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اسم المادة باللغة العربية :مقاسات
اسم المادة باللغة الإنكليزية : MODULES
اسم الحاضرة الأولى باللغة العربية: تشاكل المقاسي
اسم المحاضرة الأولى باللغة الإنكليزية: Module Homomorphisms

محتوى المحاضرة السابعة

Module Homomorphisms Fix a ring R.

Definition 5.1. Let M, N be R-modules. A function $\phi : M \to N$ is an R-module homomorphism if it is a homomorphism of abelian groups such that (for all $m \in M$, $r \in R$) $\phi(mr) = \phi(m)r$. In this case we also say ϕ is R-linear. We denote the set of these as HomR (M, N). Example 5.1. k-module homomorphisms are just k-linear maps.

Proposition 5.1. HomR (M, N) is an abelian group endowed with group addition $(\phi 1 + \phi 2)(m) = \phi 1(m) + \phi 2(m)$.

Proof. Note that addition is well defined as $\phi 1$, $\phi 2$ are R-linear. Indeed it is additive and for $m \in M$, $r \in R$ we have $(\phi 1 + \phi 2)(mr) = \phi 1(mr) + \phi 2(mr) = \phi 1(m)r + \phi 2(m)r = (\phi 1(m) + \phi 2(m))r = (\phi 1 + \phi 2)(m)r$. Exercise. Check group axioms.

Proposition 5.2. For any R-module M, there is the following isomorphism of abelian groups $\Phi : M \longrightarrow HomR (RR, M)$ $m 7\rightarrow (\lambda m : r 7\rightarrow mr)$ with inverse $\Psi : \phi 7\rightarrow \phi(1)$.

Proof. Check λm is R-linear so Φ is well defined. The distributive law implies that $\lambda m(r + r 0) = m(r + r 0) = mr + mr0 = \lambda m(r) + \lambda m(r 0)$. Associativity gives us $\lambda m(rr0) = m(rr0) = (mr)r 0 = \lambda m(r)r 0$. Φ is additive by the other distributive law. It now suffices to check that Ψ is the inverse of Φ. We observe that $(\Psi\Phi)(m) = \Psi(\lambda m) = \lambda m(1) = m1 = m$ $[(\Phi\Psi)(\phi)](r) = [\Phi(\phi(1))](r) = \phi(1)r = \phi(r)$ so $\Phi\Psi = id$.

Proposition 5.3. As for vector spaces, the composition of R-linear maps is R-linear. Proof. Exercise. Proposition 5.4. Let $\phi : M \rightarrow N$ be a homomorphism of right R-modules. Then (1) $\ker \phi \leq M$ (2) $im(\phi) \leq N$

Proof. (1) Exercise. (2) Any element in $im(\phi)$ has the form $\phi(m)$ for $m \in M$. Then for $r \in R$, see $\phi(m)r = \phi(mr) \in im(\phi)$, so $im(\phi)$ is closed under scalar multiplication. We know $im(\phi)$ is a subgroup so it must be a submodule. Proposition 5.5. Let $N \leq M$. Then there are module homomorphisms (1) The inclusion map $\iota : N ,\rightarrow M$ (2) the projection map $\pi : M \rightarrow M/N$ defined by $m 7\rightarrow m + N$.