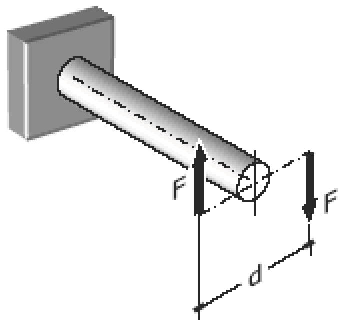
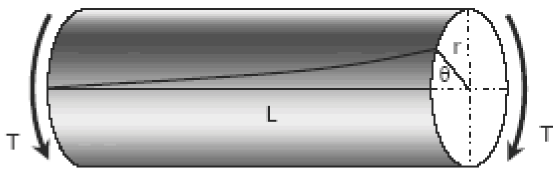
Torsion

Consider a bar to be rigidly attached at one end and twisted at the other end by a torque or twisting moment T equivalent to F × d, which is applied perpendicular to the axis of the bar, as shown in the figure. Such a bar is said to be in torsion.





TORSIONAL SHEARING STRESS, τ

For a solid or hollow circular shaft subject to a twisting moment T, the torsional shearing stress τ at a distance ρ from the center of the shaft is

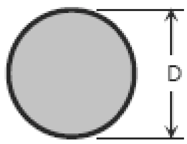
$$\tau = \frac{T\rho}{J} \text{ and } \tau_{max} = \frac{Tr}{J}$$

where J is the polar moment of inertia of the section and r is the outer radius.

For solid cylindrical shaft:

$$J = \frac{\pi}{32}D^4$$

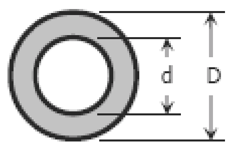
$$\tau_{max} = \frac{16T}{\pi D^3}$$



For hollow cylindrical shaft:

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\tau_{max} = \frac{16TD}{\pi(D^4 - d^4)}$$



ANGLE OF TWIST

The angle θ through which the bar length L will twist is

$$\theta = \frac{TL}{JG} \text{ in radians}$$

where T is the torque in N·mm, L is the length of shaft in mm, G is shear modulus in MPa, J is the polar moment of inertia in mm^4, D and d are diameter in mm, and r is the radius in mm.

POWER TRANSMITTED BY THE SHAFT

A shaft rotating with a constant angular velocity ω (in radians per second) is being acted by a twisting moment T. The power transmitted by the shaft is

$$P = T\omega = 2\pi T f$$

where T is the torque in N·m, f is the number of revolutions per second, and P is the power in watts.

Solved Problems in Torsion

Problem 304

A steel shaft 3 ft long that has a diameter of 4 in. is subjected to a torque of 15 kip·ft. Determine the maximum shearing stress and the angle of twist. Use $G = 12 \times 10^6$ psi.

Solution 304

$$\tau_{max} = \frac{16T}{\pi D^3} = \frac{16(15)(1000)(12)}{\pi(4^3)}$$

$$\tau_{max} = 14\,324 \text{ psi}$$

$$\tau_{max} = \mathbf{14.3\ ksi}$$

$$\theta = \frac{TL}{JG} = \frac{15(3)(1000)(12^2)}{\frac{1}{32}\pi(4^4)(12\times10^6)}$$

$$\theta = 0.0215 \text{ rad}$$

$$\theta = \mathbf{1.23°}$$

Problem 305

What is the minimum diameter of a solid steel shaft that will not twist through more than 3° in a 6-m length when subjected to a torque of 12 kN·m? What maximum shearing stress is developed? Use G = 83 GPa.

Solution 305

$$\theta = \frac{TL}{JG}$$

$$3°\left(\frac{\pi}{180°}\right) = \frac{12(6)(1000^3)}{\frac{1}{32}\pi d^4(83000)}$$

$$d = \mathbf{113.98\ mm}$$

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(12)(1000^2)}{\pi(113.98^3)}$$

$$\tau_{max} = \mathbf{41.27\ MPa}$$

Problem 306

A steel marine propeller shaft 14 in. in diameter and 18 ft long is used to transmit 5000 hp at 189 rpm. If G = 12 × 10^6 psi, determine the maximum shearing stress.

