كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

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اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنكليزية : Complex Functions

اسم الحاضرة العاشرة باللغة العربية: الدوال التوافقية

اسم المحاضرة العاشرة باللغة الإنكليزية : Harmonic Functions

Lecture 10

Harmonic Functions الدوال التوافقية

A real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy −plane if throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\mathrm{H}_{xx}(x,y) + \mathrm{H}_{yy}(x,y) = 0 \qquad \dots (1)$$

Equation (1) is said to be Laplace's equation

يقال للدالة الحقيقية $\mathrm{H}(x,y)$ ذات المتغيرين الحقيقين y,x انها توافقية في منطلق معين في المستوي xy اذا كانت لها مشتقات جزئية مستمرة اولى والثانية ضمن ذلك المنطلق وتحقق المعادلة التفاضلية الجزئية $\mathrm{H}_{xx}(x,y) + \mathrm{H}_{yy}(x,y) = 0$

Theorem 1:

If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof:

Suppose that the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D.

$\therefore \quad u_x = v_y \qquad and \qquad u_y = -v_x$

$\therefore \quad u_{xx} = v_{yx} \qquad and \qquad u_{yx} = -\, v_{xx}$

and

$\therefore \quad u_{xy} = v_{yy} \qquad and \qquad u_{yy} = -\, v_{xy}$

The continuity of the partial derivatives of u and v ensures that

$$u_{xy} = u_{yx} \qquad and \qquad v_{xy} = v_{yx}$$

$$\therefore \quad u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

$$\therefore \quad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

That is, u and v are harmonic in D.

Example: The function

$$f(z) = e^{-y} \sin x - ie^{-y} \cos x$$

is entire

$\therefore \quad f(z)$ is analytic function

$\therefore \quad u(x, y) = e^{-y} \sin x \qquad and \qquad v(x, y) = -\, e^{-y} \cos x$

are harmonic function

$\therefore \quad u_x = e^{-y} \cos x \qquad and \qquad u_y = -e^{-y} \sin x$

$\therefore \quad u_{xx} = -e^{-y} \sin x \qquad and \qquad u_{yy} = e^{-y} \sin x$

$\therefore \quad u_{xx} + u_{yy} = -e^{-y} \sin x + e^{-y} \sin x = 0$

$\therefore \quad v_x = e^{-y} \sin x \qquad and \qquad v_y = e^{-y} \cos x$

$\therefore \quad v_{xx} = e^{-y}\cos x \qquad and \qquad v_{yy} = -e^{-y}\cos x$

$\therefore \quad v_{xx} + v_{yy} = e^{-y}\cos x - e^{-y}\cos x = 0$

H.W

Show that

$f(z) = \dfrac{i}{z^2} \quad , z \neq 0$

is analytic whenever $z \neq 0$

Now,

$$f(z) = \frac{i}{z^2} = \frac{i}{z^2}.\frac{\bar{z}^2}{\bar{z}^2} = \frac{i\bar{z}^2}{(z\bar{z})^2}$$

$$= \frac{i\bar{z}^2}{|z|^4} = \frac{i\bar{z}^2}{(x^2+y^2)^2}$$

$$= \frac{i(x-iy)^2}{(x^2+y^2)^2}$$

$$= \frac{i(x^2-2ixy-y^2)}{(x^2+y^2)^2}$$

$$= \frac{ix^2 + 2xy - iy^2}{(x^2 + y^2)^2}$$

$$= \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2}$$

$\therefore \quad u(x, y) = \dfrac{2xy}{(x^2 + y^2)^2} \qquad and \qquad v(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

Show $u_{xx} + u_{yy} = 0 \qquad and \qquad v_{xx} + v_{yy} = 0$

So that u and v are harmonic in the xy −plane except the origin.