CHAPTER 2

LIMITS AND CONTINUITY النهايات والاتصال

2.1 Definition of Limit.

- Formal Definition of Limit.
- One-Sided Limits.

2.2 Limits Laws.

- Solution Algebraic of Limits
- Limits Involving Trigonometric Functions.

2.3 Limits Involving Infinity.

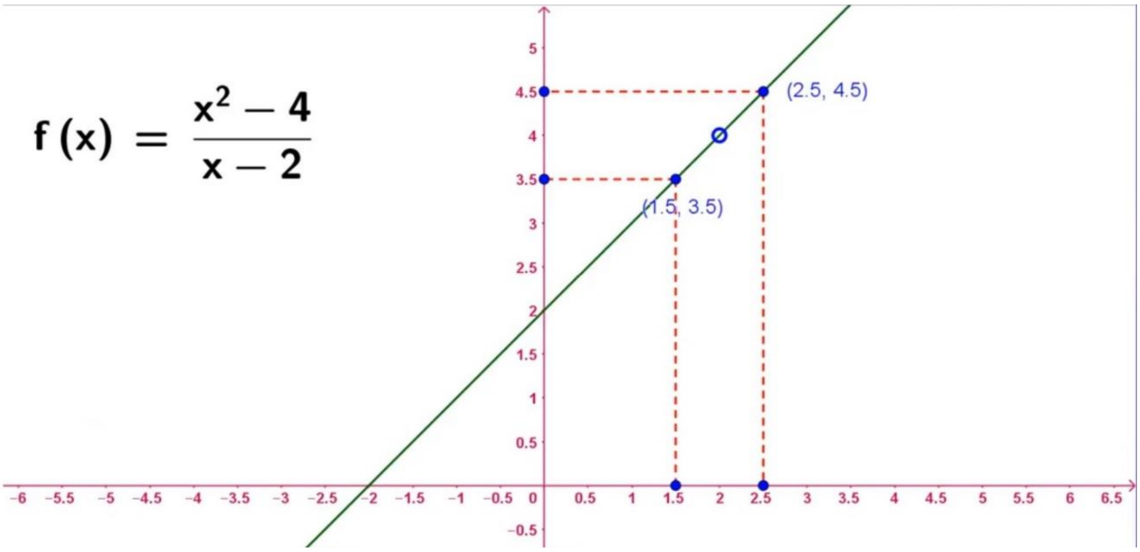
- Infinite Limits.
- Vertical Asymptote.
- Limits at Infinity.
- Horizontal Asymptote.

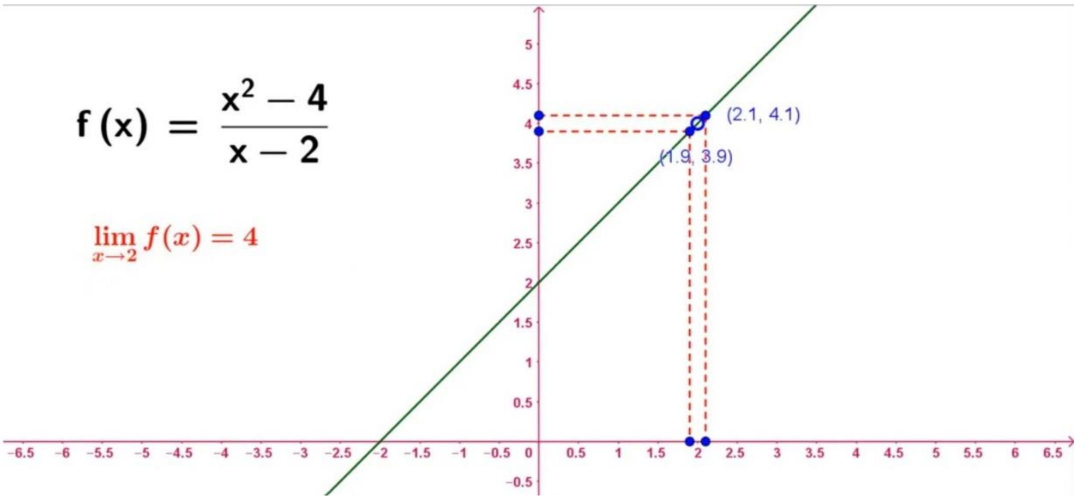
2.4 Continuity of Functions.

- Intermediate Value Theorem.

Section 2.1
DEFINITION OF LIMIT – تعريف النهاية

FORMAL DEFINITION OF LIMIT - النهاية باستخدام التعريف



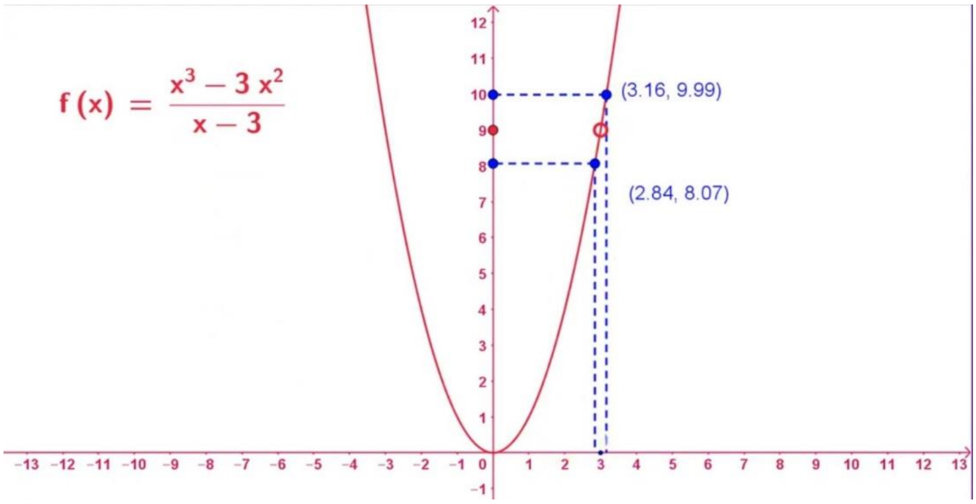


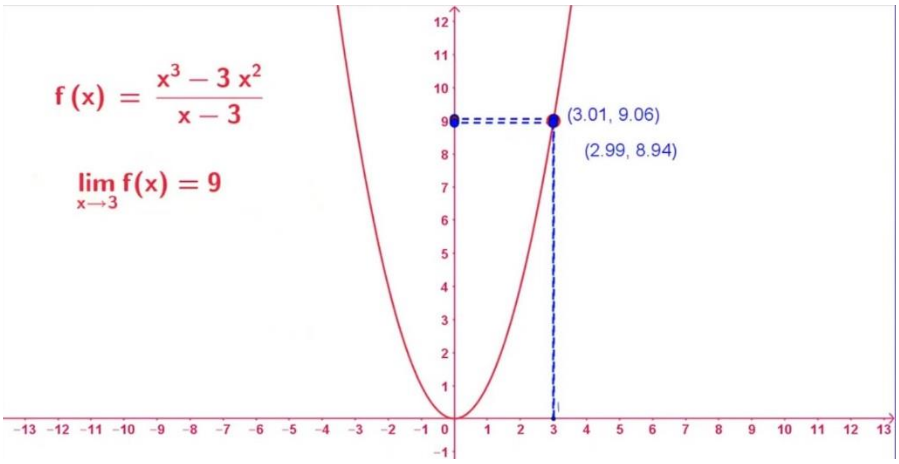
$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$$

EXAMPLE 2.1.1 Estimate the value of the following limit.

$$\lim_{x\to 3}\frac{x^3-3x^2}{x-3}$$

Solution



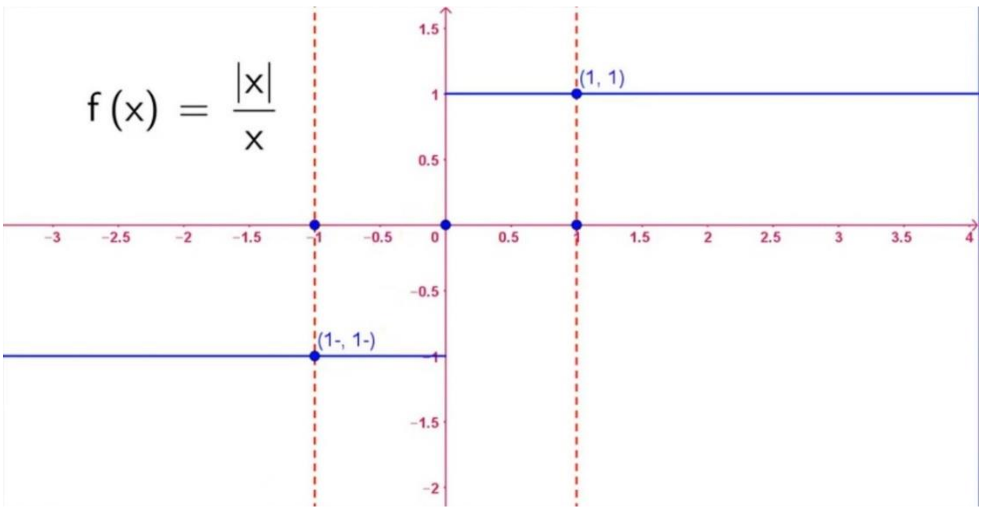


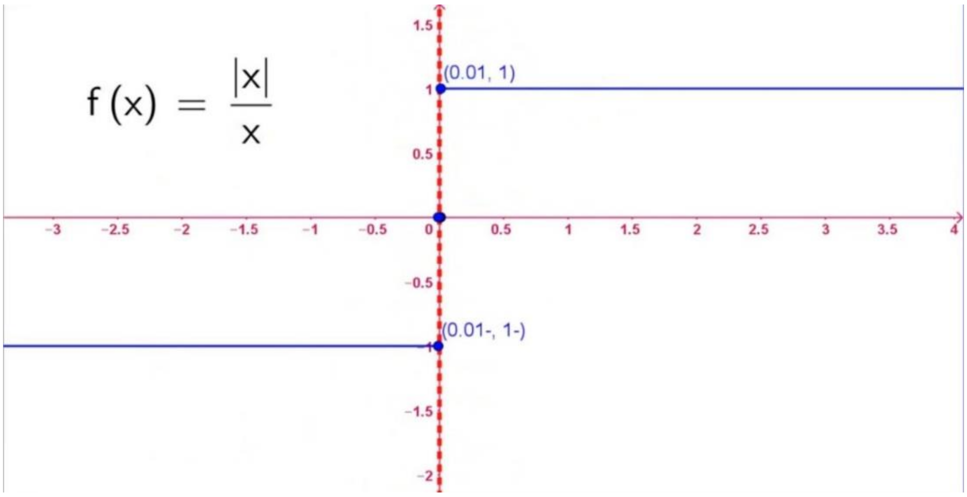
$$\lim_{x\to 3}\frac{x^3-3x^2}{x-3}=\lim_{x\to 3}\frac{x^2\,(x-3)}{x-3}=\lim_{x\to 3}x^2=9$$

EXAMPLE 2.1.2 Let

$$f(x)=\frac{|x|}{x}=\begin{cases}1, & x>0\\ -1, & x<0\end{cases}$$

Estimate the value of limit (if exists) of the function f at $x=0$.