Solution 306

$$T = \frac{P}{2\pi f} = \frac{5000(396\,000)}{2\pi(189)}$$

$$T = 1\ 667\ 337.5\ \text{lb·in}$$

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(1\,667\,337.5)}{\pi(14^3)}$$

$$\tau_{max} = \mathbf{3094.6\ psi}$$

Problem 307

A solid steel shaft 5 m long is stressed at 80 MPa when twisted through 4°. Using G = 83 GPa, compute the shaft diameter. What power can be transmitted by the shaft at 20 Hz?

Solution 307

$$\theta = \frac{TL}{JG}$$

$$4°\left(\frac{\pi}{180°}\right) = \frac{T(5)(1000)}{\frac{1}{32}\pi d^4(83\,000)}$$

$$T = 0.1138d^4$$

$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$80 = \frac{16(0.1138d^4)}{\pi d^3}$$

$$d = \mathbf{138\ mm}$$

$$T = \frac{P}{2\pi f}$$

$$0.1138d^4 = \frac{P}{2\pi(20)}$$

$$P = 14.3d^4 = 14.3(138^4)$$

$$P = 5\,186\,237\,285\ \text{N·mm/sec}$$

$$P = 5\,186\,237.28\ \text{W}$$

$$P = \mathbf{5.19\ MW}$$

Problem 308

A 2-in-diameter steel shaft rotates at 240 rpm. If the shearing stress is limited to 12 ksi, determine the maximum horsepower that can be transmitted.

Solution 308

$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$12(1000) = \frac{16T}{\pi(2^3)}$$

$$T = 18\,849.56\ \text{lb·in}$$

$$T = \frac{P}{2\pi f}$$

$$18\,849.56 = \frac{P(396\,000)}{2\pi(240)}$$

$$P = \mathbf{71.78\ hp}$$

Problem 309

A steel propeller shaft is to transmit 4.5 MW at 3 Hz without exceeding a shearing stress of 50 MPa or twisting through more than 1° in a length of 26 diameters. Compute the proper diameter if G = 83 GPa.

Solution 309

$$T = \frac{P}{2\pi f} = \frac{4.5(1\,000\,000)}{2\pi(3)}$$

$$T = 238\,732.41 \text{ N·m}$$

Based on maximum allowable shearing stress:

$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$50 = \frac{16(238732.41)(1000)}{\pi d^3}$$

$$d = 289.71 \text{ mm}$$

Based on maximum allowable angle of twist:

$$\theta = \frac{TL}{JG}$$

$$1^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{238732.41(26d)(1000)}{\frac{1}{32}\pi d^4(83\,000)}$$

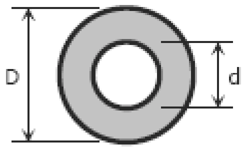
$$d = 352.08 \text{ mm}$$

Use the bigger diameter, d = **352 mm**

Problem 310

Show that the hollow circular shaft whose inner diameter is half the outer diameter has a torsional strength equal to 15/16 of that of a solid shaft of the same outside diameter.

Solution 310



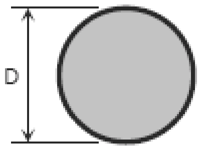
Hollow circular shaft:

$$\tau_{\text{max-hollow}} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$= \frac{16TD}{\pi[D^4 - (\frac{1}{2}D)^4]}$$

$$= \frac{16TD}{\pi(\frac{15}{16}D^4)}$$

$$= \frac{16^2 T}{15\pi D^3}$$



Solid circular shaft:

$$\tau_{\text{max-solid}} = \frac{16T}{\pi D^3}$$

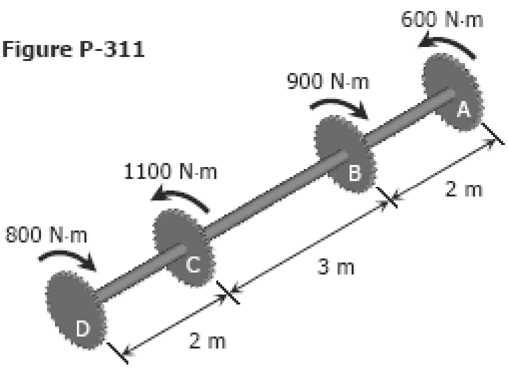
$$= \frac{15}{16}\left[\frac{16^2 T}{15\pi D^3}\right]$$

$$= \tfrac{15}{16} \times \tau_{\text{max-hollow}} \quad \textit{ok!}$$

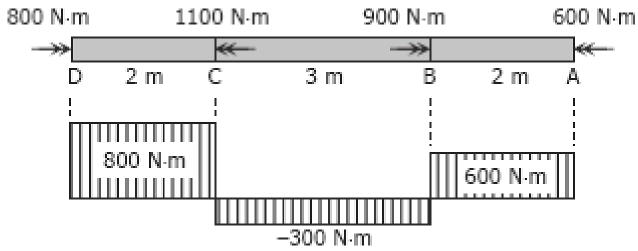
Problem 311

An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in Fig. P-311. Using G = 28 GPa, determine the relative angle of twist of gear D relative to gear A.

