كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

أستاذ المادة : م.د. مصطفى ابراهيم حميد

اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنكليزية : **Complex Functions**

اسم الحاضرة الحادية عشر باللغة العربية: المرافق التوافقي

اسم المحاضرة الحادية عشر باللغة الإنكليزية : **Harmonic Conjugate**

Lecture 11

Harmonic Conjugate المرافق التوافقي

Definition:

If two given functions u and v are harmonic functions in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations $(u_x = v_y \ \ and \ \ u_y = -v_x)$ in D, then v is said to be a harmonic conjugate of u.

(harmonic conjugate) اذا كانت v, u دالتين توافقيتين في المنطلق D فانه يقال للدالة v بالمرافق التوافقي للدالة u

Theorem:

A function $f(z) = u(x, y) + iv(x, y)$ is analytic function in a domain D if and only if v is a harmonic conjugate of u.

Proof:

Suppose that v is a harmonic conjugate of u in D.

Then u and v are harmonic functions in D and their first-order partial derivatives satisfy Cauchy-Riemann equations.

Then f is analytic in D.

Conversely,

if f is analytic in D, then u and v are harmonic functions in D and the Cauchy-Riemann equations are satisfied.

So we have v is a harmonic conjugate of u in D.

Example: Suppose that

$$u(x,y) = x^2 - y^2 \qquad and \qquad v(x,y) = 2xy$$

1. Show $u(x,y)$ and $v(x,y)$ are harmonic functions.

Solution:

$$u_x = 2x \quad , \quad u_{xx} = 2 \quad and \quad u_y = -2y \quad , \quad u_{yy} = -2$$

$$\therefore \quad u_{xx} + u_{yy} = 2 - 2 = 0$$

$$v_x = 2y \quad , \quad v_{xx} = 0 \quad and \quad v_y = 2x \quad , \quad v_{yy} = 0$$

$$\therefore \quad v_{xx} + v_{yy} = 0 + 0 = 0$$

Then $u(x,y)$ and $v(x,y)$ are harmonic functions.

2. Show v is a harmonic conjugate of u.

Solution:

$$u_x = 2x = v_y \qquad and \qquad u_y = -2y = -v_x$$

By theorem (A function $f(z) = u(x,y) + iv(x,y)$ is analytic function in a domain D if and only if v is a harmonic conjugate of u.)

$f(z) = u(x, y) + iv(x, y)$

$= (x^2 - y^2) + 2ixy$

$\therefore\ f(z) = z^2$ is analytic function.

Then v is a harmonic conjugate of u.

3. Show u cannot be a harmonic conjugate of v.

Solution:

Since the function $2xy + i(x^2 - y^2)$ is not analytic function anywhere.

Then u cannot be a harmonic conjugate of v.

Example: Show that $u(x, y)$ is harmonic function and find a harmonic conjugate $v(x, y)$ when $u(x, y) = y^3 - 3yx^2$

Solution:

$u_x = -6xy \quad , \quad u_{xx} = -6y \qquad and \qquad u_y = 3y^2 - 3x^2 \quad , \quad u_{yy} = 6y$

$u_{xx} + u_{yy} = -6y + 6y = 0$

$\therefore\ u(x, y)$ is a harmonic function in xy $-$plane.

Now, find a harmonic conjugate $v(x, y)$

Then

$u_x = v_y \qquad and \qquad u_y = -v_x$

$\therefore\ v_y(x, y) = -6xy \qquad \leftarrow y$ نكامل بالنسبة لـ

$$\int v_y(x,y)dy = \int -6xydy$$

$\Longrightarrow v(x,y) = -3xy^2 + \emptyset(x) \quad \leftarrow x$ نشتق بالنسبة لـ $\qquad , \emptyset(x)$ دالة اختيارية

$\therefore \quad v_x(x,y) = -3y^2 + \emptyset'(x)$

Since $u_y(x,y) = -v_x(x,y)$

$\therefore \quad 3y^2 - 3x^2 = -\left(-3y^2 + \emptyset'(x)\right)$

$\Longrightarrow \quad 3y^2 - 3x^2 = 3y^2 - \emptyset'(x)$

$\Longrightarrow \quad -3x^2 = -\emptyset'(x)$

$\Longrightarrow \quad \emptyset'(x) = 3x^2 \quad \leftarrow$ نكامل

$$\Longrightarrow \int \emptyset'(x)dx = \int 3x^2dx$$

$\Longrightarrow \quad \emptyset(x) = x^3 + c \qquad , c$ ثابت اختياري

$\therefore \; v(x,y) = -3xy^2 + x^3 + c$ is a harmonic conjugate of $u(x,y)$.

So that $f(z) = u(x,y) + iv(x,y)$ is analytic function.

$\therefore \quad f(z) = (y^3 - 3yx^2) + i(-3xy^2 + x^3 + c)$ is analytic function.