كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

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اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنكليزية : Complex Functions

اسم الحاضرة الثانية عشر باللغة العربية: تمارين

اسم المحاضرة الثانية عشر باللغة الإنكليزية : Exercises:

Lecture 12

Exercises:

1. Show that $u(x,y)$ is harmonic function in some domain and find a harmonic conjugate $v(x,y)$ when:
   a. $u(x,y) = 2x(1-y)$
   b. $u(x,y) = 2x - x^3 + 3xy^2$
   c. $u(x,y) = \sinh x \sin y$
   d. $u(x,y) = \dfrac{y}{(x^2+y^2)}$

Solution:

a. $u(x,y) = 2x(1-y) = 2x - 2xy$

$u_x = 2 - 2y \quad , \quad u_{xx} = 0 \qquad and \qquad u_y = -2x \quad , \quad u_{yy} = 0$

$u_{xx} + u_{yy} = 0 + 0 = 0$

$\therefore \quad u(x,y)$ is a harmonic function.

Now, find a harmonic conjugate $v(x,y)$

Then

$u_x = v_y \qquad and \qquad u_y = -v_x$

$u_x = 2 - 2y = v_y$

$\therefore \quad v_y(x,y) = 2 - 2y \qquad \leftarrow$ نكامل بالنسبة لـ y

$$\int v_y(x,y)dy = \int (2-2y)dy$$

$\Longrightarrow v(x,y) = 2y - y^2 + \emptyset(x)$ $\leftarrow$ نشتق بالنسبة لـ x

$\therefore \quad v_x(x,y) = 0 - 0 + \emptyset'(x) \Longrightarrow v_x(x,y) = \emptyset'(x)$

Since $u_y(x,y) = -v_x(x,y)$

$\therefore \; -2x = -\big(\emptyset'(x)\big)$

$\Longrightarrow \; \emptyset'(x) = 2x$ $\leftarrow$ نكامل

$$\Longrightarrow \int \emptyset'(x)dx = \int 2xdx$$

$\Longrightarrow \; \emptyset(x) = x^2 + c$

$\therefore v(x,y) = 2y - y^2 + x^2 + c = x^2 - y^2 + 2y + c$ is a harmonic conjugate of $u(x,y)$.

$d.$ $u(x,y) = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2 \quad , \quad u_{xx} = -6x \quad and \quad u_y = 6xy \quad , \quad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

$\therefore \quad u(x,y)$ is a harmonic function.

Now, find a harmonic conjugate $v(x,y)$

Then

$u_x = v_y \qquad and \qquad u_y = -v_x$

$u_x = 2 - 3x^2 + 3y^2 = v_y$

$\therefore \quad v_y(x,y) = 2 - 3x^2 + 3y^2$ $\leftarrow$ نكامل بالنسبة لـ y

$$\int v_y(x,y)dy = \int (2 - 3x^2 + 3y^2)dy$$

$\Longrightarrow v(x,y) = 2y - 3yx^2 + y^3 + \emptyset(x)$ $\leftarrow$ نشتق بالنسبة لـ x

$\therefore \quad v_x(x,y) = 0 - 6xy + 0 + \emptyset'(x) \Longrightarrow v_x(x,y) = -6xy + \emptyset'(x)$

Since $u_y(x,y) = -v_x(x,y)$

$\therefore\ 6xy = -\big(-6xy + \emptyset'(x)\big) \Longrightarrow 6xy = 6xy - \emptyset'(x)$

$\Longrightarrow \quad \emptyset'(x) = 6xy - 6xy = 0 \ \Longrightarrow \emptyset'(x) = 0$ $\leftarrow$ نكامل

$$\Longrightarrow \int \emptyset'(x)dx = \int 0dx$$

$\Longrightarrow \quad \emptyset(x) = c$

$\therefore\ v(x,y) = 2y - 3yx^2 + y^3 + c$ is a harmonic conjugate of $u(x,y)$.

c. $u(x,y) = \sinh x \sin y$

$u_x = \cosh x \sin y \quad , \quad u_{xx} = \sinh x \sin y$

and

$u_y = \sinh x \cos y \quad , \quad u_{yy} = -\sinh x \sin y$

$u_{xx} + u_{yy} = \sinh x \sin y - \sinh x \sin y = 0$

$\therefore \quad u(x,y)$ is a harmonic function.