DEFINITION 2.1.1 (Limit of a Function)

Let f be a function defined on an open interval containing a (except possibly at a itself) and let $L \in \mathbb{R}$, the statement

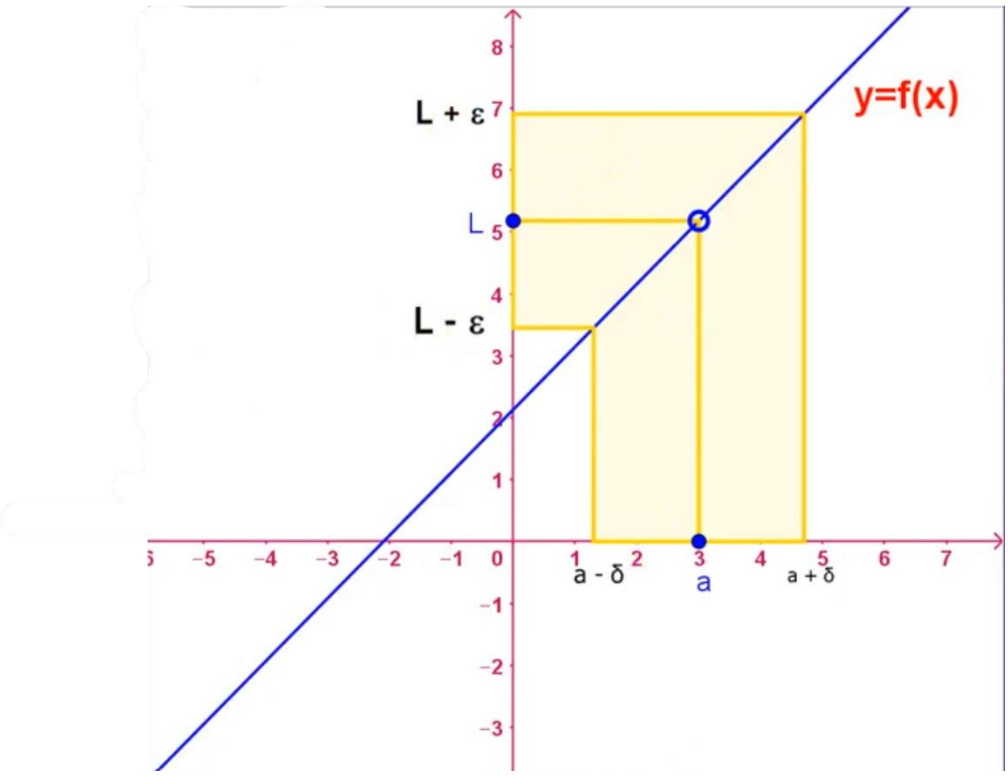
$$\lim_{x \to a} f(x) = L$$

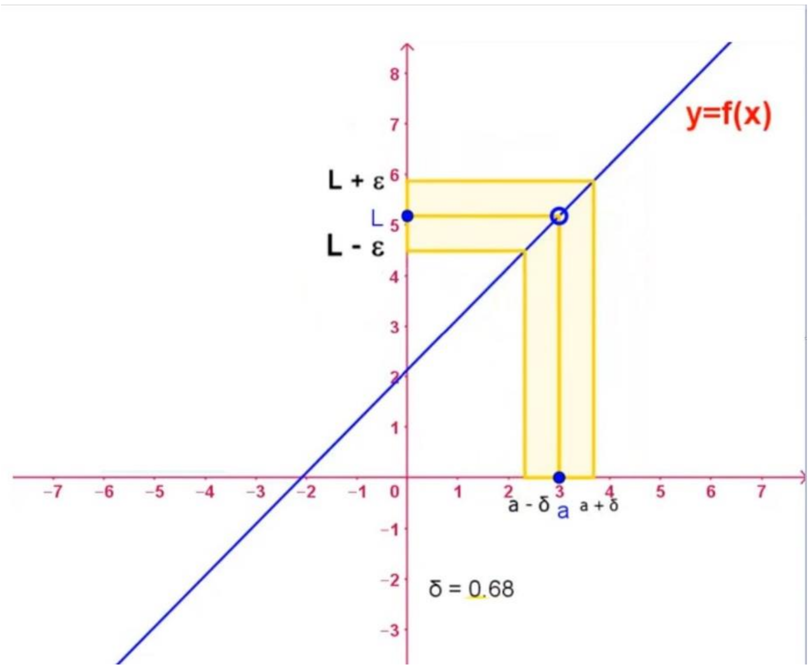
means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if} \quad 0 < \left|x - a\right| < \delta, \text{ then } \left|f\left(x\right) - L\right| < \varepsilon.$$

or

$$\text{if} \quad a - \delta < x < a + \delta, \text{ then } L - \varepsilon < f\left(x\right) < L + \varepsilon.$$





EXAMPLE 2.1.3 Using Definition of Limit, show that

$$\lim_{x \to 3} 4x = 12$$

Solution if $0 < |x-3| < \delta$, then $|4x-12| < \varepsilon$

$$|4x-12| < \varepsilon$$

$$|4(x-3)| < \varepsilon$$

$$4|x-3| < \varepsilon$$

$$|x-3| < \frac{\varepsilon}{4}$$

Let $0 < |x-3| < \frac{\varepsilon}{4} = \delta$ **than** $|4x-12| < \varepsilon$

EXAMPLE 2.1.4 Using Definition of Limit, show that

$$\lim_{x \to -2} (3x+7) = 1$$

Solution if $0 < |x+2| < \delta$, then $|(3x+7)-1| < \varepsilon$

$$|(3x+7)-1| < \varepsilon$$

$$|3x+6| < \varepsilon$$

$$|3(x+2)| < \varepsilon$$

$$3|x+2| < \varepsilon$$

$$|x+2| < \frac{\varepsilon}{3}$$

Let $0 < |x+2| < \frac{\varepsilon}{3} = \delta$ **than** $|(3x+7)-1| < \varepsilon$

A) Use definition of limit to show that $\lim_{x \to 2} (2x + 3) = 7$

Solution

$$|(2x+3) - 7| < \varepsilon$$

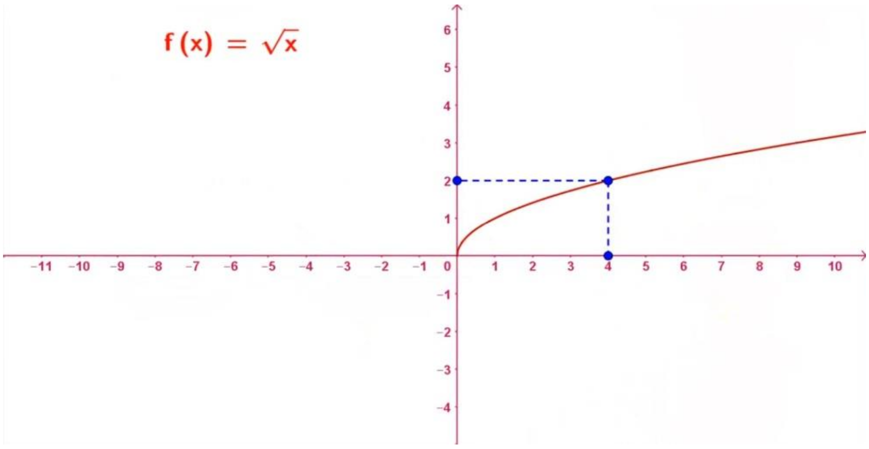
$$|2x - 4| < \varepsilon$$

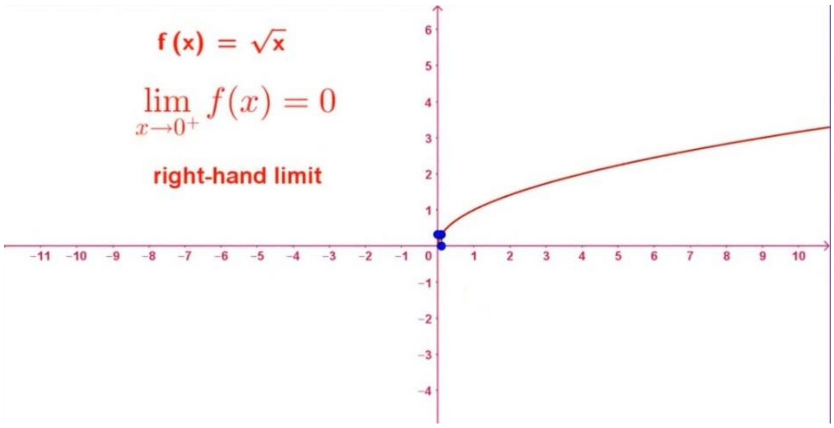
$$|2(x - 2)| < \varepsilon$$

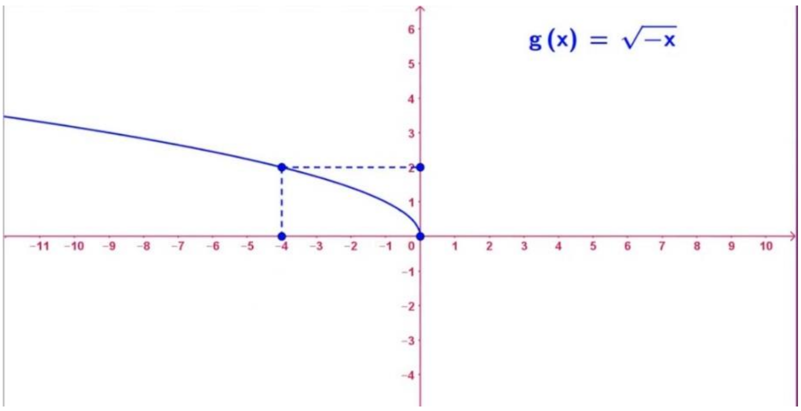
$$|x - 2| < \frac{\varepsilon}{2}$$

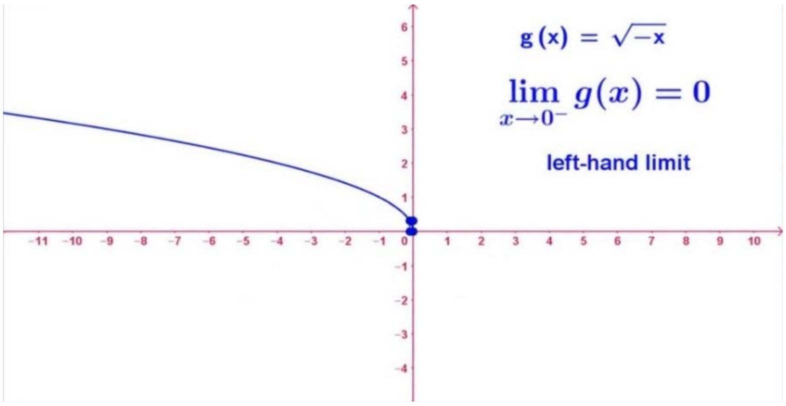
Let $0 < |x - 2| < \frac{\varepsilon}{2} = \delta$ than $|(2x+3) - 7| < \varepsilon$

