Stresses in Beams

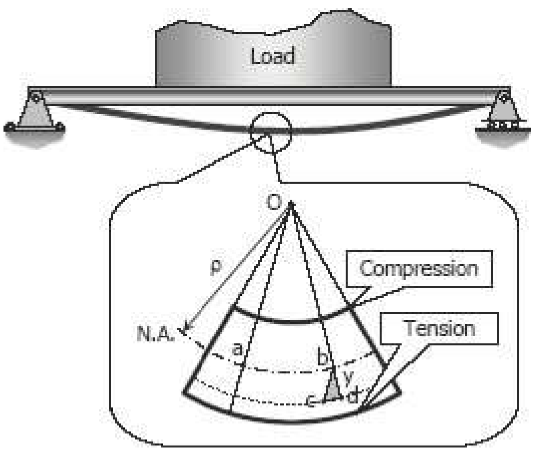
Forces and couples acting on the beam cause bending (flexural stresses) and shearing stresses on any cross section of the beam and deflection perpendicular to the longitudinal axis of the beam. If couples are applied to the ends of the beam and no forces act on it, the bending is said to be pure bending. If forces produce the bending, the bending is called ordinary bending.

ASSUMPTIONS

In using the following formulas for flexural and shearing stresses, it is assumed that a plane section of the beam normal to its longitudinal axis prior to loading remains plane after the forces and couples have been applied, and that the beam is initially straight and of uniform cross section and that the moduli of elasticity in tension and compression are equal.

Flexure Formula

Stresses caused by the bending moment are known as flexural or bending stresses. Consider a beam to be loaded as shown.



Consider a fiber at a distance y from the neutral axis, because of the beam's curvature, as the effect of bending moment, the fiber is stretched by an amount of cd. Since the curvature of the beam is very small, bcd and Oba are considered as similar triangles. The strain on this fiber is

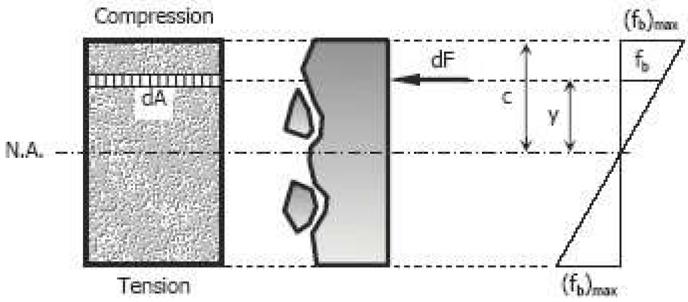
$$\varepsilon = \frac{cd}{ab} = \frac{y}{\rho}$$

By Hooke's law, $\varepsilon = \sigma / E$, then

$$\frac{\sigma}{E} = \frac{y}{\rho};\ \sigma = \frac{y}{\rho}E$$

which means that the stress is proportional to the distance y from the neutral axis.

For this chapter, the notation f_b will be used instead of σ, to denote flexural stresses.



Considering a differential area dA at a distance y from N.A., the force acting over the area is

$$dF = f_b\, dA = \frac{y}{\rho} E\, dA = \frac{E}{\rho} y\, dA$$

The resultant of all the elemental moment about N.A. must be equal to the bending moment on the section.

$$M = \int y\, dF = \int y \frac{E}{\rho} y\, dA$$

$$M = \frac{E}{\rho} \int y^2\, dA$$

but $\int y^2\, dA = I,$ then

$$M = \frac{EI}{\rho} \text{ or } \rho = \frac{EI}{M}$$

substituting $\rho = Ey / f_b$

$$\frac{Ey}{f_b} = \frac{EI}{M}$$

then

$$f_b = \frac{My}{I}$$

and

$$(f_b)_{max} = \frac{Mc}{I}$$

The bending stress due to beams curvature is

$$f_b = \frac{Mc}{I} = \frac{\frac{EI}{\rho}c}{I}$$

$$f_b = \frac{Ec}{\rho}$$

The beam curvature is:

$$k = 1 / \rho$$

where ρ is the radius of curvature of the beam in mm (in), M is the bending moment in N·mm (lb·in), f_b is the flexural stress in MPa (psi), I is the centroidal moment of inertia in mm^4 (in^4), and c is the distance from the neutral axis to the outermost fiber in mm (in).

