كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

أستاذ المادة : م.د. مصطفى ابراهيم حميد

اسم المادة بالغة العربية : الدوال المركبة

اسم المادة باللغة الإنكليزية : **Complex Functions**

اسم الحاضرة السابعة باللغة العربية: معادلتا كوشي ريمان

اسم المحاضرة السابعة باللغة الإنكليزية : **Cauchy Riemann Equations**

Lecture 7

Cauchy Riemann Equations معادلتا كوشي ريمان

Theorem: Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad and \qquad u_y = -v_x$$

there. Also, $f'(z)$ can be written

$$f'(z) = u_x + iv_x$$

Where these partial derivatives are to be evaluated at (x_0, y_0).

Example: Use Cauchy-Riemann equations to find $f'(z)$ where $f(z) = z^2$

Solution:

We know that the function $f(z)$ is differentiable everywhere

$$f(z) = z^2 = (x + iy)(x + iy)$$

$$= (x^2 - y^2) + 2ixy$$

$$u(x, y) = x^2 - y^2 \qquad and \qquad v(x, y) = 2xy$$

$$u_x = 2x \qquad and \qquad v_x = 2y$$

$$u_y = -2y \qquad and \qquad v_y = 2x$$

$$u_x = 2x = v_y \qquad and \qquad u_y = -2y = -v_x$$

$$f'(z) = u_x + iv_x$$
$$= 2x + 2iy$$
$$= 2(x + iy) = 2z$$

Theorem: Let the function $f(z) = u(x,y) + iv(x,y)$ be defined at some ϵ neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

a. The first- order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood.
b. Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad and \qquad u_y = -v_x \quad at \quad (x_0, y_0).$$

Then $f'(z_0)$ exists and $f'(z_0) = u_x + iv_x$ where the right-hand side is to be evaluated at (x_0, y_0)

Example: Use Cauchy-Riemann equations to discuss the derivative of the function $f(z) = |z|^2$

Solution:

$$f(z) = |z|^2$$
$$f(z) = \left|\sqrt{x^2 + y^2}\right|^2$$
$$f(z) = x^2 + y^2$$
$$\Longrightarrow \quad f(z) = x^2 + y^2 + i.0$$

$$u(x,y) = x^2 + y^2 \qquad and \qquad v(x,y) = 0$$

$$u_x = 2x \qquad and \qquad v_x = 0$$

$$u_y = 2y \qquad and \qquad v_y = 0$$

$if \quad u_x = v_y \quad \Longrightarrow 2x = 0 \quad \Longrightarrow x = 0$

$if \quad u_y = -v_x \quad \Longrightarrow 2y = 0 \quad \Longrightarrow y = 0$

$\therefore \quad f'(z)$ does not exist at any nonzero point

$$f'(z) = u_x + iv_x$$

$$= 2x + i.0$$

$$f'(z) = 2x$$

$if \quad z = 0 \quad \Longrightarrow \; f'(0) = 2(0) = 0$

اي ان $f'(z)$ موجودة فقط عندما تكون $z = 0$ وقيمتها هي $f'(0) = 0$

Exercises

1. Use Cauchy-Riemann equations to show that $f'(z)$ does not exist at any point if

a. $f(z) = \bar{z}$

b. $f(z) = z - \bar{z}$

c. $f(z) = 2x + ixy^2$

d. $f(z) = e^x e^{-iy}$

Solution:

a. $f(z) = \bar{z} = x - iy$

$u(x, y) = x \qquad and \qquad v(x, y) = -y$

If $\quad u_x = v_y \Longrightarrow 1 = -1$

The Cauchy-Riemann equations are not satisfied anywhere.

b. $f(z) = z - \bar{z} = x + iy - x + iy$

$f(z) = 2iy$

$u(x, y) = 0 \qquad and \qquad v(x, y) = 2y$

If $\quad u_x = v_y \Longrightarrow 0 = 2$

The Cauchy-Riemann equations are not satisfied anywhere.

c. $f(z) = 2x + ixy^2$

$u(x, y) = 2x \qquad and \qquad v(x, y) = xy^2$

If $\quad u_x = v_y \quad \Longrightarrow 2 = 2xy$

$\Longrightarrow xy = 1$

If $\quad u_y = -v_x \Longrightarrow 0 = -y^2$

$\Longrightarrow y = 0$

Substituting $y = 0$ into $xy = 1$ we have $1 = 0$

The Cauchy-Riemann equations do not hold anywhere.