ONE - SIDED LIMITS - النهاية في أتجاه واحد (اليمنى ،و اليسرى)









THEOREM 2.1.1

Let $L \in \mathbb{R}$, then

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

DEFINITION (One Sided-Limit) تعريف النهاية في أتجاه واحد

DEFINITION 2.1.2 (One Sided-Limit)

a. Right-Hand Limit (Limit from the Right)

Let f be a function defined on an open interval (a, c). Then

$$\lim_{x\to a^+} f(x) = L$$

means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } \ a < x < a + \delta, \text{ then } \left| f(x) - L \right| < \varepsilon .$$

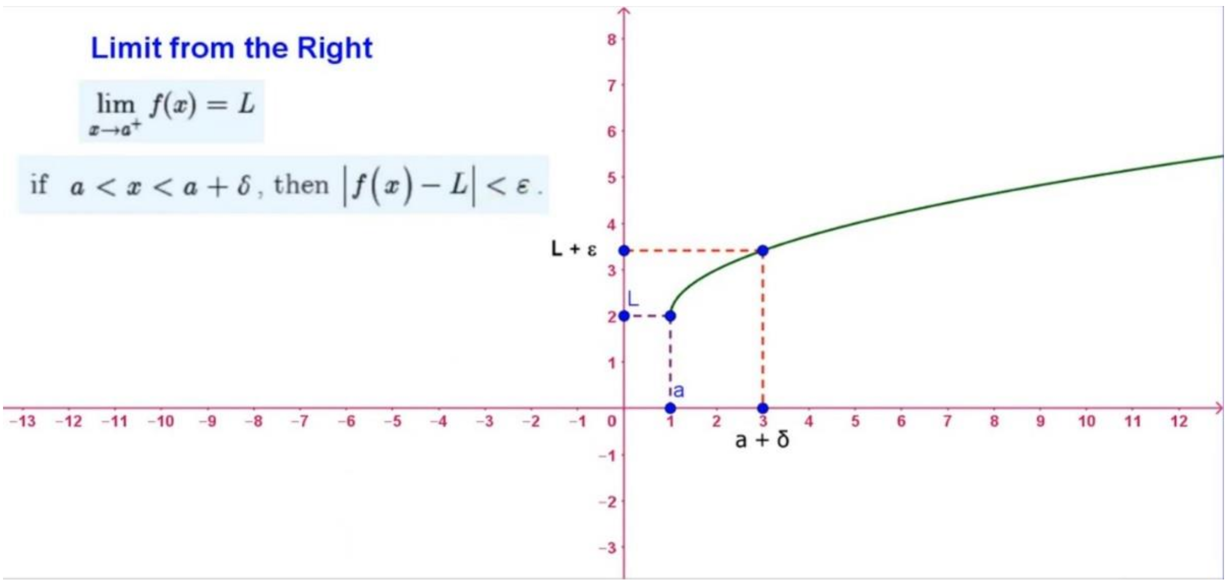
b. Left-Hand Limit (Limit from the Left)

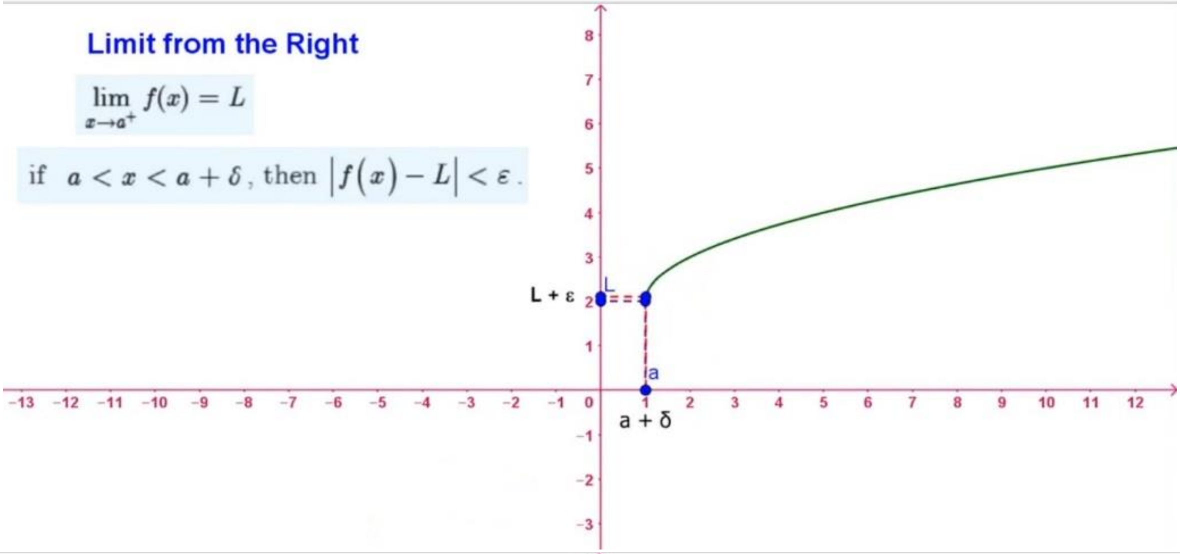
Let f be a function defined on an open interval (c, a). Then

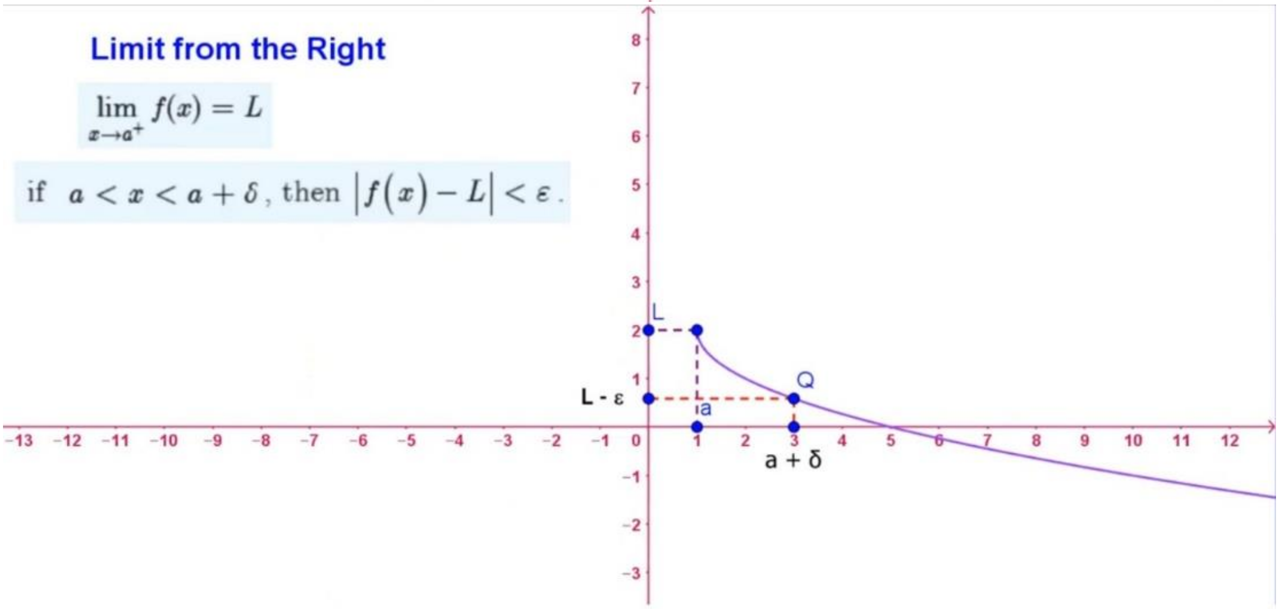
$$\lim_{x\to a^-} f(x) = L$$

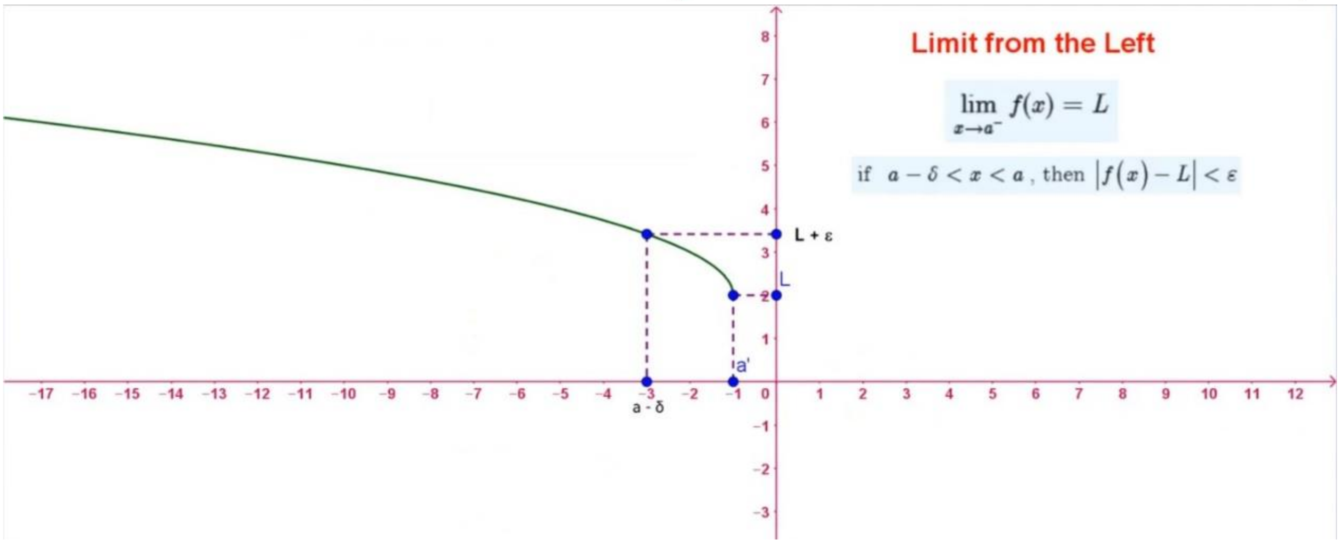
means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

