



كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

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اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنجليزية : Complex Functions

اسم الحاضرة السادسة باللغة العربية: الاستمرارية

اسم المحاضرة السادسة باللغة الإنجليزية : Continuity

Lecture 6

Continuity الاستمرارية

A function f is continuous at a point z_0 if all three of the following conditions are satisfied:

1. $\lim_{z \rightarrow z_0} f(z)$ exists

2. $f(z_0)$ exists

3. $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

Example: If

$$f(z) = \begin{cases} z^2 & , \quad z \neq i \\ 0 & , \quad z = i \end{cases}$$

Then show that the function $f(z)$ is not continuous at the point $z = i$

Solution:

1. $\lim_{z \rightarrow i} f(z) = \lim_{z \rightarrow i} z^2 = (i)^2 = -1$

2. $f(i) = 0$

3. $\lim_{z \rightarrow i} f(z) = -1 \neq 0 = f(i)$

$\therefore \lim_{z \rightarrow i} f(z) \neq f(i)$

$\therefore f(z)$ is not continuous at $z = i$

Derivatives المشتقات

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0 . The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists

if $\Delta z = z - z_0$, $z \neq z_0$ then $f'(z_0)$ can be write as:

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Example: Suppose that $f(z) = z^2$. At any point z

Solution: Then

$$\begin{aligned} f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ \Rightarrow f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} \end{aligned}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\Delta z(2z + \Delta z)}{\Delta z}$$

$$\therefore f'(z) = \lim_{\Delta z \rightarrow 0} (2z + \Delta z) = 2z$$

$$\therefore f'(z) = 2z$$

Example: Let $w = f(z) = \bar{z}$. Prove that f has no derivative

Solution:

$$\begin{aligned} \frac{dw}{dz} &= f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

Now, consider $\Delta z = \Delta x + i\Delta y$ so

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

When $\Delta z \rightarrow 0$ on the real axis, then

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z \rightarrow 0} \frac{\Delta x - 0}{\Delta x + 0}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\Delta x}{\Delta z} = 1$$

When $\Delta z \rightarrow 0$ on the imaginary axis, then

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} &= \lim_{\Delta z \rightarrow 0} \frac{0 - i\Delta y}{0 + i\Delta y} \\ &= \lim_{\Delta z \rightarrow 0} \frac{-i\Delta y}{i\Delta y} = \lim_{\Delta z \rightarrow 0} \frac{-1}{1} = -1 \end{aligned}$$

$\therefore f$ has no derivative because the limit does not exist

H.W.

1. Consider the real-valued function $f(z) = |z|^2$. Show that $f'(z)$ does not exist when $z \neq 0$
2. Prove that $\frac{dw}{dz} = 2z + 3i$ if $w = z^2 + 3iz$

Differentiation Formulas قواعد الاشتقاق

If $f(z)$ is a complex function, then the derivative of a function f at a point z is denoted by either

$$\frac{d}{dz}f(z) \quad or \quad f'(z)$$

Let c be a complex constant, and let f be a function whose derivative exists at a point z . Then

1. $\frac{d}{dz}c = 0$

2. $\frac{d}{dz}z = 1$

3. $\frac{d}{dz}[cf(z)] = cf'(z)$

4. $\frac{d}{dz}z^n = nz^{n-1}, \quad n \in \mathbb{Z}^+$

5. If the derivative of g exist at a point z , then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$$

6. If the derivative of g exist at a point z , then

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + g(z)f'(z)$$

7. If the derivative of g exist at a point z , then

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}, \quad g(z) \neq 0$$

Example: If $f(z) = (2z^2 + i)^5$, then find $f'(z)$

Solution:

$$f'(z) = 5(2z^2 + i)^{5-1}(4z + 0)$$

$$f'(z) = 5(2z^2 + i)^4(4z)$$

$$f'(z) = 20z(2z^2 + i)^4$$

H.W.

By using differentiation formulas find $f'(z)$ for each the following complex functions:

1. $f(z) = 3z^2 - 2z + 4$

2. $f(z) = (1 - 4z^2)^3$

3. $f(z) = \frac{z - 1}{2z + 1} , z \neq 0$

4. $f(z) = \frac{(1 + z^2)^4}{z^2} , z \neq 0$

5. Use the definition of derivative to prove that

$$\frac{dw}{dz} = -\frac{1}{z^2} \quad \text{when } w = \frac{1}{z}, z \neq 0$$