SECTION MODULUS

In the formula

$$(f_b)_{max} = \frac{Mc}{I} = \frac{M}{I/c},$$

the ratio I/c is called the section modulus and is usually denoted by S with units of mm^3 (in^3). The maximum bending stress may then be written as

$$(f_b)_{max} = \frac{M}{S}$$

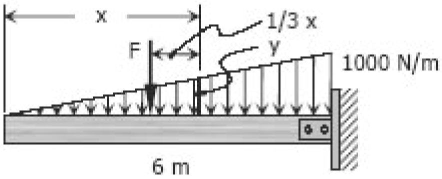
This form is convenient because the values of S are available in handbooks for a wide range of standard structural shapes.

Solved Problems in Flexure Formula

Problem 503

A cantilever beam, 50 mm wide by 150 mm high and 6 m long, carries a load that varies uniformly from zero at the free end to 1000 N/m at the wall. (a) Compute the magnitude and location of the maximum flexural stress. (b) Determine the type and magnitude of the stress in a fiber 20 mm from the top of the beam at a section 2 m from the free end.

Solution 503



$$M = F(\tfrac{1}{3}x)$$

$$\frac{y}{x} = \frac{1000}{6}$$

$$y = \tfrac{500}{3}x$$

$$F = \tfrac{1}{2}xy$$

$$F = \tfrac{1}{2}x(\tfrac{500}{3}x)$$

$$F = \tfrac{250}{3}x^2$$

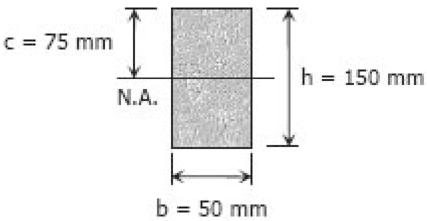
thus $M = \tfrac{250}{3}x^2(\tfrac{1}{3}x)$

$$M = \tfrac{250}{9}x^3$$

(a) The maximum moment occurs at the support (the wall) or at $x = 6$ m.

$$M = \tfrac{250}{9}x^3 = \tfrac{250}{9}(6^3)$$

$$= 6000 \text{ N·m}$$

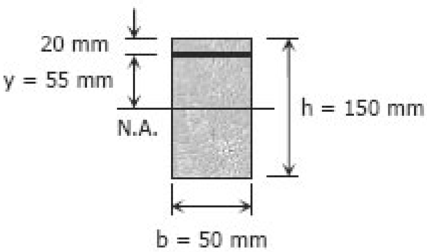


$$(f_b)_{max} = \frac{Mc}{I} = \frac{Mc}{\frac{bh^3}{12}}$$

$$(f_b)_{max} = \frac{6000(1000)(75)}{\frac{50(150)^3}{12}}$$

$$(f_b)_{max} = \mathbf{32\ MPa}$$

(b) At a section 2 m from the free end or at $x = 2$ m at fiber 20 mm from the top of the beam:



$$M = \tfrac{250}{9}x^3 = \tfrac{250}{9}(2)^3$$

$$M = \tfrac{2000}{9} \text{ N·m}$$

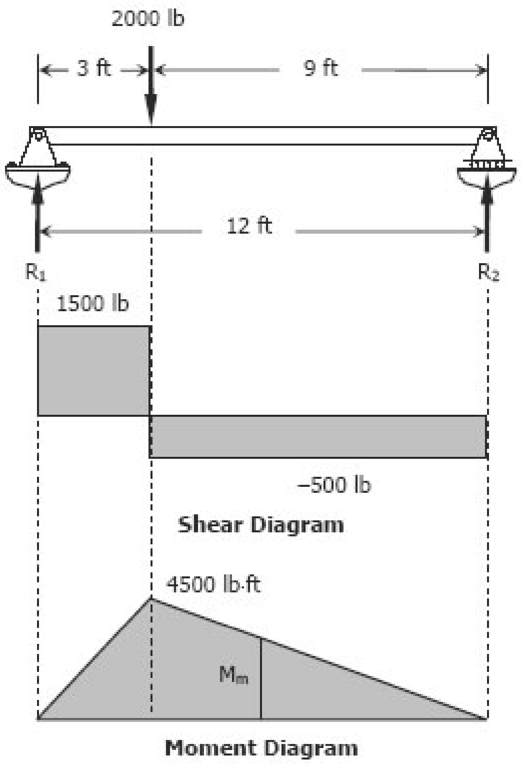
$$f_b = \frac{My}{I} = \frac{(\frac{2000}{9})(1000)(55)}{\frac{50(150)^3}{12}}$$

$$f_b = 0.8691 \text{ MPa} = \mathbf{869.1\ kPa}$$

Problem 504

A simply supported beam, 2 in wide by 4 in high and 12 ft long is subjected to a concentrated load of 2000 lb at a point 3 ft from one of the supports. Determine the maximum fiber stress and the stress in a fiber located 0.5 in from the top of the beam at midspan.