d. $f(z) = e^x e^{-iy}$

$f(z) = e^x(\cos(y) + i\sin(-y))$

$f(z) = e^x(\cos y - i\sin y)$

$u(x, y) = e^x \cos y \qquad and \qquad v(x, y) = -e^x \sin y$

If $\quad u_x = v_y \quad \Longrightarrow e^x \cos y = -e^x \cos y$

$$\Longrightarrow e^x \cos y + e^x \cos y = 0$$

$$\Longrightarrow 2e^x \cos y = 0$$

$$\Longrightarrow \cos y = 0 \quad \Longrightarrow y = \frac{\pi}{2} + n\pi \ , \qquad n = 0, \pm 1, \dots$$

If $\quad u_y = -v_x \ \Longrightarrow -e^x \sin y = e^x \sin y$

$$\Longrightarrow -e^x \sin y - e^x \sin y = 0$$

$$\Longrightarrow -2e^x \sin y = 0$$

$$\Longrightarrow 2e^x \sin y = 0$$

$$\Longrightarrow \sin y = 0 \Longrightarrow y = n\pi \ , \qquad n = 0, \pm 1, \dots$$

$$\therefore \ \ y = \frac{\pi}{2} + n\pi \ \ , \ \ y = n\pi \ , \ \ but \ \ \frac{\pi}{2} + n\pi \neq n\pi \qquad n = 0, \pm 1, \dots$$

The Cauchy-Riemann equations do not hold anywhere.

2. Use Cauchy-Riemann equations to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$ when

a. $f(z) = iz + 2$

b. $f(z) = e^{-x}e^{-iy}$

c. $f(z) = z^3$ H.W

d. $f(z) = \cos x \cosh y - i \sin x \sinh y$ H.W.

Solution:

a. $f(z) = iz + 2$

$f(z) = i(x + iy) + 2$

$f(z) = ix - y + 2$

$f(z) = (2 - y) + ix$

$u(x,y) = 2 - y \qquad and \qquad v(x,y) = x$

$u_x = v_y \quad \Longrightarrow 0 = 0$

$u_y = -v_x \;\Longrightarrow -1 = -1$

$\therefore \quad f'(z)$ exist everywhere

$\therefore \quad f'(z) = u_x + iv_x$

$\therefore \; f'(z) = 0 + i.1 = i$

Now, show that $f''(z)$

$f'(z) = u_1 + iv_1 = 0 + i.1$

$u_1 = 0 \qquad and \qquad v_1 = 1$

$(u_1)_x = (v_1)_y \quad \Longrightarrow 0 = 0$

$(u_1)_y = -(v_1)_x \quad \Longrightarrow 0 = 0$

$\therefore \quad f''(z)$ exist everywhere

$\therefore \quad f''(z) = (u_1)_x + i(v_1)_x = 0 + i.0 = 0$

b. $f(z) = e^{-x}e^{-iy}$

$f(z) = e^{-x}(\cos(y) + i\sin(-y))$

$f(z) = e^{-x}(\cos y - i\sin y) = e^{-x}\cos y - ie^{-x}\sin y$

$u(x,y) = e^{-x}\cos y \qquad and \qquad v(x,y) = -e^{-x}\sin y$

$u_x = -e^{-x}\cos y \quad , \quad v_x = e^{-x}\sin y$

$u_y = -e^{-x}\sin y \quad , \quad v_y = -e^{-x}\cos y$

$\therefore \quad u_x = -e^{-x}\cos y = v_y$

and $\therefore \quad u_y = -e^{-x}\sin y = -v_x$

$\therefore \quad f'(z)$ exist everywhere

$\therefore \quad f'(z) = u_x + iv_x$

$\therefore \quad f'(z) = -e^{-x}\cos y + i\, e^{-x}\sin y$

Now, show that $f''(z)$

$f'(z) = u_1(x, y) + iv_1(x, y)$

$u_1 = -e^{-x}\cos y \qquad and \qquad v_1 = e^{-x}\sin y$

$(u_1)_x = e^{-x}\cos y \quad , \quad (v_1)_x = -e^{-x}\sin y$

$(u_1)_y = e^{-x}\sin y \quad , \quad (v_1)_y = e^{-x}\cos y$

$\therefore \quad (u_1)_x = e^{-x}\cos y = (v_1)_y$

$\therefore \quad (u_1)_y = e^{-x}\sin y = -(v_1)_x$

$\therefore \quad f''(z)$ exist everywhere

$$
\begin{aligned}
\therefore \quad f''(z) &= (u_1)_x + i(v_1)_x \\
&= e^{-x}\cos y + i(-e^{-x}\sin y) \\
&= e^{-x}(\cos y + ie^{-x}\sin(-y)) \\
&= e^{-x}e^{-iy} = f(z)
\end{aligned}
$$

Then

$\therefore \quad f''(z) = f(z) = e^{-x}e^{-iy}$