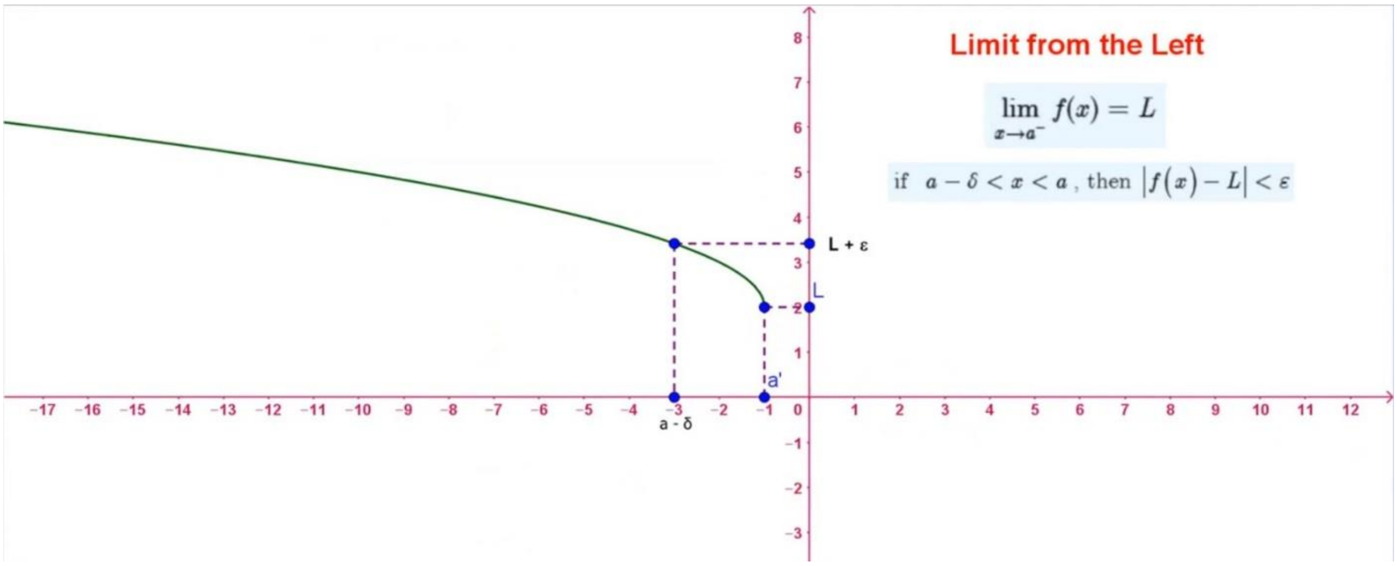
$$\text{if } \ a - \delta < x < a, \text{ then } \left| f(x) - L \right| < \varepsilon .$$

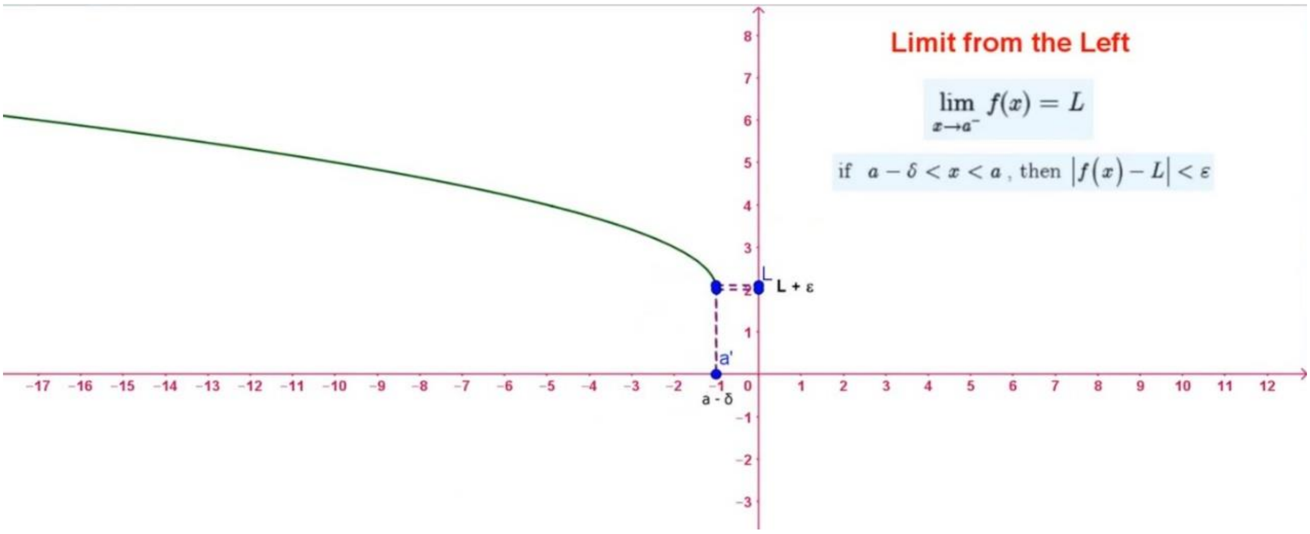


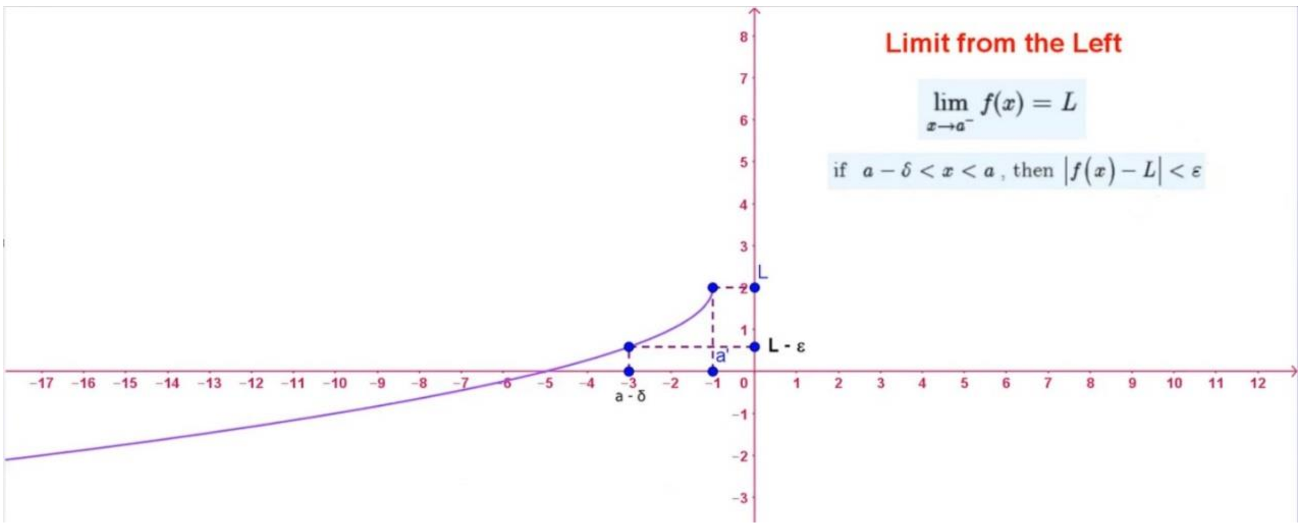

Limit from the Right
$\lim_{x \to a^+} f(x) = L$
if $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.
L + ε
L
a
a + δ


Limit from the Right
$\lim_{x \to a^+} f(x) = L$
if $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.
L
L - ε
Q
a
a + δ


Limit from the Left
$\lim_{x \to a^-} f(x) = L$
if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$
L + ε
L
a'
a - δ


Limit from the Left
$\lim_{x \to a^-} f(x) = L$
if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$
L + ε
L
a'
a - δ


Limit from the Left
$\lim_{x \to a^-} f(x) = L$
if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$
L
L + ε
a'
a - δ


Limit from the Left
$\lim_{x \to a^-} f(x) = L$
if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$
L
L - ε
a'
a - δ

EXAMPLE 2.1.6 Using definition of one sided limit, show that

a. $\lim_{x\to 2^+} \sqrt{x-2} = 0$

Solution if $2 < x < 2+\delta$, then $\left|\sqrt{x-2}-0\right| = \left|\sqrt{x-2}\right| < \varepsilon$

Let $2 < x < 2+\delta \Rightarrow 0 < x-2 < \delta \Rightarrow \sqrt{x-2} < \sqrt{\delta} \Rightarrow |\sqrt{x-2}\,| < \sqrt{\delta}$

choosing $\sqrt{\delta} \le \varepsilon \Rightarrow |\sqrt{x-2}\,| < \sqrt{\delta} \le \varepsilon$

b. $\lim_{x\to 2^-} \sqrt{2-x} = 0$

Solution if $2-\delta < x < 2$, then $\left|\sqrt{2-x}-0\right| = \left|\sqrt{2-x}\right| < \varepsilon$

Let $2-\delta < x < 2 \Rightarrow -\delta < x-2 < 0 \Rightarrow \delta > 2-x > 0 \Rightarrow \sqrt{2-x} < \sqrt{\delta} \Rightarrow |\sqrt{2-x}\,| < \sqrt{\delta}$

choosing $\sqrt{\delta} \le \varepsilon \Rightarrow |\sqrt{2-x}\,| < \sqrt{\delta} \le \varepsilon$

RELATED PROBLEM 3 Using definition of one sided limit, show that

a. $\lim_{x\to \frac{1}{3}^+} \sqrt{x-\frac{1}{3}} = 0$

b. $\lim_{x\to \frac{1}{3}^-} \sqrt{\frac{1}{3}-x} = 0$

Answer

a. Hint: Choose $\sqrt{\delta} \le \varepsilon$.

b. Hint: Choose $\sqrt{\delta} \le \varepsilon$.