Solution 504



$\Sigma M_{R2} = 0$

$12R_1 = 9(2000)$

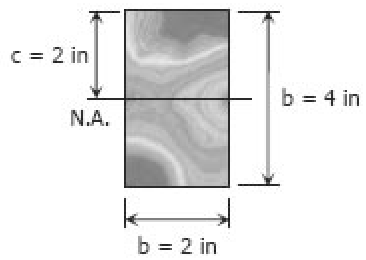
$R_1 = 1500 \text{ lb}$

$\Sigma M_{R1} = 0$

$12R_2 = 3(2000)$

$R_2 = 500 \text{ lb}$

Maximum fiber stress:



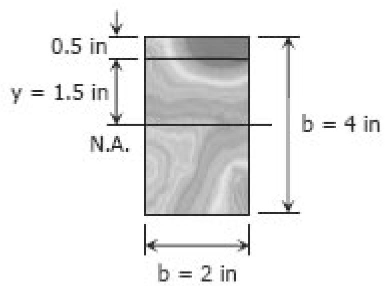
$$(f_b)_{max} = \frac{Mc}{I} = \frac{4500(12)(2)}{\frac{2(4)^3}{12}}$$

$(f_b)_{max} = \mathbf{10{,}125 \text{ psi}}$

Stress in a fiber located 0.5 in from the top of the beam at midspan:

$$\frac{M_m}{6} = \frac{4500}{9}$$

$M_m = 3000 \text{ lb·ft}$



$$f_b = \frac{My}{I}$$

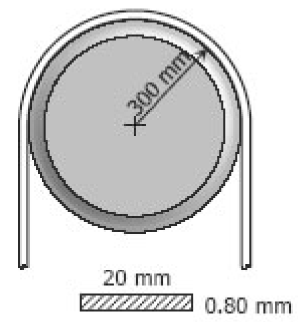
$$f_b = \frac{3000(12)(1.5)}{\frac{2(4^3)}{12}}$$

$f_b = \mathbf{5{,}062.5 \text{ psi}}$

Problem 505

A high strength steel band saw, 20 mm wide by 0.80 mm thick, runs over pulleys 600 mm in diameter. What maximum flexural stress is developed? What minimum diameter pulleys can be used without exceeding a flexural stress of 400 MPa? Assume E = 200 GPa.

Solution 505



Flexural stress developed:

$$M = \frac{EI}{\rho}$$

$$f_b = \frac{Mc}{I} = \frac{(EI/\rho)c}{I}$$

$$f_b = \frac{Ec}{\rho} = \frac{200000(0.80/2)}{300}$$

$$f_b = \mathbf{266.67\ MPa}$$

Minimum diameter of pulley:

$$f_b = \frac{Ec}{\rho}$$

$$400 = \frac{200000(0.80/2)}{\rho}$$

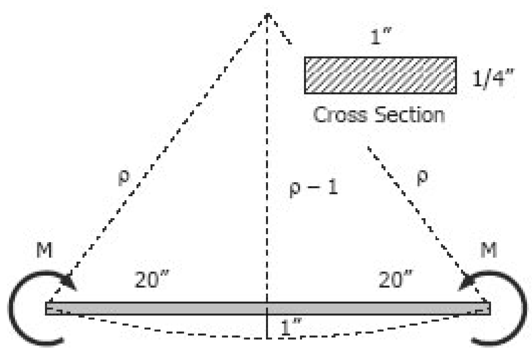
$\rho = 200$ mm

diameter, d = **400 mm**

Problem 506

A flat steel bar, 1 inch wide by ¼ inch thick and 40 inches long, is bent by couples applied at the ends so that the midpoint deflection is 1.0 inch. Compute the stress in the bar and the magnitude of the couples. Use E = 29 × 10^6 psi.

