



كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

أستاذ المادة : م.د. مصطفى ابراهيم حميد

اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنجليزية : Complex Functions

اسم الحاضرة الثامنة باللغة العربية: معادلتا كوشي ريمان في الاحاديث القطبية

اسم المحاضرة الثامنة باللغة الإنجليزية : Cauchy Riemann Equations in Polar Coordinates

Lecture 8

Cauchy Riemann Equations in Polar Coordinates

معادلتا كوشي ريمان في الاحاديث القطبية

Theorem: Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be defined at some ϵ neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that

- a. The first- order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood.
- b. Those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r$$

of the Cauchy-Riemann equations at (r_0, θ_0)

Then $f'(z_0)$ exists and

$$f'(z_0) = e^{-i\theta}(u_r + iv_r) \quad \text{at } (r_0, \theta_0)$$

Example: Use Cauchy-Riemann equations in polar form to find $f'(z)$ where

$$f(z) = \frac{1}{z}$$

Solution:

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \quad , \quad z \neq 0$$

$$f(z) = \frac{1}{r}(\cos \theta - i \sin \theta)$$

$$u(r, \theta) = \frac{1}{r} \cos \theta \quad \text{and} \quad v(r, \theta) = -\frac{1}{r} \sin \theta$$

$$u_r = -\frac{1}{r^2} \cos \theta \quad \text{and} \quad v_r = \frac{1}{r^2} \sin \theta$$

$$u_\theta = -\frac{1}{r} \sin \theta \quad \text{and} \quad v_\theta = -\frac{1}{r} \cos \theta$$

$$r u_r = r \left(-\frac{1}{r^2} \cos \theta \right) = -\frac{1}{r} \cos \theta = v_\theta$$

$$\therefore r u_r = v_\theta$$

$$u_\theta = -\frac{1}{r} \sin \theta = -r v_r = -r \left(\frac{1}{r^2} \sin \theta \right) = -\frac{1}{r} \sin \theta$$

$$\therefore u_\theta = -r v_r$$

$\therefore f'(z)$ exists when $z \neq 0$

$$f'(z) = e^{-i\theta} (u_r + i v_r)$$

$$= e^{-i\theta} \left(-\frac{1}{r^2} \cos \theta + i \frac{1}{r^2} \sin \theta \right)$$

$$= -\frac{1}{r^2} e^{-i\theta} (\cos \theta - i \sin \theta)$$

$$= -\frac{1}{r^2} e^{-i\theta} e^{-i\theta}$$

$$\therefore f'(z) = -\frac{1}{r^2} e^{-2i\theta} = -\frac{1}{r^2 e^{2i\theta}} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

Example: Use Cauchy-Riemann equations to show that when α is a fixed real number, the function

$$f(z) = \sqrt[3]{r} e^{\frac{i\theta}{3}}, \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has a derivative everywhere in its domain of definition

Solution:

$$f(z) = \sqrt[3]{r} e^{\frac{i\theta}{3}} = \sqrt[3]{r} \left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)$$

$$= \sqrt[3]{r} \cos \frac{\theta}{3} + i \sqrt[3]{r} \sin \frac{\theta}{3}$$

$$u(r, \theta) = \sqrt[3]{r} \cos \frac{\theta}{3} \quad \text{and} \quad v(r, \theta) = \sqrt[3]{r} \sin \frac{\theta}{3}$$

$$u_r = \frac{1}{3} r^{\frac{1}{3}-1} \cos \frac{\theta}{3} = \frac{1}{3} r^{\frac{-2}{3}} \cos \frac{\theta}{3} \quad \text{and} \quad v_r = \frac{1}{3} r^{\frac{-2}{3}} \sin \frac{\theta}{3}$$

$$u_\theta = -\frac{1}{3} \sqrt[3]{r} \sin \frac{\theta}{3} \quad \text{and} \quad v_\theta = \frac{1}{3} \sqrt[3]{r} \cos \frac{\theta}{3}$$

$$r u_r = r \left(\frac{1}{3} r^{\frac{-2}{3}} \cos \frac{\theta}{3} \right) = \frac{1}{3} r^{\frac{1}{3}} \cos \frac{\theta}{3} = v_\theta$$

$$\therefore r u_r = v_\theta$$

$$u_\theta = -\frac{1}{3} \sqrt[3]{r} \sin \frac{\theta}{3} = -r v_r = -r \left(\frac{1}{3} r^{\frac{-2}{3}} \sin \frac{\theta}{3} \right) = -\frac{1}{3} \sqrt[3]{r} \sin \frac{\theta}{3}$$

