



كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

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اسم المادة باللغة العربية : الدوال المركبة

اسم المادة باللغة الإنجليزية : Complex Functions

اسم الحاضرة الرابعة عشر باللغة العربية: الدوال المثلثية

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## Lecture 14

### Trigonometric Functions الدوال المثلثية

From the Euler's formula, we have

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

For every real number  $x$ .

Hence,

$$e^{ix} - e^{-ix} = \cos x + i \sin x - \cos x + i \sin x$$

$$e^{ix} - e^{-ix} = i \sin x + i \sin x = 2i \sin x$$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \dots (1)$$

and

$$e^{ix} + e^{-ix} = \cos x + i \sin x + \cos x - i \sin x$$

$$e^{ix} + e^{-ix} = i \sin x + i \sin x = 2 \cos x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \dots (2)$$

It is therefore, natural to define the sin and cosine functions of a complex variable  $z$  as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \dots (3)$$

These functions are entire since they are linear combinations of the entire

functions  $e^{iz}$  and  $e^{-iz}$ . The derivatives are

$$\frac{d}{dz} e^{iz} = ie^{iz} \quad \text{and} \quad \frac{d}{dz} e^{-iz} = -ie^{-iz}.$$

From equations (3), we find

$$\frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z$$

### خواص الدالة المثلثية

1.  $\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z$

2.  $e^{iz} = \cos z + i \sin z$

3.  $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

4.  $\sin 2z = 2 \sin z \cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z$

$$5. \quad \sin\left(z + \frac{\pi}{2}\right) = \cos z \quad \text{and} \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z$$

$$6. \quad \sin^2 z + \cos^2 z = 1$$

$$7. \quad \sin(z + 2\pi) = \sin z \quad \text{and} \quad \sin(z + \pi) = -\sin z$$

$$8. \quad \cos(z + 2\pi) = \cos z \quad \text{and} \quad \cos(z + \pi) = -\cos z$$

When  $y$  is any real number, then

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

and said to be hyperbolic functions.

$$9. \quad \sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y$$

10. if  $z = x + iy$ , then

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\ &= \frac{e^{ix} \cdot e^{-y} - e^{-ix} \cdot e^y}{2i} \\ &= \frac{1}{2i} e^{-y} (\cos x + i \sin x) - \frac{1}{2i} e^y (\cos x - i \sin x) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2i} e^{-y} \cos x + \frac{1}{2} e^{-y} \sin x - \frac{1}{2i} e^y \cos x + \frac{1}{2} e^y \sin x \\
&= \sin x \left( \frac{1}{2} e^{-y} + \frac{1}{2} e^y \right) + \cos x \left( \frac{1}{2i} e^{-y} - \frac{1}{2i} e^y \right) \\
&= \sin x \left( \frac{e^y + e^{-y}}{2} \right) - \frac{1}{i} \cos x \left( \frac{e^y - e^{-y}}{2} \right) \\
(-i^2) \cdot \sin z &= (-i^2) \cdot \sin x \cosh y - (-i^2) \cdot \frac{1}{i} \cos x \sinh y \\
\Rightarrow \sin z &= \sin x \cosh y + i \cos x \sinh y
\end{aligned}$$

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11.  $\cos z = \cos x \cosh y - i \sin x \sinh y$

The Properties (10) and (11) tells us that the real and imaginary components of  $\sin(z)$  and  $\cos(z)$  can be displayed in terms of those hyperbolic functions.

12.  $|\sin z|^2 = \sin^2 x + \sinh^2 y$

13.  $|\cos z|^2 = \cos^2 x + \sinh^2 y$