



كلية : التربية للعلوم الصرفة

القسم او الفرع : الفيزياء

المرحلة: الثالثة

أستاذ المادة : م.د. مصطفى ابراهيم حميد

اسم المادة باللغة العربية : نظرية المجاميع

اسم المادة باللغة الإنجليزية : Set Theory

اسم الحاضرة الثامنة باللغة العربية: أمثلة ٢ على سلاسل فورييه

اسم المحاضرة الثامنة باللغة الإنجليزية : Examples2 of Fourier Series :

## Lecture 8

Example:(3) Compute the Fourier Series of the even square wave function of period  $2L$

$$f(t) = \begin{cases} -1 & -L \leq t < -\frac{L}{2} \\ 1 & -\frac{L}{2} \leq t < \frac{L}{2} \\ -1 & \frac{L}{2} \leq t < L \end{cases}$$

Solution:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi}{L}\right)t + b_n \sin\left(\frac{n\pi}{L}\right)t \right)$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(t) dt$$

$$a_0 = \frac{1}{L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} f(t) dt + \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(t) dt + \frac{1}{L} \int_{\frac{L}{2}}^L f(t) dt$$

$$a_0 = \frac{1}{L} \int_{-L}^{-\frac{L}{2}} (-1) dt + \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} (1) dt + \frac{1}{L} \int_{\frac{L}{2}}^L (-1) dt$$

$$a_0 = \left[ \frac{1}{L}(-t) \right]_L^{-\frac{L}{2}} + \left[ \frac{1}{L}(t) \right]_{-\frac{L}{2}}^{\frac{L}{2}} - \left[ \frac{1}{L}(t) \right]_{\frac{L}{2}}^L$$

$$a_0 = -\frac{1}{L} \left[ -\frac{L}{2} + L \right] + \frac{1}{L} \left[ \frac{L}{2} + \frac{L}{2} \right] - \frac{1}{L} \left[ L - \frac{L}{2} \right]$$

$$a_0 = -\frac{1}{L} \cdot \frac{L}{2} + \frac{1}{L} \cdot L - \frac{1}{L} \cdot \frac{L}{2}$$

$$a_0 = -\frac{1}{2} + 1 - \frac{1}{2} = \frac{-1 + 2 - 1}{2} = \frac{-2 + 2}{2} = \frac{0}{2} = 0$$

$$\therefore a_0 = 0$$

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cdot \cos\left(\frac{n\pi}{L}\right) t dt , \quad \text{for } n = 1, 2, 3, \dots$$

$$a_n = \frac{1}{L} \int_{-L}^{-\frac{L}{2}} f(t) \cos\left(\frac{n\pi}{L}\right) t dt + \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(t) \cos\left(\frac{n\pi}{L}\right) t dt + \frac{1}{L} \int_{\frac{L}{2}}^L f(t) \cos\left(\frac{n\pi}{L}\right) t dt$$

$$a_n = \frac{1}{L} \int_{-L}^{-\frac{L}{2}} (-1) \cos\left(\frac{n\pi}{L}\right) t dt + \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} (1) \cos\left(\frac{n\pi}{L}\right) t dt + \frac{1}{L} \int_{\frac{L}{2}}^L (-1) \cos\left(\frac{n\pi}{L}\right) t dt$$

$$a_n = \frac{1}{L} \left\{ - \left[ \frac{L}{n\pi} \cdot \sin\left(\frac{n\pi}{L}\right) t \right]_{-L}^{-\frac{L}{2}} + \left[ \frac{L}{n\pi} \cdot \sin\left(\frac{n\pi}{L}\right) t \right]_{-\frac{L}{2}}^{\frac{L}{2}} - \left[ \frac{L}{n\pi} \cdot \sin\left(\frac{n\pi}{L}\right) t \right]_{\frac{L}{2}}^L \right\}$$

$$a_n = \frac{1}{L} \cdot \frac{L}{n\pi} \left\{ -\sin\left(\frac{-n\pi}{2}\right) + \sin(-n\pi) + \sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right) - \sin(n\pi) + \sin\left(\frac{n\pi}{2}\right) \right\}$$

$$a_n = \frac{1}{n\pi} \left\{ \sin\left(\frac{n\pi}{2}\right) + 0 + \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) - 0 + \sin\left(\frac{n\pi}{2}\right) \right\}$$

$$a_n = \frac{1}{n\pi} \left\{ \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) \right\}$$

$$a_n = \frac{1}{n\pi} \left\{ 4 \cdot \sin\left(\frac{n\pi}{2}\right) \right\} = \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) , \quad n = 1, 2, 3, \dots$$