Section 2.2
LIMITS LAWS قوانين النهايات

الحل الجبري لنهايات - Solution algebraic of limits

THEOREM 2.2.1

Let $a, c \in \mathbb{R}$. Then

a. $\lim_{x\to a} c = c$ $\lim_{x\to 5} 7 = 7$

b. $\lim_{x\to a} x = a$ $\lim_{x\to 5} x = 5$

THEOREM 2.2.2

Let $a, c \in \mathbb{R}$. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

a. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ Addition Law

b. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ Subtraction Law

c. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ Product Law

d. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \ne 0$ Quotient Law

e. $\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$ Scalar Multiplication Law

EXAMPLE 2.2.6 Find the following limits (if exist)

Solution

a. $\lim_{x\to1}\frac{x^2-x}{x-1}=\frac{0}{0}$

$$\lim_{x\to1}\frac{x^2-x}{x-1}=\lim_{x\to1}\frac{x(x-1)}{(x-1)}=\lim_{x\to1}x=1$$

b. $\lim_{x\to-3}\frac{x^2+2x-3}{x^3+27}=\frac{0}{0}$

$$\lim_{x\to-3}\frac{x^2+2x-3}{x^3+27}=\lim_{x\to-3}\frac{(x+3)(x-1)}{(x+3)(x^2-3x+9)}=\lim_{x\to-3}\frac{(x-1)}{(x^2-3x+9)}=\frac{-4}{27}$$

c. $\lim_{x\to2}\frac{x^2-4}{x^3-8}=\frac{0}{0}$

$$\lim_{x\to2}\frac{x^2-4}{x^3-8}=\lim_{x\to2}\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}=\lim_{x\to2}\frac{(x+2)}{(x^2+2x+4)}=\frac{4}{12}=\frac{1}{3}$$

d. $\lim_{h\to\frac{1}{2}}\frac{\frac{1}{h}-2}{h-\frac{1}{2}}=\frac{0}{0}$

$$\lim_{h\to\frac{1}{2}}\frac{\frac{1}{h}-2}{h-\frac{1}{2}}=\lim_{h\to\frac{1}{2}}\frac{\frac{1-2h}{h}}{\frac{2h-1}{2}}=\lim_{h\to\frac{1}{2}}\frac{\frac{-(2h-1)}{h}}{\frac{2h-1}{2}}=\lim_{h\to\frac{1}{2}}\frac{\frac{-1}{h}}{\frac{1}{2}}=\frac{\frac{-1}{\frac{1}{2}}}{\frac{1}{2}}=\frac{\frac{-2}{1}}{\frac{1}{2}}=-4$$

e. $\lim_{h\to0}\frac{(3+h)^2-9}{h}=\frac{0}{0}$

$$\lim_{h\to0}\frac{(3+h)^2-9}{h}=\lim_{h\to0}\frac{(3+h-3)(3+h+3)}{h}=\lim_{h\to0}\frac{h(6+h)}{h}=\lim_{h\to0}(6+h)=6$$

طريقة أخرى

$$\lim_{h\to0}\frac{(3+h)^2-9}{h}=\lim_{h\to0}\frac{9+6h+h^2-9}{h}=\lim_{h\to0}\frac{6h+h^2}{h}=\lim_{h\to0}\frac{h(6+h)}{h}=6$$

EXAMPLE 2.2.7 Find the following limits (if exist)

Solution

a. $\lim_{t\to0}\frac{\sqrt{t^2+9}-3}{t}=\frac{0}{0}$

$$\lim_{t\to0}\frac{\sqrt{t^2+9}-3}{t}=\lim_{t\to0}\frac{\sqrt{t^2+9}-3}{t}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}=\lim_{t\to0}\frac{t^2+9-9}{t(\sqrt{t^2+9}+3)}$$

$$=\lim_{t\to0}\frac{t^2}{t(\sqrt{t^2+9}+3)}=\lim_{t\to0}\frac{t}{(\sqrt{t^2+9}+3)}=\frac{0}{6}=0$$

b. $\lim_{x\to 2}\frac{x-2}{4-\sqrt{x^2+12}} = \frac{0}{0}$

$$\lim_{x\to 2}\frac{x-2}{4-\sqrt{x^2+12}} = \lim_{x\to 2}\frac{x-2}{4-\sqrt{x^2+12}}\cdot\frac{4+\sqrt{x^2+12}}{4+\sqrt{x^2+12}}$$

$$= \lim_{x\to 2}\frac{(x-2)(4+\sqrt{x^2+12})}{16-(x^2+12)} = \lim_{x\to 2}\frac{(x-2)(4+\sqrt{x^2+12})}{4-x^2}$$

$$= \lim_{x\to 2}\frac{(x-2)(4+\sqrt{x^2+12})}{(2-x)(2+x)} = \lim_{x\to 2}\frac{(x-2)(4+\sqrt{x^2+12})}{-(x-2)(2+x)}$$

$$= \lim_{x\to 2}\frac{(4+\sqrt{x^2+12})}{-(2+x)} = -2$$

C) Evaluate each of the following limits (if exist):

Solution

3) $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x^2-9} = \frac{0}{0}$

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x^2-9} = \lim_{x\to 3}\frac{\sqrt{x+1}-2}{(x-3)(x+3)}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$= \lim_{x\to 3}\frac{x+1-4}{(x-3)(x+3)\sqrt{x+1}+2} = \lim_{x\to 3}\frac{1}{(x+3)\sqrt{x+1}+2} = \frac{1}{24}$$

قوانين مهم حفظها و تطبيقها

$x^2 - y^2 = (x-y)(x+y)$

$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$

$(x \pm y)^2 = x^2 \pm 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

THEOREM 2.2.6 (The Sandwich (Squeeze) Theorem) نظرية السندويش

If $g(x) \le f(x) \le h(x)$ for every x in an open interval containing a real number c (except possibly at c itself), and

$$\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)\ \text{, then } \lim_{x\to c} f(x) = L.$$

EXAMPLE 2.2.9 Use Sandwich Theorem to evaluate

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right], x \ne 0$$

Solution

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \Rightarrow -x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} x^2 = 0 \quad and \quad \lim_{x\to 0} -x^2 = 0 \quad than \quad \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$$

HOMEWORK

5) $\lim_{x\to 0} x^2 \cos\left(\frac{3}{x}\right)$

THEOREM 2.2.7

a. $\lim_{\theta\to 0} \sin\theta = 0$

b. $\lim_{\theta\to 0} \cos\theta = 1$

COROLLARY 2.2.1

a. $\lim_{x\to a} \sin x = \sin a$

b. $\lim_{x\to a} \cos x = \cos a$

COROLLARY 2.2.2

a. $\lim_{x\to a} \tan x = \tan a$, $a \ne \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

b. $\lim_{x\to a} \cot x = \cot a$, $a \ne n\pi, n \in \mathbb{Z}$

c. $\lim_{x\to a} \sec x = \sec a$, $a \ne \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$

d. $\lim_{x\to a} \csc x = \csc a$, $a \ne n\pi, n \in \mathbb{Z}$

When $a = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$, then

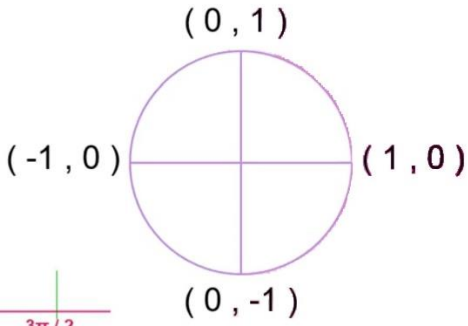
$$\lim_{x\to a^{\pm}} \tan x = \lim_{x\to a\pm} \sec x = \infty \text{ or } -\infty$$

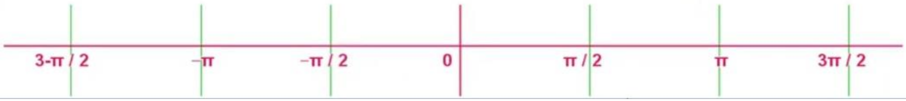
Similarly, when $a = n\pi$, n $\in \mathbb{Z}$, then

$$\lim_{x\to a^{\pm}} \cot x = \lim_{x\to a^{\pm}} \csc x = \infty \text{ or } -\infty$$

$$\lim_{x\to -\frac{\pi}{2}^-} \tan x = \lim_{x\to -\frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \infty$$

$$\lim_{x\to \pi^+} \csc x = \lim_{x\to \pi^+} \frac{1}{\sin x} = -\infty$$





EXAMPLE 2.2.10 Find the following limits (if exist)

Solution

a. $\lim_{x\to 0} \sec x = \lim_{x\to 0} \frac{1}{\cos x} = 1$

b. $\lim_{x\to \pi} \left(x\cos x\right) = \left(\lim_{x\to \pi} x\right)\left(\lim_{x\to \pi} \cos x\right) = \pi(-1) = -\pi$

c. $\lim_{x\to \frac{\pi}{4}} \sin^2 x = \left(\lim_{x\to \frac{\pi}{4}} \sin x\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$

THEOREM 2.2.8

a. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

b. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

EXAMPLE

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3.\sin(3x)}{3\,x} = 3$$

$$\lim_{x\to 0} \frac{\sin(3x)}{5x} = \lim_{x\to 0} \frac{3.\sin(3x)}{5.3\,x} = \frac{3}{5}$$

COROLLARY 2.2.3

Let a and b be two real numbers such that both different from zero. Then

a. $\lim_{x\to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}$

b. $\lim_{x\to 0} \frac{\tan(ax)}{bx} = \frac{a}{b}$

c. $\lim_{x\to 0} \frac{x}{\sin x} = 1$

EXAMPLE 2.2.11 Find

a. $\lim_{x\to 0} \frac{\sin(5x)}{3x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{\sin(5x)}{3x} = \frac{5}{3}$

b. $\lim_{\theta\to 0} \frac{\tan\theta}{2\theta} = \frac{0}{0}$

$\lim_{\theta\to 0} \frac{\tan\theta}{2\theta} = \frac{1}{2}$

c. $\lim_{t\to 0} \frac{\sin(2t)+1-\cos t}{3t} = \frac{0}{0}$

$$\lim_{t\to 0} \frac{\sin(2t)+1-\cos t}{3t} = \lim_{t\to 0} \frac{\sin(2t)}{3t} + \frac{1-\cos t}{3t}$$

$$= \lim_{t\to 0} \frac{\sin(2t)}{3t} + \frac{1}{3} . \frac{1-\cos t}{t} = \frac{2}{3} + \frac{1}{3} . 0 = \frac{2}{3}$$

d. $\lim_{x\to 0} \frac{\tan(2x)}{\sqrt{3x+1}-1} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{\tan(2x)}{\sqrt{3x+1}-1} = \lim_{x\to 0} \frac{\tan(2x)}{\sqrt{3x+1}-1} . \frac{\sqrt{3x+1}+1}{\sqrt{3x+1}+1}$$

$$= \lim_{x\to 0} \frac{\tan(2x)\sqrt{3x+1}+1}{(3x+1)-1} = \lim_{x\to 0} \frac{\tan(2x)\sqrt{3x+1}+1}{3x}$$

$$= \lim_{x\to 0} \frac{\tan(2x)}{3x} . \sqrt{3x+1}+1 = \frac{2}{3} . 2 = \frac{4}{3}$$

e. $\lim_{x\to 6} \frac{\sin(x-6)}{x^2-5x-6} = \frac{0}{0}$

$$\lim_{x\to 6} \frac{\sin(x-6)}{x^2-5x-6} = \lim_{x\to 6} \frac{\sin(x-6)}{(x-6)(x+1)} = \lim_{x\to 6} \frac{\sin(x-6)}{(x-6)} \cdot \frac{1}{(x+1)} = 1 \cdot \frac{1}{7} = \frac{1}{7}$$

f. $\lim_{x\to \pi} \frac{\sin x}{x-\pi} = \frac{0}{0}$

Let y = x - π ⟹ x = y + π

if x ⟶ π than y ⟶ 0

$$\lim_{x\to \pi} \frac{\sin x}{x-\pi} = \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} \frac{\sin(y)\cos(\pi) + \sin(\pi)\cos(y)}{y}$$

$$= \lim_{y\to 0} \frac{\sin(y)(-1) + 0 \cdot \cos(y)}{y} = \lim_{y\to 0} \frac{-\sin(y)}{y} = -1$$

