



كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

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اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنجليزية : Complex Functions

اسم الحاضرة الثالثة عشر باللغة العربية: الدوال الاولية

اسم المحاضرة الثالثة عشر باللغة الإنجليزية : Elementary Functions

Lecture 13

Elementary Functions الدوال الأولية

The Exponential Function الدالة الأسية

$$e^z = e^x \cdot e^{iy} , \quad z = x + iy \quad \dots (1)$$

where

$$e^{iy} = \cos y + i \sin y \quad (\text{Euler's formula})$$

خواص الدالة الأسية

1. $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$
2. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$
3. $\frac{1}{e^z} = e^{-z} \quad \text{using (2.) and the fact that } e^0 = 1$
4. $\frac{d}{dz} e^z = e^z$

everywhere in the z -plane. Note that the differentiability of e^z for all z tells us that e^z is entire. It is also true that $e^z \neq 0$ for any complex number z .

الدالة الأسية بالصيغة القطبية

$$e^z = e^x \cdot e^{iy}$$

if $e^x = \rho$ and $y = \emptyset$

$$\Rightarrow e^z = \rho \cdot e^{i\emptyset}$$

* $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

Since

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i}$$

$$= e^z \cdot (\cos 2\pi + i \sin 2\pi)$$

$$= e^z \cdot (1)$$

$$\Rightarrow e^{z+2\pi i} = e^z$$

We find that e^z is periodic with a pure imaginary period of $2\pi i$

اي ان $e^{z+2\pi i} = e^z$ تعني ان الدالة الأسية e^z هي دورية بدوره تخيلية بحثة مقدار دورانها $.2\pi i$

Example: $e^{i\pi+2n\pi i} = e^{i\pi}$

Solution:

$$\begin{aligned} e^{i\pi+2n\pi i} &= e^{i\pi} \cdot e^{2n\pi i} \\ &= (\cos \pi + i \sin \pi)(\cos 2n\pi + i \sin 2n\pi) \\ &= (-1 + 0)(1 + 0) \\ &= (-1)(1) = -1 = e^{i\pi} \\ \Rightarrow e^{i\pi+2n\pi i} &= e^{i\pi} \quad , \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Example: Find $z = x + iy$ such that $e^z = 1 + i$

Solution:

$$\begin{aligned} e^z = e^x \cdot e^{iy} &= \sqrt{2} e^{i\frac{\pi}{4}} \\ \Rightarrow e^x &= \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \quad , \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow \ln(e^x) &= \ln \sqrt{2} \\ &= \ln(2)^{\frac{1}{2}} = \frac{1}{2} \ln(2) \\ \Rightarrow x &= \frac{1}{2} \ln(2) \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \\ \Rightarrow z = x + iy &= \frac{1}{2} \ln(2) + \left(2n + \frac{1}{4}\right) \pi i \quad , \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Example: Show that

$$1. \quad \exp(2 \pm 3\pi i) = -e^2$$

Solution:

$$\begin{aligned}\exp(2 \pm 3\pi i) &= e^2 \cdot e^{\pm 3\pi i} \\ &= e^2(\cos(\pm 3\pi) + i \sin(\pm 3\pi)) \\ &= e^2(\cos(3\pi) + i \cdot 0) \\ &= e^2(-1) = -e^2\end{aligned}$$

$$2. \quad \exp\left(\frac{2 + \pi i}{4}\right) = \sqrt{\frac{e}{2}}(1 + i)$$

Solution:

$$\begin{aligned}\exp\left(\frac{2 + \pi i}{4}\right) &= \exp\left(\frac{1}{2}\right) \cdot \exp\left(\frac{\pi}{4}i\right) \\ &= \sqrt{e} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ &= \sqrt{e} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \\ &= \frac{\sqrt{e}}{\sqrt{2}} (1 + i)\end{aligned}$$

$$= \sqrt{\frac{e}{2}}(1 + i)$$

3. $\exp(z + \pi i) = -\exp(z)$

Solution:

$$\begin{aligned}\exp(z + \pi i) &= \exp(z) \cdot \exp(\pi i) \\ &= \exp(z)(\cos \pi + i \sin \pi) \\ &= \exp(z)(-1 + i \cdot 0) \\ &= -\exp(z)\end{aligned}$$

H.W.

1. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.
2. Use the Cauchy-Riemann equations to show that function $f(z) = \exp(\bar{z})$ is not analytic anywhere.
3. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative ?
4. Find all values of z such that
 - a. $e^z = -2$

b. $e^z = 1 + \sqrt{3}i$

c. $\exp(2z - 1) = 1$