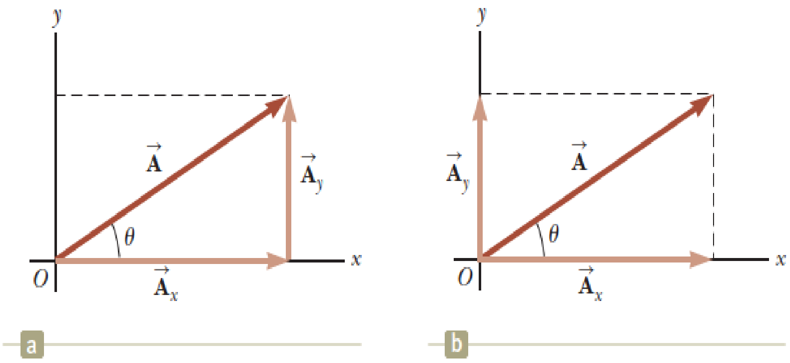
Components of a Vector and Unit Vectors

The graphical method of adding vectors is not recommended whenever high accuracy is required or in three-dimensional problems. In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes. These projections are called the components of the vector or its rectangular components.



$A_x = A \cos \theta$

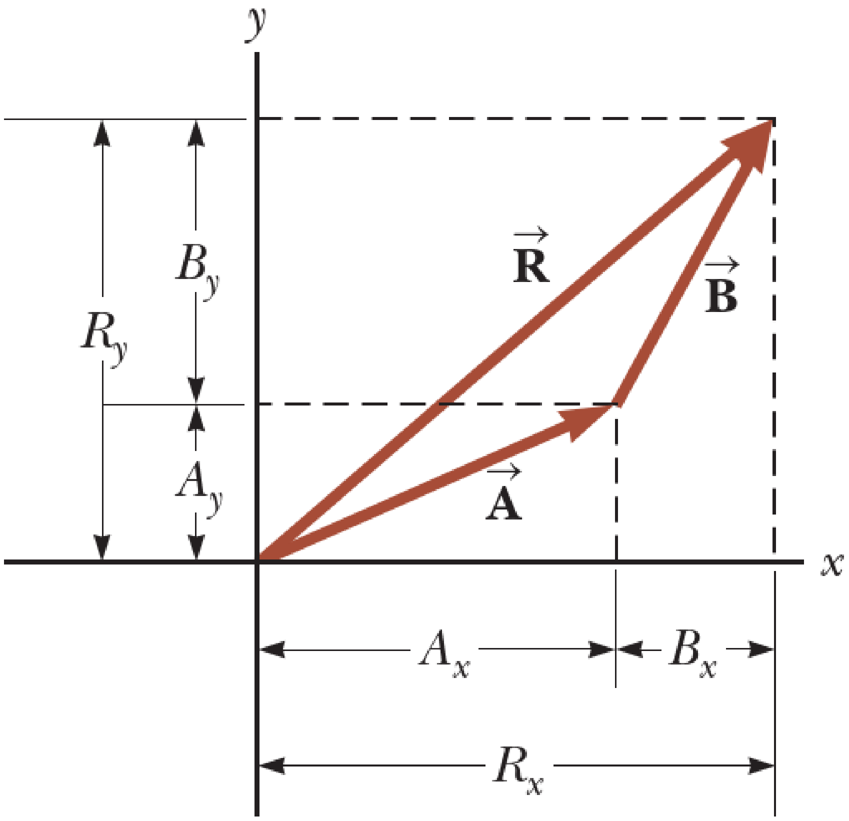
$A_y = A \sin \theta$

Unit vector

*A unit vector is a dimensionless vector having a magnitude of exactly 1.

*Unit vectors are used to specify a given direction and have no other physical significance.

* We shall use the symbols i, j, and k⁄ to represent unit vectors pointing in the positive x, y, and z directions, respectively.



The magnitude of each
Consider a vector $\vec{A}$
uct of the component
which lies on the x ax
ponent vector of mag
notation for the vector

The resultant vector $\vec{\mathbf{R}}$ is

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}})$$

or, rearranging terms,

$$\vec{\mathbf{R}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$$

Because $\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$, we see that the components of the resultant vector ar

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$

At times, we need to consider situations involving motion in three component directions. The extension of our methods to three-dimensional vectors is straightforward. If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ both have *x*, *y*, and *z* components, they can be expressed in the form

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

The sum of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{R}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$$

Notice that Equation 3.19 differs from Equation 3.13: in Equation 3.19, the resultant vector also has a *z* component $R_z = A_z + B_z$. If a vector $\vec{\mathbf{R}}$ has *x*, *y*, and *z* components, the magnitude of the vector is $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$. The angle θ_x that $\vec{\mathbf{R}}$ makes with the *x* axis is found from the expression $\cos\theta_x = R_x/R$, with similar expressions for the angles with respect to the *y* and *z* axes.

The extension of our method to adding more than two vectors is also straightforward using the component method. For example, $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = (A_x + B_x + C_x)\hat{\mathbf{i}} + (A_y + B_y + C_y)\hat{\mathbf{j}} + (A_z + B_z + C_z)\hat{\mathbf{k}}$.