Figure P-311



Solution 311



$$\theta = \frac{TL}{JG}$$

Rotation of D relative to A:

$$\theta_{D/A} = \frac{1}{JG}\sum TL$$

$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(50^4)(28000)}[800(2) - 300(3) + 600(2)]\ (100^2)$$

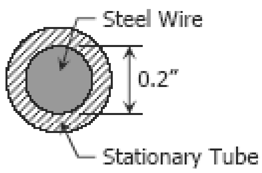
$$\theta_{D/A} = 0.1106 \text{ rad}$$

$$\theta_{D/A} = 6.34°$$

Problem 312

A flexible shaft consists of a 0.20-in-diameter steel wire encased in a stationary tube that fits closely enough to impose a frictional torque of 0.50 lb·in/in. Determine the maximum length of the shaft if the shearing stress is not to exceed 20 ksi. What will be the angular deformation of one end relative to the other end? G = 12 × 10^6 psi.

Solution 312



$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$20(1000) = \frac{16T}{\pi(0.20)^3}$$

$$T = 10\pi \text{ lb·in}$$

$$L = \frac{T}{0.50 \text{ lb}\cdot\text{in/in}} = \frac{10\pi \text{ lb}\cdot\text{in}}{0.50 \text{ lb}\cdot\text{in/in}}$$

$$L = 20\pi \text{ in} = \mathbf{62.83 \text{ in}}$$

$$\theta = \frac{TL}{JG}$$

If $\theta = d\theta$, $T = 0.5L$ and $L = dL$

$$\int d\theta = \frac{1}{JG}\int_0^{20\pi} (0.5L)dL$$

$$\theta = \frac{1}{JG}\left[\frac{0.5L^2}{2}\right]_0^{20\pi} = \frac{1}{JG}[0.25(20\pi)^2 - 0.25(0)^2]$$

$$\theta = \frac{100\pi^2}{\frac{1}{32}\pi(0.20^4)(12\times10^6)}$$

$$\theta = 0.5234 \text{ rad} = \mathbf{30°}$$

Problem 313

Determine the maximum torque that can be applied to a hollow circular steel shaft of 100-mm outside diameter and an 80-mm inside diameter without exceeding a shearing stress of 60 MPa or a twist of 0.5 deg/m. Use G = 83 GPa.

Solution 313

Based on maximum allowable shearing stress:

$$\tau_{max} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$60 = \frac{16T(100)}{\pi(100^4 - 80^4)}$$

$$T = 6\,955\,486.14 \text{ N·mm}$$

$$T = 6\,955.5 \text{ N·m}$$

Based on maximum allowable angle of twist:

$$\theta = \frac{TL}{JG}$$

$$0.5°\left(\frac{\pi}{180°}\right) = \frac{T(1000)}{\frac{1}{32}\pi(100^4 - 80^4)(83\,000)}$$

$$T = 4\,198\,282.97 \text{ N·mm}$$

$$T = 4\,198.28 \text{ N·m}$$

Use the smaller torque, T = **4 198.28 N·m**

Problem 314

The steel shaft shown in Fig. P-314 rotates at 4 Hz with 35 kW taken off at A, 20 kW removed at B, and 55 kW applied at C. Using G = 83 GPa, find the maximum shearing stress and the angle of rotation of gear A relative to gear C.

