CH4 Summary

The Mean Value Theorem (M.V.T.):

- ***Rolle's Theorem:***

  1. *f* Continuous on [a, b]
  2. *f* differentiable on (a, b)
  3. *f* (a) = *f* (b)

$$\exists \; c \in (a, b) \; \text{such that} \; f'(c) = 0$$

- ***The Mean Value Theorem (M. V. T.):***

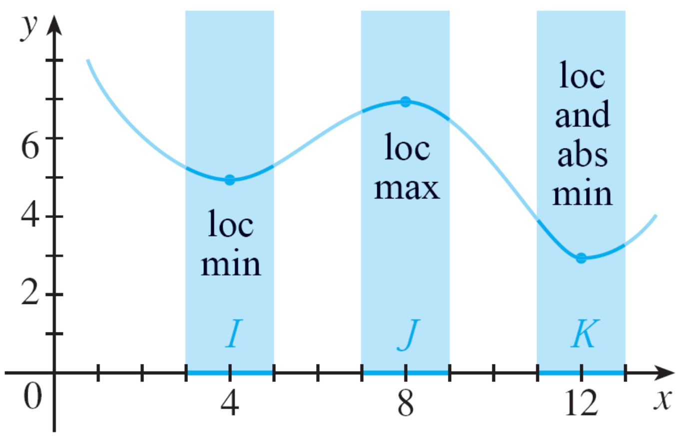
  1. *f* Continuous on [a, b]
  2. *f* differentiable on (a, b)

$$\exists \; c \in (a, b) \; \text{such that} \; f'(c) = \frac{f(b) - f(a)}{b - a}$$

Maximum and Minimum Values:

Definition: *if* ***c*** *in Domain* ***D*** *of function* ***f*** *Then:*

- *Local Maximum Value if:*
  $f(c) \geq f(x)$ *when x is near c.*
- *Local Minimum Value if:*
  $f(c) \leq f(x)$ *when x is near c.*
- *Absolute Maximum Value if:*
  $f(c) \geq f(x)$ *for all x in D.*
- *Absolute Minimum Value if:*
  $(c) \leq f(x)$ *for all x in D.*



Absolute = Global
Extreme Value: Maximum or Minimum value.

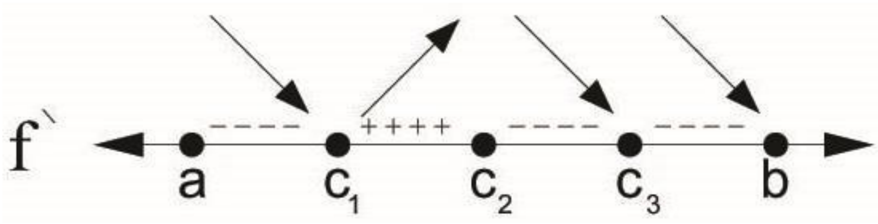
Remark:

Absolute max: largest value between Local max & end point (max).

Absolute min: smallest value between Local min & end point (min).

How Derivatives Affect the Shape of a graph:

Let ***f(x)*** *has critical points* c_1, c_2, c_3 *in figure below defined on* **[*a, b*]**:



- ***The First Derivative Test:***

Def: we called c Critical point if:

1) *It is defined on* $f(c)$.
2) $f'(c) = 0$ *or* $f'(c)$ *D. N. E*
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- $f(c_1)$ *local min,* $f(c_2)$ *local max.*
- ***f(c3)*** *not extreme (no max, no min)*
- ***f(a)*** *end point (max),* ***f(b)*** *end point (min).*

- ***Increasing and Decreasing Test:***

- $f'(x) > 0$ *on an interval,* f *increasing on that interval.*
- $f'(x) < 0$ *on an interval,* f *decreasing on that interval.*

f *decreasing on* $(a, c_1) \cup (c_2, b)$

f *increasing on* (c_1, c_2)

- ***The Second Derivative Test:***

Def: we called c Critical point if:

1) *It is defined on* $f(c)$.
2) $f''(c) = 0$ *or* $f''(c)$ *D. N. E*
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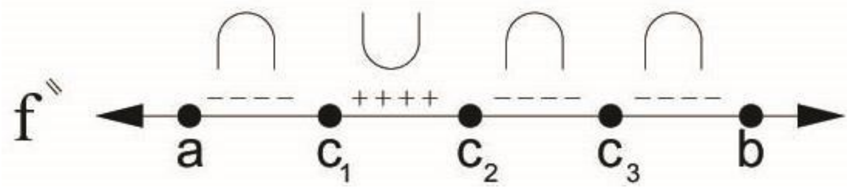
- ***Concavity Test:***



- $f''(x) > 0$ on an interval, f concaves up on that interval.
- $f''(x) < 0$ on an interval, f concaves down on that interval.

- ***Inflection point:***

$(c_1, f(c_1))$,	**$(c_2, f(c_2))$**

1) *It is defined on f (c).*
2) *Change the concavity.*

Summary of Curve Sketching:

لرسم المنحنى البياني للدالة $y = f(x)$ نتبع ما يلي:

1. إيجاد المجال.
2. نعين ما يلي ان وجدت:

X-intercept at y = 0
Y-intercept at x = 0
Horizontal asymptote → $\lim_{x\to\mp\infty} f(x)$
Vertical asymptote هي اصفار المقام

3. نوجد المشتقة الأولى ونساويها بالصفر ونحدد منها فترات التزايد والتناقص والقيم القصوى (Maximum and Minimum).
4. نوجد المشتقة الثانية ونساويها بالصفر ونحدد منها فترات التقعر للأعلى وللأسفل ونقاط الانعطاف.
5. نعين على الرسم القيم القصوى ونقاط الانعطاف ثم نصل بين المقاطع والقيم القصوى ونقاط الانعطاف مع مراعاة التزايد والتناقص والتقعر معا.

Example 1: Let $\boldsymbol{f(x) = \frac{4-x^2}{x^2-9}}$

a) *Study the symmetry of the curve and find the x–intercepts and y–intercepts, if any.*

b) *Find the vertical and horizontal asymptotes for the graph of **f**, if any.*

c) *Given that* $\boldsymbol{f'(x) = \frac{10x}{(x^2-9)^2}}$

  i) *Find the intervals on which **f** is increasing and the intervals on which **f** is decreasing.*

  ii) *Find the local maximum and minimum values of **f**, if any.*

d) *Given that* $\boldsymbol{f''(x) = \frac{-30(x^2+3)}{(x^2-9)^3}}$

  i) *Find the intervals on which the graph of **f** is concave upward and the intervals on which the graph of **f** is concave downward.*

  ii) *Find the points of inflection, if any.*

e) *Sketch the graph of **f**.*

Solution:

The domain of the function f(x) is all numbers except those that make the denominator equal to zero.

$$x^2 - 9 = 0 \implies x^2 = 9 \Rightarrow x = \mp 3$$

$$\therefore D_f = \mathcal{R} - \{-3, 3\}$$

(a) X-intercept: put y=0,

$$y = \frac{4 - x^2}{x^2 - 9} \implies 0 = \frac{4 - x^2}{x^2 - 9} \implies x^2 = 4 \implies x = \pm 2$$

X-intercept: put y=0,

$$y = \frac{4 - 0^2}{0^2 - 9} \implies y = \frac{4}{-9}$$

(b) Find the vertical and horizontal asymptote:

To find the horizontal asymptote:

$$x^2 - 9 = 0 \implies x^2 = 9 \Rightarrow x = \mp 3$$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{\boldsymbol{4 - x^2}}{\boldsymbol{x^2 - 9}} = \frac{\boldsymbol{4 - 3^2}}{\boldsymbol{3^2 - 9}} = \infty$$

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{\boldsymbol{4 - x^2}}{\boldsymbol{x^2 - 9}} = \frac{\boldsymbol{4 - (-3)^2}}{\boldsymbol{(-3)^2 - 9}} = -\infty$$