Example

Find the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lying in the *xy* plane and given by

$$\vec{\mathbf{A}} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \quad \text{and} \quad \vec{\mathbf{B}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})$$

solution

$$\vec{\mathbf{R}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} = (2.0 + 2.0)\hat{\mathbf{i}} + (2.0 - 4.0)\hat{\mathbf{j}}$$
$$= 4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}}$$

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}} = \sqrt{(4.0)^2 + (-2.0)^2} = \sqrt{20} = 4.5$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0}{4.0} = -0.50$$

θ = -27

This answer is correct if we interpret it to mean 27° clockwise from the x axis. Our standard form has been to quote the angles measured counterclockwise from the 1x axis, and that angle for this vector is

$\theta = 360 - 27$

$= 333°$.

Example

A particle undergoes three consecutive displacements: $\Delta\vec{\mathbf{r}}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})$ cm, $\Delta\vec{\mathbf{r}}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})$ cm, and $\Delta\vec{\mathbf{r}}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$ cm. Find unit-vector notation for the resultant displacement and its magnitude.

solution

To find the resultant displacement, add the three vectors:

$$\Delta\vec{\mathbf{r}} = \Delta\vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}_2 + \Delta\vec{\mathbf{r}}_3$$
$$= (15 + 23 - 13)\hat{\mathbf{i}}\text{ cm} + (30 - 14 + 15)\hat{\mathbf{j}}\text{ cm} + (12 - 5.0 + 0)\hat{\mathbf{k}}\text{ cm}$$
$$= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\text{ cm}$$

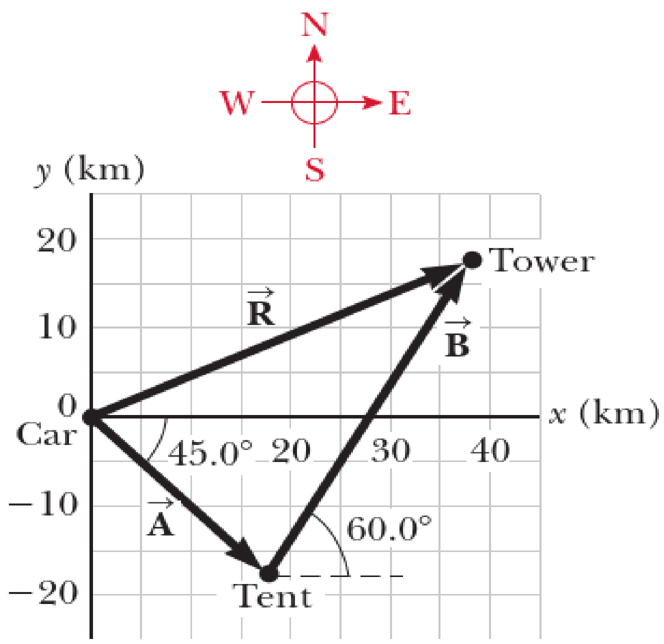
Find the magnitude of the resultant vector:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$= \sqrt{(25\text{ cm})^2 + (31\text{ cm})^2 + (7.0\text{ cm})^2} = 40\text{ cm}$$

Example

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.



Solution

$$A_x = A \cos(-45.0°) = (25.0 \text{ km})(0.707) = 17.7 \text{ km}$$

$$A_y = A \sin(-45.0°) = (25.0 \text{ km})(-0.707) = -17.7 \text{ km}$$

$$B_x = B \cos 60.0° = (40.0 \text{ km})(0.500) = 20.0 \text{ km}$$

$$B_y = B \sin 60.0° = (40.0 \text{ km})(0.866) = 34.6 \text{ km}$$

(B) Determine the components of the hiker's resultant displacement $\vec{\mathbf{R}}$ for the trip. Find an expression for $\vec{\mathbf{R}}$ in terms of unit vectors.

solution

$$R_x = A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = \boxed{37.7\text{ km}}$$

$$R_y = A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = \boxed{17.0\text{ km}}$$

$$\vec{\mathbf{R}} = \boxed{(37.7\hat{\mathbf{i}} + 17.0\hat{\mathbf{j}})\text{ km}}$$

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$\vec{A} = 5\hat{i} + 2\hat{j}$, $\vec{B} = -3\hat{i} - 5\hat{j}$, and $\vec{E} = 2\vec{A} + 3\vec{B}$

a) write vector $\vec{E}$ in component form

b) Draw a coordinate system and on it show vectors $\vec{A}$, $\vec{B}$, and $\vec{E}$

c) what are the magnitude and direction of vector $\vec{E}$

Solution

(a) $\vec{E} = 2(5\hat{i} + 2\hat{j}) + 3(-3\hat{i} - 5\hat{j})$

$= 10\hat{i} + 4\hat{j} - 3\hat{i} - 15\hat{j}$

$\vec{E} = 1\hat{i} - 11\hat{j}$

(c)

$E = \sqrt{1^2 + (-11)^2}$

$= \sqrt{122}$

$= 11.05$

$\theta = \tan^{-1} \frac{E_y}{E_x}$

$= \tan^{-1} \frac{-11}{1}$

$= -84.8$

$\varnothing = 360 - 84.8$

$= 275.19°$

(b)