RELATED PROBLEM 6 Find the following limits

c. $\lim_{x\to 1} \frac{\sin(x-1)}{\sqrt{x+3}-2} = \frac{0}{0}$

$$\lim_{x\to 1} \frac{\sin(x-1)}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{\sin(x-1)}{\sqrt{x+3}-2} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1} \frac{\sin(x-1)\sqrt{x+3}+2}{x+3-4} = \lim_{x\to 1} \frac{\sin(x-1)}{x-1} \cdot \sqrt{x+3}+2 = 4$$

d. $\lim_{\theta\to 0} \frac{1-\cos\theta}{5\theta^2} = \frac{0}{0}$

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{5\theta^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{5\theta^2} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta\to 0} \frac{1-\cos^2\theta}{5\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right)^2 \cdot \frac{1}{5(1+\cos\theta)} = \frac{1}{10}$$

e. $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{1-\tan x} = \frac{0}{0}$

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{1-\tan x} = \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{1 - \frac{\sin x}{\cos x}} = \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\frac{\cos x - \sin x}{\cos x}}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-(\cos x - \sin x)}{\frac{\cos x - \sin x}{\cos x}} = \lim_{x\to\frac{\pi}{4}} \frac{-1}{\frac{1}{\cos x}} = \lim_{x\to\frac{\pi}{4}} -\cos x = -\frac{\sqrt{2}}{2}$$

f. $\lim_{\theta\to 0} \frac{\sin(\tan 3\theta)}{\theta} = \frac{0}{0}$

$$\lim_{\theta\to 0} \frac{\sin(\tan 3\theta)}{\theta} = \lim_{\theta\to 0} \frac{\sin(\tan 3\theta)}{\theta} \cdot \frac{3\tan 3\theta}{3\tan 3\theta} = \lim_{\theta\to 0} \frac{\sin(\tan 3\theta)}{\tan 3\theta} \cdot \frac{3\tan 3\theta}{3\theta} = 3$$

EXAMPLE 2.2.12 Let $f(x) = \begin{cases} \frac{\tan(ax)}{x}, & x < 0 \\ 2(x-1) + a^2, & x \geq 0 \end{cases}$. Find the value of a such that $\lim_{x\to 0} f(x)$ exists.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\tan(ax)}{x} = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(2(x-1) + a^2\right) = -2 + a^2$$

$$a = -2 + a^2$$

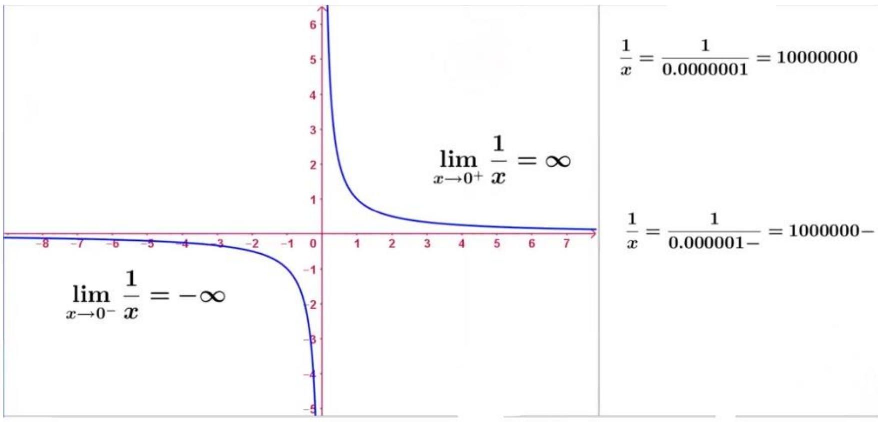
$$\Rightarrow a^2 - a - 2 = 0$$

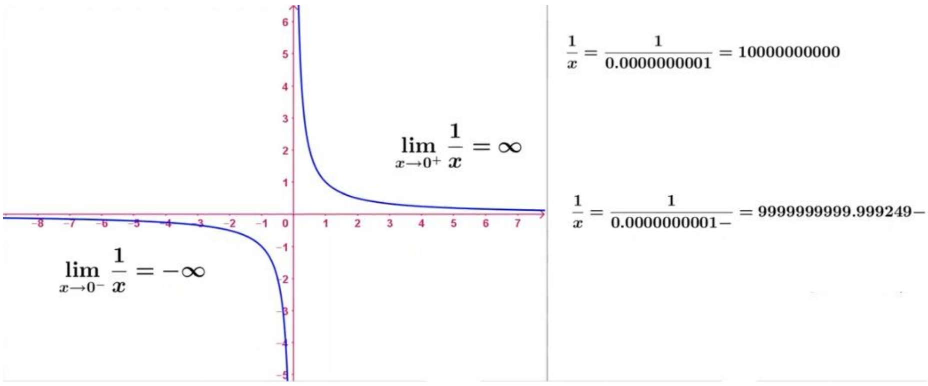
$$(a - 2)(a + 1) = 0$$

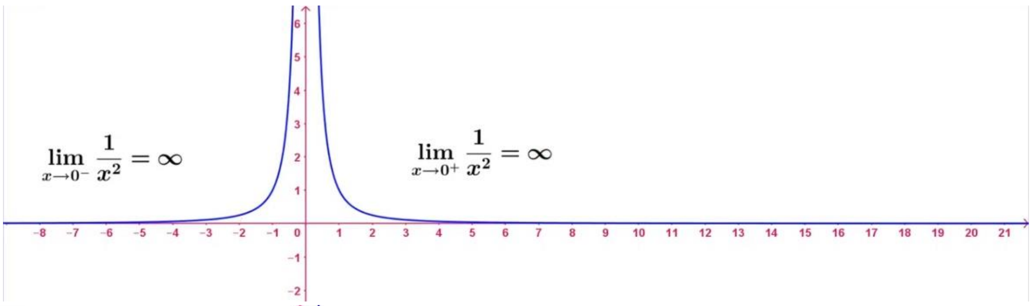
$$a=2 \text{ or } a = -1$$

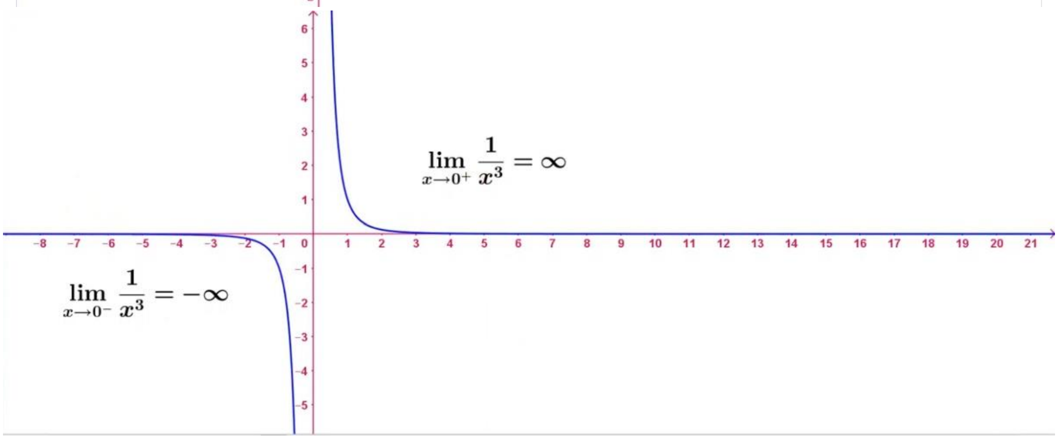
Section 2.3
LIMITS INVOLVING INFINITY – النهايات باستخدام ما لانهاية

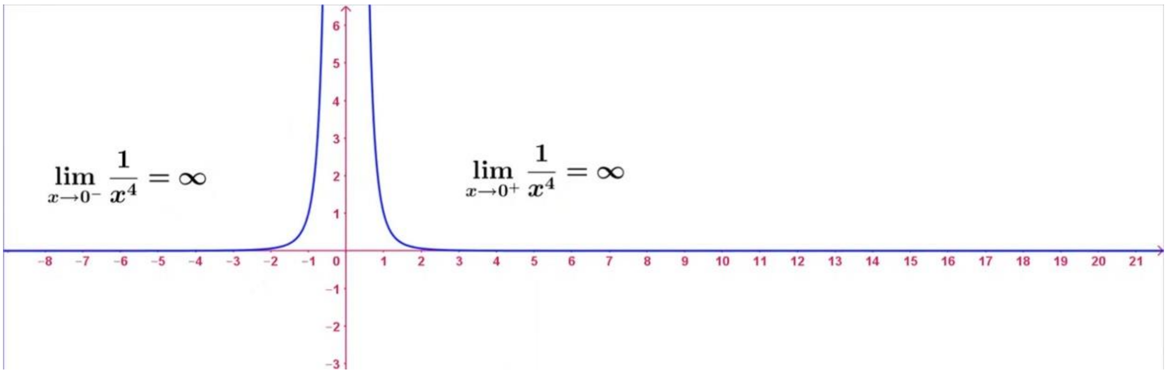
INFINITE LIMITS - نهاية الدالة ما لانهاية