$\therefore$ *There are vertical asymptotes at x=3 and x = -3.*

To find the horizontal asymptote:

$$\lim_{x\to\infty} \frac{4-x^2}{x^2-9} = \lim_{x\to\infty} \frac{x^2\left(\frac{4}{x^2}-1\right)}{x^2\left(1-\frac{9}{x^2}\right)} = \frac{\left(\frac{4}{\infty^2}-1\right)}{\left(1-\frac{9}{\infty^2}\right)} = -1$$

$$\lim_{x\to-\infty} \frac{4-x^2}{x^2-9} = \lim_{x\to-\infty} \frac{x^2\left(\frac{4}{x^2}-1\right)}{x^2\left(1-\frac{9}{x^2}\right)} = \frac{\left(\frac{4}{(-\infty)^2}-1\right)}{\left(1-\frac{9}{(-\infty)^2}\right)} = -1$$

$\therefore$ *There is a horizontal asymptote at x=-1.*

(c) Given $f'(x) = \frac{10x}{(x^2-9)^2}$

$$f'(x) = 0 \implies \frac{10x}{(x^2-9)^2} = 0 \implies 10x = 0 \implies x = 0$$

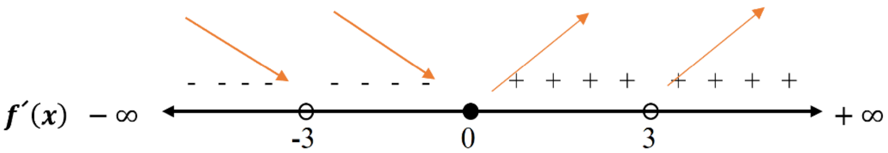
$$f'(x) \text{ D.N.E.} \implies (x^2-9)^2 = 0 \implies x^2 - 9 = 0 \implies x = \pm 3$$

$$f'(-4) = \frac{10(-4)}{((-4)^2-9)^2} = \frac{-40}{49}$$

$$f'(-1) = \frac{10(-1)}{((-1)^2-9)^2} = \frac{-10}{64}$$

$$f'(1) = \frac{10(1)}{((1)^2-9)^2} = \frac{10}{64}$$

$$f'(4) = \frac{10(4)}{((4)^2-9)^2} = \frac{40}{49}$$



$$f \text{ decreasing on} (-\infty, -3) \cup (-3, 0)$$

$$f \text{ increasing on} (0, 3) \cup (3, \infty)$$

$$f(0) = \frac{4-0^2}{0^2-9} = \frac{-4}{9} \qquad \text{local minimum}$$

(d) Given $f''(x) = \frac{-30(x^2+3)}{(x^2-9)^3}$

$$f''(x) = 0 \implies \frac{-30(x^2+3)}{(x^2-9)^3} = 0 \implies$$

$$-30(x^2+3) = 0 \implies x \pm \sqrt{-3} \qquad \text{Rejected}$$

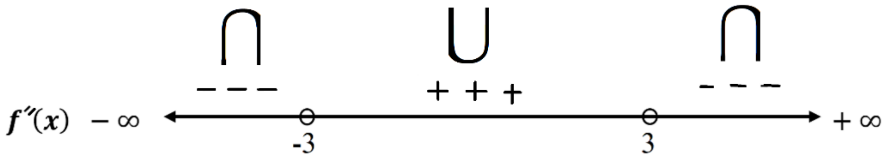
$$f''(x)\ D.N.E. \implies (x^2-9)^3 = 0 \implies \sqrt[3]{(x^2-9)^3} = \sqrt[3]{0}$$

$$\implies x^2 - 9 = 0 \implies x = \pm 3$$

$$f''(-4) = \frac{-30((-4)^2+3)}{((-4)^2-9)^3} = \frac{-570}{343}$$

$$f''(0) = \frac{-30(0^2+3)}{(0^2-9)^3} = \frac{-90}{-729} = \frac{90}{729}$$

$$f''(4) = \frac{-30((4)^2+3)}{((4)^2-9)^3} = \frac{-570}{343}$$



f concaves down on $(-\infty, -3) \cup (3, \infty)$

f concaves up on $(-3, 3)$

$f(-3), f(3)$ undefined.

$\therefore$ There is no influctions point.