If  $n = 1$

$$\Rightarrow a_1 = \frac{4}{\pi} \sin\left(\frac{\pi}{2}\right) = \frac{4}{\pi} (1) = \frac{4}{\pi}$$

If  $n = 2$

$$\Rightarrow a_2 = \frac{4}{(2)\pi} \sin\left(\frac{2\pi}{2}\right) = \frac{2}{\pi} \sin(\pi) = \frac{2}{\pi} (0) = 0$$

If  $n = 3$

$$\Rightarrow a_3 = \frac{4}{(3)\pi} \sin\left(\frac{3\pi}{2}\right) = \frac{4}{3\pi} (-1) = -\frac{4}{3\pi}$$

If  $n = 4$

$$\Rightarrow a_4 = \frac{4}{(4)\pi} \sin\left(\frac{4\pi}{2}\right) = \frac{1}{\pi} \sin(2\pi) = 0$$

If  $n = 5$

$$\Rightarrow a_5 = \frac{4}{(5)\pi} \sin\left(\frac{5\pi}{2}\right) = \frac{4}{5\pi}(1) = \frac{4}{5\pi}$$

If  $n = 6$

$$\Rightarrow a_6 = \frac{4}{(6)\pi} \sin\left(\frac{6\pi}{2}\right) = \frac{2}{3\pi} \sin(3\pi) = 0$$

If  $n = 7$

$$\Rightarrow a_7 = \frac{4}{(7)\pi} \sin\left(\frac{7\pi}{2}\right) = \frac{4}{7\pi}(-1) = -\frac{4}{7\pi}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(t) \cdot \sin\left(\frac{n\pi}{L}\right) t dt = 0 \quad , \quad n = 1, 2, 3, \dots$$

$f(t) \cdot \sin\left(\frac{n\pi}{L}\right) t$  is the product of an even function and odd function and hence is odd, which implies by theorem page (14) part (5), that the integral is zero.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi}{L}\right) t + b_n \sin\left(\frac{n\pi}{L}\right) t \right)$$

$$a_0 = 0 \quad \text{and} \quad b_n = 0$$

$$f(t) = \frac{0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi}{L}\right) t + (0) \sin\left(\frac{n\pi}{L}\right) t \right)$$

$$f(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) t$$

$$\begin{aligned} f(t) &\sim \frac{4}{\pi} \cos\left(\frac{\pi}{L}\right) t + (0) \cos\left(\frac{2\pi}{L}\right) t + \left(\frac{-4}{3\pi}\right) \cos\left(\frac{3\pi}{L}\right) t + (0) \cos\left(\frac{4\pi}{L}\right) t \\ &\quad + \left(\frac{4}{5\pi}\right) \cos\left(\frac{5\pi}{L}\right) t + (0) \cos\left(\frac{6\pi}{L}\right) t + \left(\frac{-4}{7\pi}\right) \cos\left(\frac{7\pi}{L}\right) t + \dots \end{aligned}$$

$$f(t) \sim \frac{4}{\pi} \cos\left(\frac{\pi}{L}\right) t - \frac{4}{3\pi} \cos\left(\frac{3\pi}{L}\right) t + \frac{4}{5\pi} \cos\left(\frac{5\pi}{L}\right) t - \frac{4}{7\pi} \cos\left(\frac{7\pi}{L}\right) t + \dots$$

$$f(t) \sim \frac{4}{\pi} \left( \cos\left(\frac{\pi}{L}\right) t - \frac{1}{3} \cos\left(\frac{3\pi}{L}\right) t + \frac{1}{5} \cos\left(\frac{5\pi}{L}\right) t - \frac{1}{7} \cos\left(\frac{7\pi}{L}\right) t + \dots \right)$$

### Exercises:

1- Compute the Fourier Series of the even triangular wave function of period  $2\pi$  given by:

$$f(t) = \begin{cases} -t & -\pi \leq t < 0 \\ t & 0 \leq t < \pi \end{cases}$$

### Ans:

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \frac{\cos(7t)}{7^2} + \dots \right)$$

2- Compute the Fourier Series of the function

$$f(t) = t - t^2 \quad \text{from } t = -\pi \text{ to } t = \pi$$

Ans:

$$t - t^2 \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt) - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$$