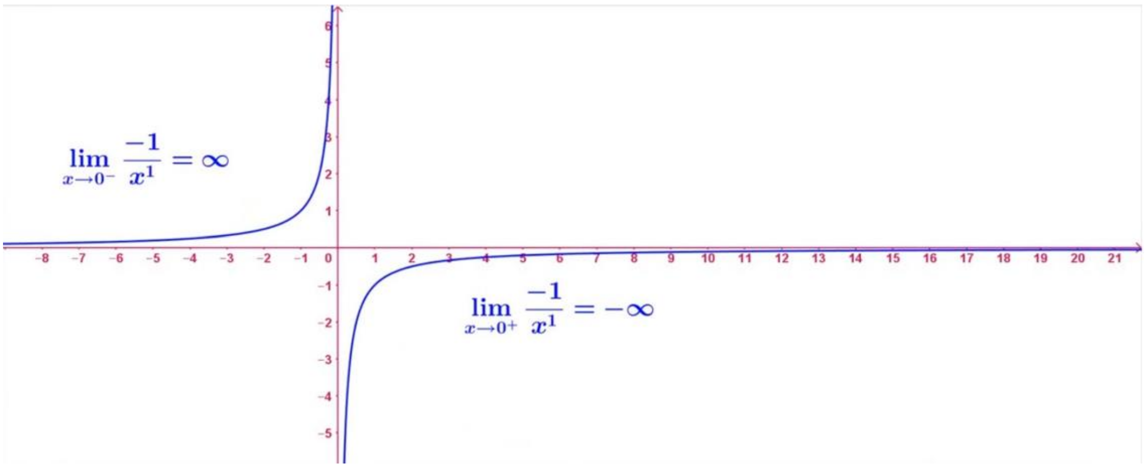


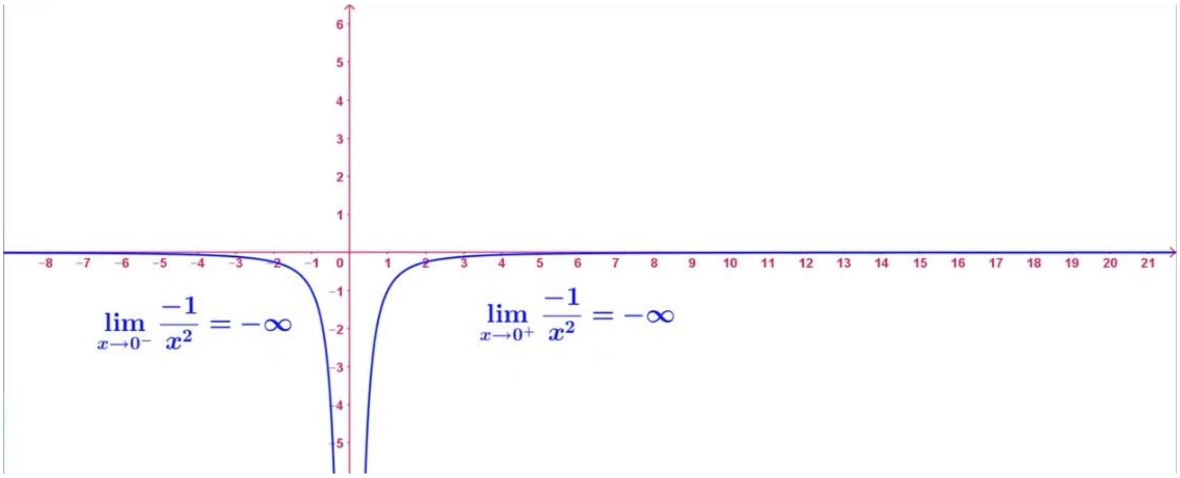


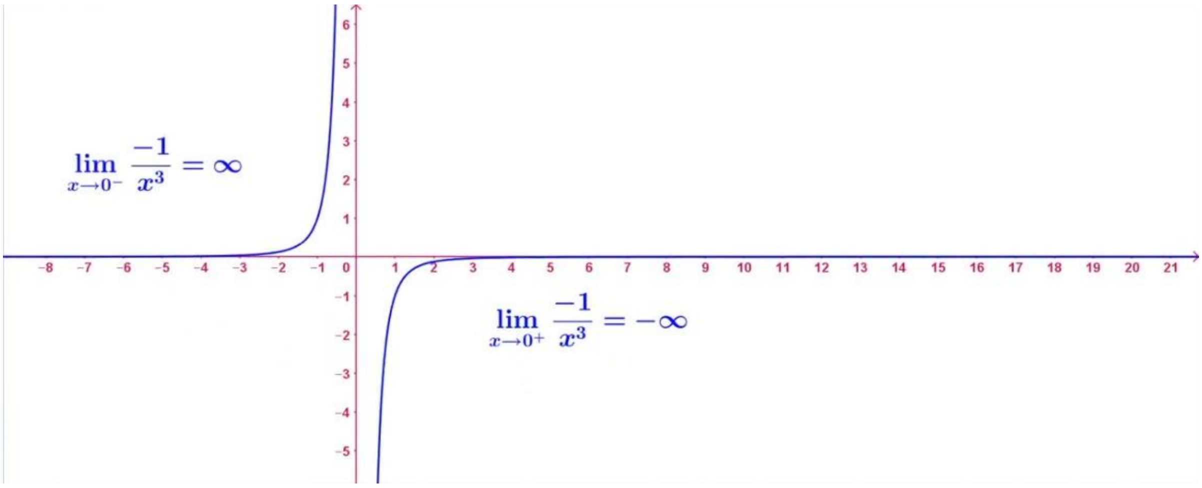


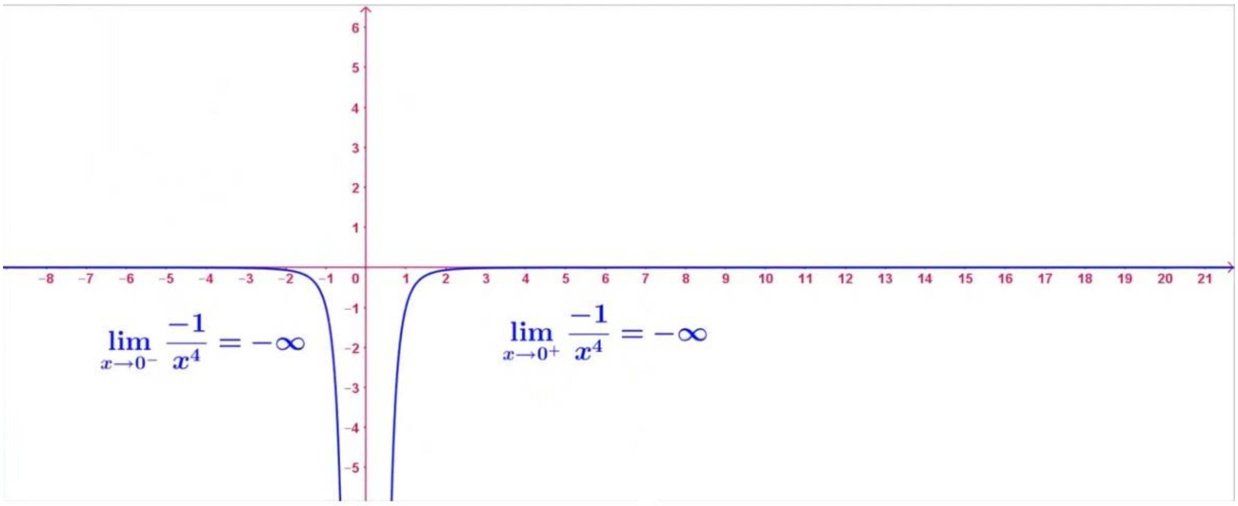



$\lim_{x\to 0^-} \frac{1}{x^4} = \infty$
$\lim_{x\to 0^+} \frac{1}{x^4} = \infty$


$\lim_{x\to 0^-} \frac{-1}{x^1} = \infty$
$\lim_{x\to 0^+} \frac{-1}{x^1} = -\infty$


$\lim_{x\to 0^-} \frac{-1}{x^2} = -\infty$
$\lim_{x\to 0^+} \frac{-1}{x^2} = -\infty$


$\lim_{x\to 0^-} \frac{-1}{x^3} = \infty$
$\lim_{x\to 0^+} \frac{-1}{x^3} = -\infty$



EXAMPLE 2.3.1 For the function f in Figure 2.3.5, determine the following limits.

a. $\lim_{x\to -3^-} f(x)$ b. $\lim_{x\to -3^+} f(x)$ c. $\lim_{x\to -3} f(x)$

d. $\lim_{x\to 0^-} f(x)$ e. $\lim_{x\to 0^+} f(x)$ f. $\lim_{x\to 0} f(x)$

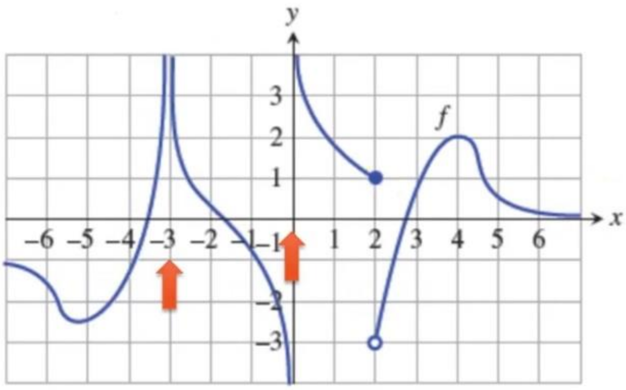


Figure 2.3.5

EXAMPLE 2.3.1 For the function f in Figure 2.3.5, determine the following limits.

a. $\lim_{x\to -3^-} f(x) = \infty$ b. $\lim_{x\to -3^+} f(x) = \infty$ c. $\lim_{x\to -3} f(x) = \infty$

d. $\lim_{x\to 0^-} f(x) = -\infty$ e. $\lim_{x\to 0^+} f(x) = \infty$ f. $\lim_{x\to 0} f(x)$ **does not exists**

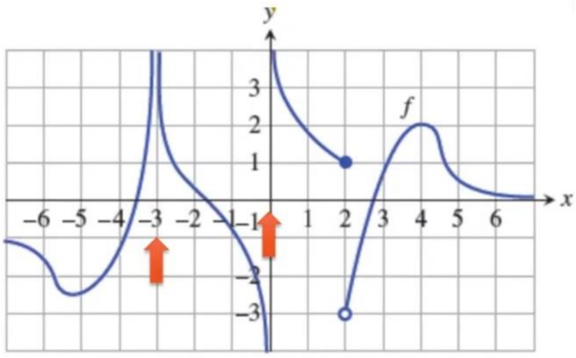
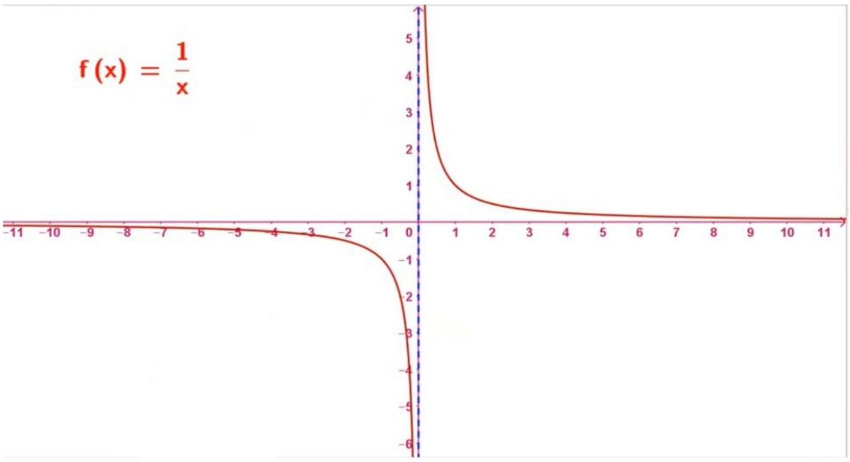