Now, find a harmonic conjugate $v(x,y)$

Then

$u_x = v_y \qquad and \qquad u_y = -v_x$

$u_x = \cosh x \sin y = v_y$

$\therefore \quad v_y(x,y) = \cosh x \sin y \qquad \leftarrow$ نكامل بالنسبة لـ y

$$\int v_y(x,y)dy = \int (\cosh x \sin y)dy$$

$\Longrightarrow v(x,y) = -\cosh x \cos y + \emptyset(x) \qquad \leftarrow$ نشتق بالنسبة لـ x

$\therefore \quad v_x(x,y) = -\sinh x \cos y + \emptyset'(x)$

Since $\quad u_y(x,y) = -v_x(x,y)$

$\therefore \ \sinh x \cos y = -\big(-\sinh x \cos y + \emptyset'(x)\big)$

$\Longrightarrow \quad \emptyset'(x) = \sinh x \cos y - \sinh x \cos y = 0 \ \Longrightarrow \emptyset'(x) = 0 \quad \leftarrow$ نكامل

$$\Longrightarrow \int \emptyset'(x)dx = \int 0dx$$

$\Longrightarrow \quad \emptyset(x) = c$

$\therefore \ v(x,y) = -\cosh x \cos y + c$ is a harmonic conjugate of $u(x,y)$.

d. $u(x,y) = \dfrac{y}{(x^2+y^2)}$

Show that $\quad u_{xx} + u_{yy} = 0 \qquad$ (H.W.)

Now, find a harmonic conjugate $v(x,y)$

Then

$$u_x = \frac{0 - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = v_y$$

and

$$u_y = \frac{(x^2+y^2).1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = -v_x$$

$$u_x = \frac{-2xy}{(x^2+y^2)^2} = v_y$$

$$\therefore \quad v_y(x,y) = \frac{-2xy}{(x^2+y^2)^2} \qquad \leftarrow y \text{ نكامل بالنسبة لـ}$$

$$\int v_y(x,y)dy = -x\int 2y(x^2+y^2)^{-2}dy$$

$$\Longrightarrow v(x,y) = -x.\frac{(x^2+y^2)^{-1}}{-1} + \emptyset(x)$$

$$\Longrightarrow \; v(x,y) = \frac{x}{(x^2+y^2)} + \emptyset(x) \quad \leftarrow x \text{ نشتق بالنسبة لـ}$$

$$\therefore \quad v_x(x,y) = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2} + \emptyset'(x)$$

$$\Longrightarrow \; v_x(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} + \emptyset'(x)$$

Since $u_y(x,y) = -v_x(x,y)$

$$\therefore \; \frac{x^2-y^2}{(x^2+y^2)^2} = -\left(\frac{y^2-x^2}{(x^2+y^2)^2} + \emptyset'(x)\right)$$

$$\Longrightarrow \; \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \emptyset'(x)$$

$$\Longrightarrow \; \emptyset'(x) = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0 \;\Longrightarrow \emptyset'(x) = 0 \quad \leftarrow \text{نكامل}$$

$$\Longrightarrow \int \emptyset'(x)dx = \int 0dx$$

$\Longrightarrow \quad \emptyset(x) = c$

$\therefore \; v(x,y) = \dfrac{x}{(x^2+y^2)} + c$ is a harmonic conjugate of $u(x,y)$.

2. Suppose that v and V are harmonic conjugates of u in a domain D.

Solution:

$u_x = v_y \qquad and \qquad u_y = -v_x$

$u_x = V_y \qquad and \qquad u_y = -V_x$

If $w = v - V$, then

$w_x = v_x - V_x = -u_y - (-u_y)$

$w_x = -u_y + u_y = 0 \;\Longrightarrow\; w_x = 0$

$\therefore \quad w_x(x,y) = 0 \qquad \leftarrow$ نكامل بالنسبة لـ x

$$\int w_x(x,y)dx = \int 0dx$$

$\Longrightarrow w(x,y) = c$

$\therefore \; v(x,y) - V(x,y) = c$ are harmonic conjugates of $u(x,y)$.

3. Suppose that u and v are harmonic conjugates of each other in a domain D.

Solution:

$u_x = v_y \qquad and \qquad u_y = -v_x$

$v_x = u_y \qquad and \qquad v_y = -u_x$

$\therefore\ v_y + v_y = u_x - u_x = 0$

$\Longrightarrow\ 2v_y = 0\ \Longrightarrow\ v_y(x,y) = 0 \qquad \leftarrow$ نكامل بالنسبة لـ y

$$\int v_y(x,y)dy = \int 0dy$$

$\Longrightarrow v(x,y) = c_1$

and $u_y + u_y = v_x - v_x = 0$

$\Longrightarrow\ 2u_y = 0\ \Longrightarrow\ u_y(x,y) = 0 \qquad \leftarrow$ نكامل بالنسبة لـ y

$$\int u_y(x,y)dy = \int 0dy$$

$\Longrightarrow u(x,y) = c_2$

$\therefore\ u(x,y) + v(x,y) = c_1 + c_2$ are harmonic conjugates of each other in a domain D.