divisible module, then there exists x belong to M such that f(a)=ax. Now let r belong to I. So rf(a)=f(ra)=af(r) =arx.