Solution 506



$(\rho - 1)^2 + 20^2 = \rho^2$

$\rho^2 - 2\rho + 1 + 400 = \rho^2$

$2\rho = 401$

$\rho = 200.5$ in

$$M = \frac{EI}{\rho}$$

$$f_b = \frac{Mc}{I} = \frac{(EI/\rho)c}{I}$$

$$f_b = \frac{Ec}{\rho} = \frac{(29\times10^6)(1/8)}{200.5}$$

f_b = 18 079.8 psi

f_b = **18.1 ksi**

$$M = \frac{EI}{\rho} = \frac{(29\times10^6)\frac{1(1/4)^3}{12}}{200.5}$$

M = **188.3 lb·in**

Problem 507

In a laboratory test of a beam loaded by end couples, the fibers at layer AB in Fig. P-507 are found to increase 60×10^{-3} mm whereas those at CD decrease 100×10^{-3} mm in the 200-mm-gage length. Using E = 70 GPa, determine the flexural stress in the top and bottom fibers.

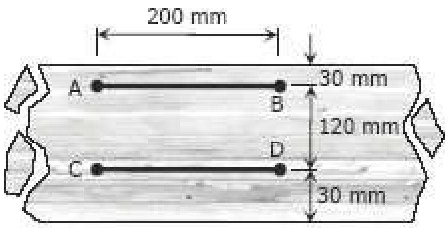
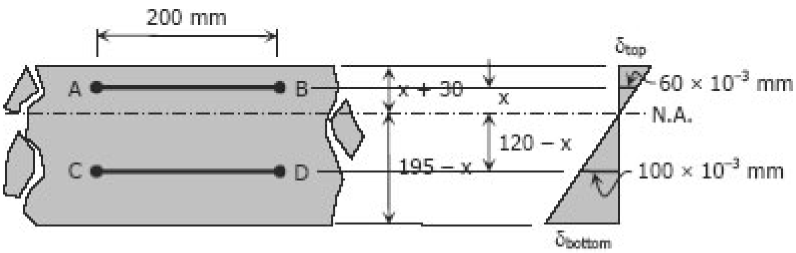