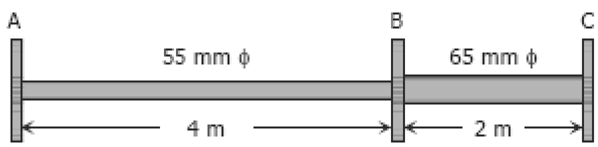


Figure P-314

Solution 314

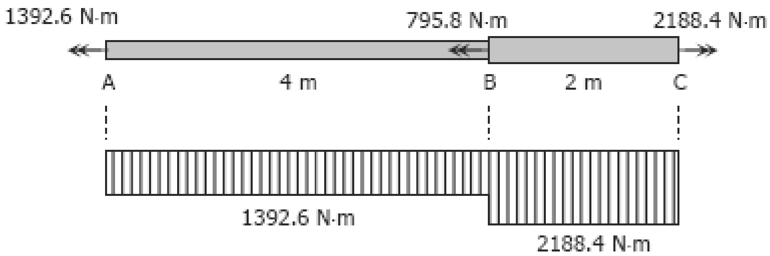
$$T = \frac{P}{2\pi f}$$

$$T_A = \frac{-35(1000)}{2\pi(4)} = -1392.6 \text{ N·m}$$

$$T_B = \frac{-20(1000)}{2\pi(4)} = -795.8 \text{ N·m}$$

$$T_C = \frac{55(1000)}{2\pi(4)} = 2188.4 \text{ N·m}$$

Relative to C:



$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{AB} = \frac{16(1392.6)(1000)}{\pi(55^3)} = 42.63 \text{ MPa}$$

$$\tau_{BC} = \frac{16(2188.4)(1000)}{\pi(65^3)} = 40.58 \text{ MPa}$$

$$\therefore \quad \tau_{max} = \tau_{AB} = 42.63 \text{ MPa}$$

$$\theta = \frac{TL}{JG}$$

$$\theta_{A/C} = \frac{1}{G}\sum\frac{TL}{J}$$

$$\theta_{A/C} = \frac{1}{83\,000}\left[\frac{1392.6(4)}{\frac{1}{32}\pi(55^4)} + \frac{2188.4(2)}{\frac{1}{32}\pi(65^4)}\right](1000^2)$$

$$\theta_{A/C} = 0.104\,796\,585 \text{ rad}$$

$$\theta_{A/C} = 6.004°$$

Problem 315

A 5-m steel shaft rotating at 2 Hz has 70 kW applied at a gear that is 2 m from the left end where 20 kW are removed. At the right end, 30 kW are removed and another 20 kW leaves the shaft at 1.5 m from the right end. (a) Find the uniform shaft diameter so that the shearing stress will not exceed 60 MPa. (b) If a uniform shaft diameter of 100 mm is specified, determine the angle by which one end of the shaft lags behind the other end. Use G = 83 GPa.

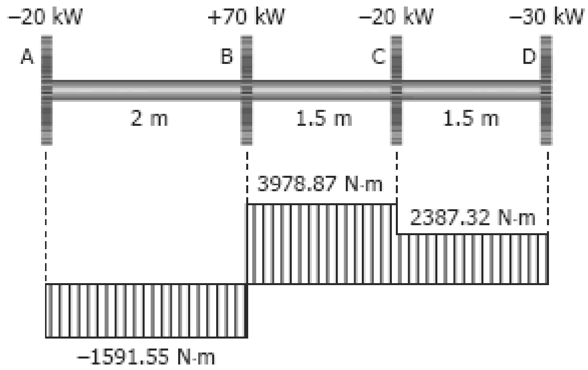
Solution 315

$$T = \frac{P}{2\pi f}$$

$$T_A = T_C = \frac{-20(1000)}{2\pi(2)} = -1591.55 \text{ N·m}$$

$$T_B = \frac{70(1000)}{2\pi(2)} = 5570.42 \text{ N·m}$$

$$T_D = \frac{-30(1000)}{2\pi(2)} = -2387.32 \text{ N·m}$$



Part (*a*)

$$\tau_{max} = \frac{16T}{\pi d^3}$$

For *AB*: $60 = \dfrac{16(1591.55)(1000)}{\pi d^3}$

$d = 51.3$ mm

For *BC*: $60 = \dfrac{16(3978.87)(1000)}{\pi d^3}$

$d = 69.6$ mm

For *CD*: $60 = \dfrac{16(2387.32)(1000)}{\pi d^3}$

$d = 58.7$ mm

Use d = **69.6 mm**

Part (*b*)

$$\theta = \frac{TL}{JG}$$

$$\theta_{D/A} = \frac{1}{JG}\sum TL$$

$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(100^4)(83000)}[-1591.55(2) + 3978.87(1.5) + 2387.32(1.5)](1000^2)$$

$\theta_{D/A} = 0.007\,813$ rad

$\theta_{D/A} =$ **0.448°**