$$\therefore u_\theta = -r v_r$$

$\therefore f'(z)$ exists when $z \neq 0$

$$\begin{aligned}
f'(z) &= e^{-i\theta}(u_r + i\nu_r) \\
&= e^{-i\theta} \left(\frac{1}{3} r^{\frac{-2}{3}} \cos \frac{\theta}{3} + \frac{i}{3} r^{\frac{-2}{3}} \sin \frac{\theta}{3} \right) \\
&= \frac{1}{3} r^{\frac{-2}{3}} e^{-i\theta} \left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right) \\
&= \frac{1}{3} \frac{1}{[\sqrt[3]{r}]^2} e^{-i\theta} e^{i\frac{\theta}{3}} \\
&= \frac{1}{3} \frac{1}{[\sqrt[3]{r}]^2} e^{\frac{-2}{3}i\theta} \\
\\
&= \frac{1}{3} \frac{1}{[\sqrt[3]{r}]^2} \frac{1}{\left[e^{\frac{i\theta}{3}}\right]^2} \\
\\
\therefore f'(z) &= \frac{1}{3} \frac{1}{[\sqrt[3]{r} e^{\frac{i\theta}{3}}]^2} = \frac{1}{3} \frac{1}{[f(z)]^2}
\end{aligned}$$

Examples: Where $f'(z)$ exists and find its value when:

1. $f(z) = \frac{1}{z}$

Solution:

$$f(z) = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

$$f(z) = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}$$

$$\therefore u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{-y}{x^2 + y^2}$$

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y \quad \text{and} \quad u_y = \frac{-2xy}{(x^2 + y^2)^2} = -v_x , \quad x^2 + y^2 \neq 0$$

$f'(z)$ exists when $z \neq 0$

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \frac{2xy}{(x^2 + y^2)^2} \\ &= \frac{-(x^2 - 2ixy - y^2)}{(x^2 + y^2)^2} \\ &= -\frac{(x - iy)^2}{(x^2 + y^2)^2} = -\frac{(\bar{z})^2}{(z\bar{z})^2} = \frac{1}{z^2} \quad , z \neq 0 \end{aligned}$$

2. $f(z) = x^2 + iy^2$

Solution:

$$f(z) = x^2 + iy^2$$

$$\therefore u(x, y) = x^2 \quad \text{and} \quad v(x, y) = y^2$$

$$u_x = 2x , \quad v_x = 0 \quad \text{and} \quad u_y = 0 , \quad v_y = 2y$$

$$u_x = v_y \Rightarrow 2x = 2y \Rightarrow x = y \quad \text{and} \quad u_y = -v_x \Rightarrow 0 = 0$$

So $f'(z)$ exists only when $x = y$, and we find that

$$\begin{aligned} f'(x + ix) &= u_x(x, x) + iv_x(x, x) \\ &= 2x + i \cdot 0 = 2x \end{aligned}$$

3. $f(z) = zIm(z)$

Solution:

$$f(z) = zIm(z)$$

$$f(z) = (x + iy) \cdot y$$

$$f(z) = xy + iy^2$$

$$\therefore u(x, y) = xy \quad \text{and} \quad v(x, y) = y^2$$

$$u_x = y \quad , \quad v_x = 0 \quad \text{and} \quad u_y = x \quad , \quad v_y = 2y$$

$$u_x = v_y \Rightarrow y = 2y \Rightarrow 2y - y = 0 \Rightarrow y = 0 \quad \text{and} \quad u_y = -v_x \Rightarrow x = 0$$

$f'(z)$ exists only when $x = 0$ and $y = 0$

Hence $f'(z)$ exists only when $z = 0$

$$\begin{aligned} f'(0) &= u_x(0,0) + iv_x(0,0) \\ &= 0 + i \cdot 0 = 0 \end{aligned}$$

H.W.

Use Cauchy-Riemann equations in polar coordinates to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

a. $f(z) = \frac{1}{z^4} , \quad z \neq 0$

b. $f(z) = \sqrt{r} e^{\frac{i\theta}{2}} , \quad r > 0 , \quad \alpha < \theta < \alpha + 2\pi$

c. $f(z) = e^{-\theta} \cos(\ln r) + i e^{-\theta} \sin(\ln r) , \quad r > 0 , \quad 0 < \theta < 2\pi$