Figure P-507

Solution 507



$$\frac{x}{60\times10^{-3}} = \frac{120-x}{100\times10^{-3}}$$

$x = 0.6(120 - x)$

$x + 0.6x = 0.6(120)$

$1.6x = 72$

x = 45 mm

$$\frac{\delta_{top}}{x+30} = \frac{60\times10^{-3}}{x}$$

$$\delta_{top} = \frac{60\times10^{-3}}{45}(45+30)$$

δ_{top} = 0.1 mm lengthening

$$\frac{\delta_{bottom}}{195-x} = \frac{100\times10^{-3}}{120-x}$$

$$\delta_{bottom} = \frac{100\times10^{-3}}{120-45}(195-45)$$

δ_{bottom} = 0.2 mm shortening

From Hooke's Law

$$f_b = E\varepsilon$$

$$f_b = \frac{E\delta}{L}$$

$$(f_b)_{top} = \frac{70000(0.1)}{200}$$

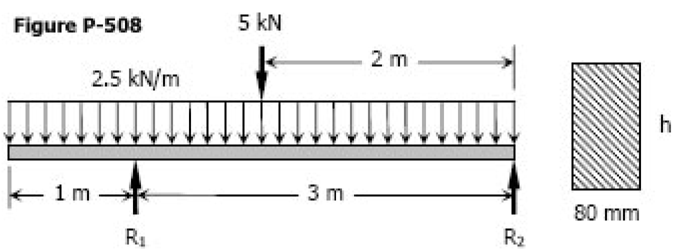
= **35 MPa tension**

$$(f_b)_{bottom} = \frac{70000(0.2)}{200}$$

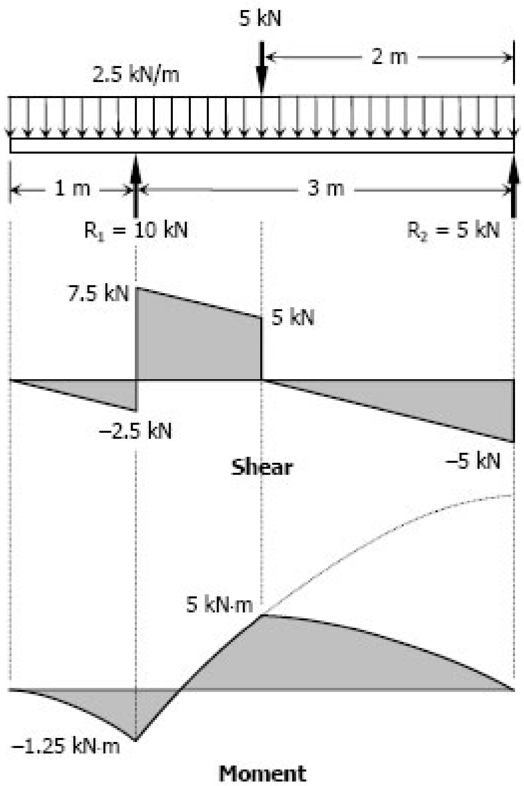
= **70 MPa compression**

Problem 508

Determine the minimum height h of the beam shown in Fig. P-508 if the flexural stress is not to exceed 20 MPa.



Solution 508



$\Sigma M_{R2} = 0$

$3R_1 = 2(5) + 2(2.5)(4)$

$R_1 = 10\text{ kN}$

$\Sigma M_{R1} = 0$

$3R_2 = 1(5) + 1(2.5)(4)$

$R_2 = 5\text{ kN}$

$$f_b = \frac{Mc}{I}$$

Where:

$f_b = 20\text{ MPa}$

$M = 5\text{ kN·m}$

$= 5(1000)^2\text{ N·mm}$

$c = \frac{1}{2}h$

$I = \dfrac{bh^3}{12} = \dfrac{80h^3}{12}$

$= \frac{20}{3}h^3$

Thus,

$$20 = \frac{5(1000)^2(\frac{1}{2}h)}{\frac{20}{3}h^3}$$

$h^2 = 18\,750$

$h = 137\text{ mm}$

Problem 509

A section used in aircraft is constructed of tubes connected by thin webs as shown in Fig. P-509. Each tube has a cross-sectional area of 0.20 in2. If the average stress in the tubes is no to exceed 10 ksi, determine the total uniformly distributed load that can be supported in a simple span 12 ft long. Neglect the effect of the webs.

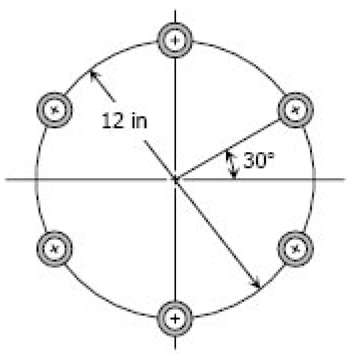
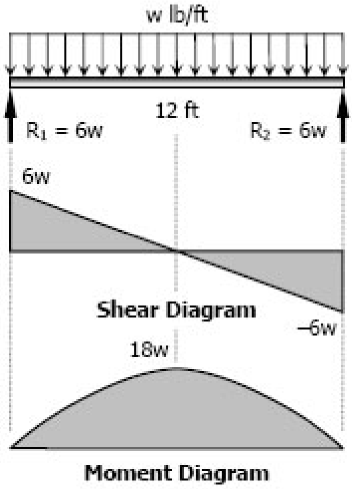


Figure P-509

Solution 509



$R_1 = R_2 = \frac{1}{2}(12)(w)$
$R_1 = R_2 = 6w$

$f_b = 10$ ksi $= 10{,}000$ psi
$M = 18w$ lb·ft
$c = 6$

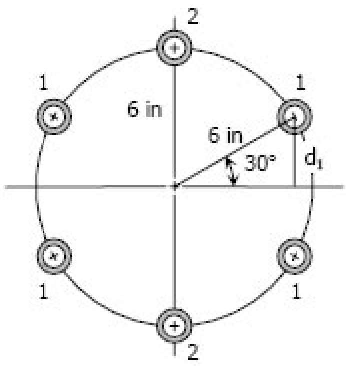
Centroidal moment of inertia of one tube:

$A = \pi r^2 = 0.20$
$r = 0.2523$ in → hollow portion of the tube was neglected

$$\bar{I}_x = \frac{\pi r^4}{4} = \frac{\pi(0.2523)^4}{4}$$
$\bar{I}_x = 0.0032 \text{ in}^4$

Moment of inertia at the center of the section:

$d_1 = 6 \sin 30° = 3$ in



$I_1 = \bar{I}_x + Ad_1^2$
$I_1 = 0.0032 + 0.2(3^2)$
$I_1 = 1.8 \text{ in}^4$

$I_2 = \bar{I}_x + Ad_2^2$
$I_2 = 0.0032 + 0.2(6^2)$
$I_2 = 7.2 \text{ in}^4$

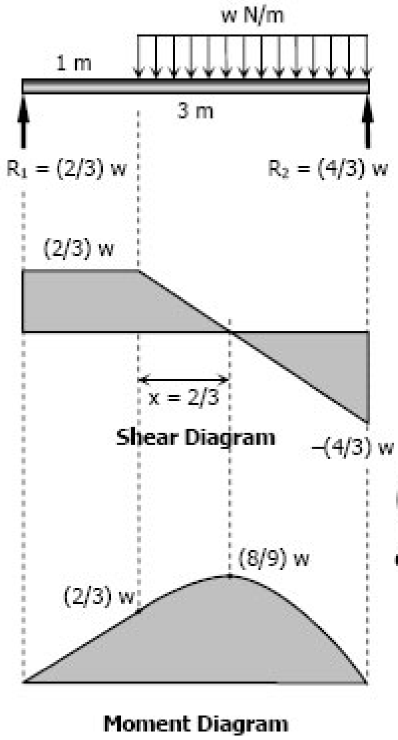
$I = 4I_1 + 2I_2 = 4(1.8) + 2(7.2)$
$I = 21.6 \text{ in}^4$

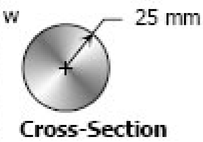
$$f_b = \frac{Mc}{I}$$
$$10{,}000 = \frac{18w(12)(6)}{21.6}$$
$w =$ **166.7 lb/ft**

Problem 510

A 50-mm diameter bar is used as a simply supported beam 3 m long. Determine the largest uniformly distributed load that can be applied over the right two-thirds of the beam if the flexural stress is limited to 50 MPa.

Solution 510





$\Sigma M_{R1} = 0$

$3R_2 = 2w(2)$

$R_2 = \frac{4}{3}w$

$\Sigma M_{R2} = 0$

$3R_1 = 2w(1)$

$R_1 = \frac{2}{3}w$

$$(f_b)_{max} = \frac{Mc}{I}$$

where $(f_b)_{max} = 50 \text{ MPa}$

$M = \frac{8}{9}w \text{ N·m}$

$c = 25 \text{ mm}$

$I = \frac{\pi r^4}{4} = \frac{\pi(25)^4}{4}$

$I = 97656.25\pi \text{ mm}^4$

$$50 = \frac{\frac{8}{9}w(1000)(25)}{97656.25\,\pi}$$

$w = \mathbf{690.29 \text{ N/m}}$

Problem 511

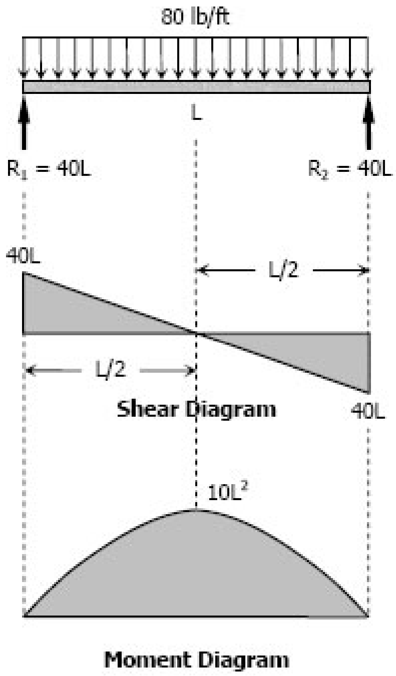
A simply supported rectangular beam, 2 in wide by 4 in deep, carries a uniformly distributed load of 80 lb/ft over its entire length. What is the maximum length of the beam if the flexural stress is limited to 3000 psi?

Solution 511

By symmetry:

$R_1 = R_2 = \frac{1}{2}(80L)$

$R_1 = R_2 = 40L$



$$(f_b)_{max} = \frac{Mc}{I}$$

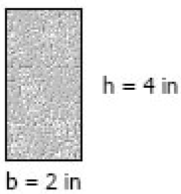
where $(f_b)_{max} = 3000 \text{ psi}$

$M = 10L^2 \text{ lb·ft}$

$c = h / 2 = 2 \text{ in}$

$I = \frac{bh^3}{12} = \frac{2(4)^3}{12}$

$= \frac{32}{3} \text{ in}^4$



$$3000 = \frac{10L^2(12)(2)}{32/3}$$

$L = \mathbf{11.55 \text{ ft}}$