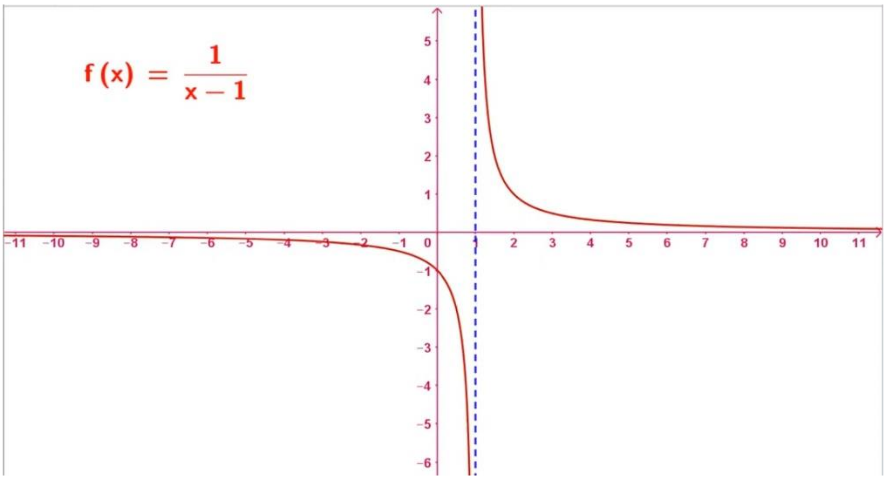
Figure 2.3.5

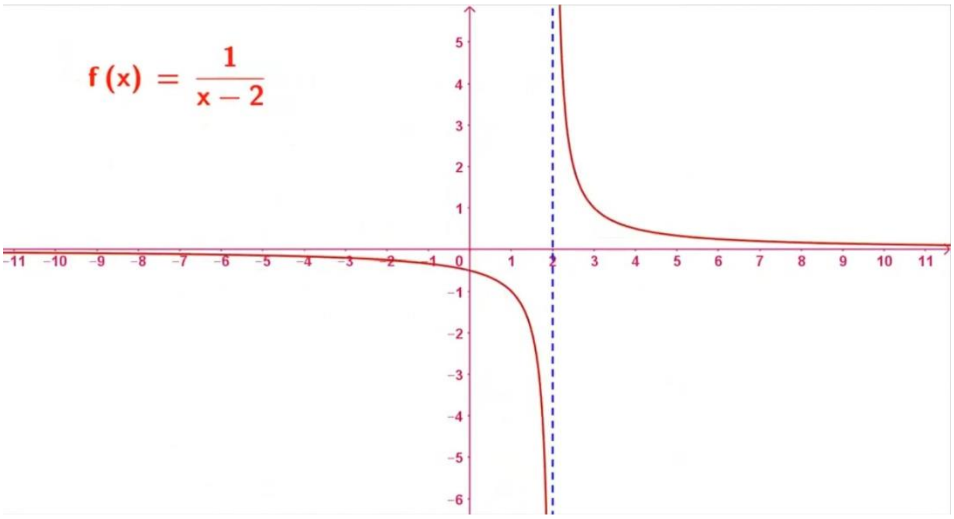
VERTICAL ASYMPTOTE خط التقارب الرأسي

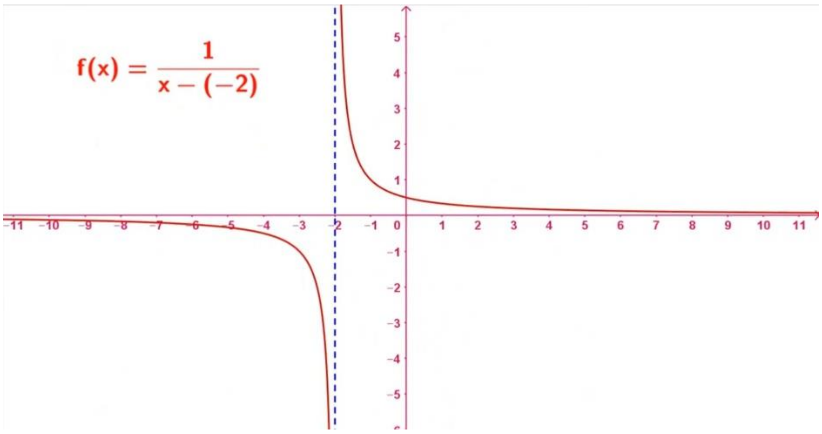
DEFINITION 2.3.2

If a function f approaches ∞ or $-\infty$ as x approaches a from the right or the left, then, the line $x = a$ is called *a vertical asymptote* for the graph of f.









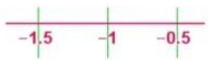
EXAMPLE 2.3.2 Determine all vertical asymptotes of the graph of the following functions

حدد الخطوط التقاربية الرأسية لمنحنى الدوال التالية

a. $f(x) = \dfrac{1}{2x+2}$

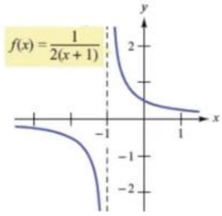
Solution : Let 2x+2 =0 ⟹ 2x =-2 ⟹ x =-1

$$\lim_{x\to -1^+} \frac{\mathbf{1}}{\mathbf{2x+2}} = \lim_{x\to -1^+} \frac{\mathbf{1}}{\mathbf{2(x+1)}} = \infty$$



$$\lim_{x\to -1^-} \frac{\mathbf{1}}{\mathbf{2x+2}} = \lim_{x\to -1^-} \frac{\mathbf{1}}{\mathbf{2(x+1)}} = -\infty$$

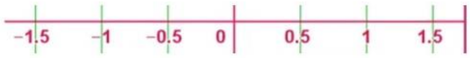
the line $x = -1$ is a vertical asymptote for f



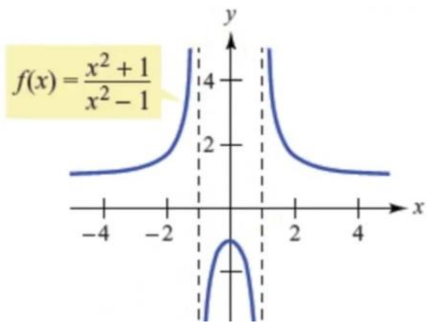
b. $f(x) = \dfrac{x^2+1}{x^2-1}$

Solution : Let x^2-1 =0 ⟹ (x-1)(x+1)=0 ⟹ x=1 , x=-1

$$\lim_{x\to 1^+} \frac{x^2+1}{x^2-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{x^2+1}{x^2-1} = -\infty$$



the line $x = 1$ is a vertical asymptote for f .



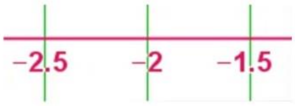
$$\lim_{x\to -1^+} \frac{x^2+1}{x^2-1} = -\infty \quad \text{and} \quad \lim_{x\to -1^-} \frac{x^2+1}{x^2-1} = \infty$$

the line $x = -1$ is a vertical asymptote for f .

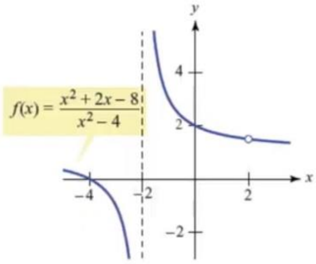
c. $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$

Solution :

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4} = \frac{(x-2)(x+4)}{(x-2)(x+2)} = \frac{x+4}{x+2}$$



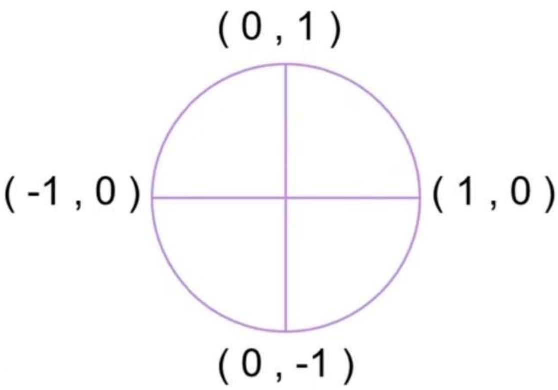
Let x+2 =0 ⟹ x = -2



$$\lim_{x \to -2^+} \frac{x+4}{x+2} = \infty$$

$$\lim_{x \to -2^-} \frac{x+4}{x+2} = -\infty$$

the line $x = -2$ is a vertical asymptote for f.



Example 2.3.3 Determine all vertical asymptotes of the graph of f (x) = tan x

$$\tan x = \frac{\sin x}{\cos x}$$

$\cos x = 0$ ⟹ $x = \dfrac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

f (x) = tan x has vertical asymptotes at $x = \dfrac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

RELATED PROBLEM 1 Determine all vertical asymptotes of the graph of the following functions

a. $f(x) = \dfrac{1}{3x - 9}$

b. $f(x) = \dfrac{x^2 + 4}{x^2 - 8}$

c. $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

d. $f(x) = \dfrac{1 - 5x}{|x| + 3}$

e. $f(x) = \cot x$

Answers

a. $x = 3$

b. $x = -2\sqrt{2},\ x = 2\sqrt{2}$

c. $x = -1$

d. No vertical asymptotes

e. $x = n\pi,\ n \in \mathbb{Z}$

LIMITS AT INFINITY - النهاية عند ما لانهاية

REMARK

a. Limits laws introduced in Section 2.2 still hold for limits involving infinity.

b. Let $p(x)$ be a polynomial of degree n, $n \geq 1$, and let a_n be the coefficient of x^n in $p(x)$. Then:

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} \left(a_n x^n\right)$$

a. $\lim_{x\to\infty} \left(-2x^3 - 5x^2 + 6x - 3\right) = \lim_{x\to\infty} \left(-2x^3\right) = -\infty$.

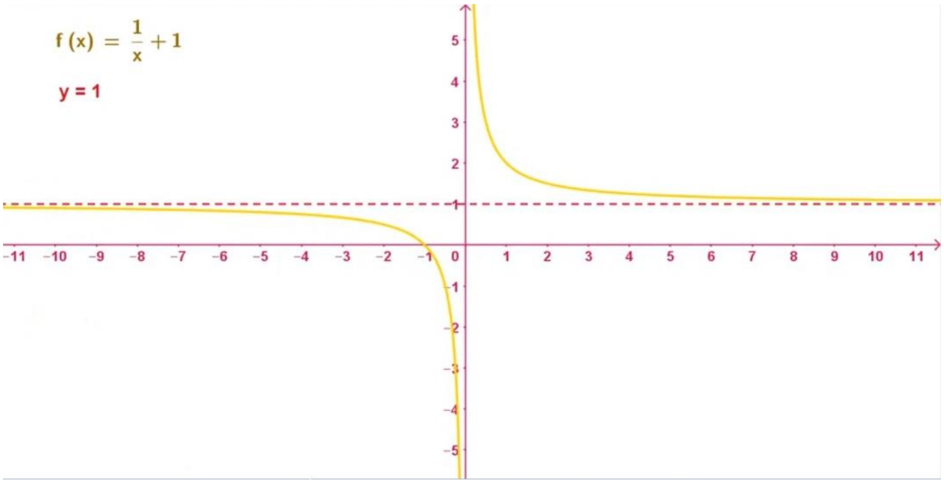
b. $\lim_{x\to-\infty} \left(7x^3 - 5x^7 + 6x^5 - 3\right) = \lim_{x\to-\infty} \left(-5x^7\right) = \infty$.

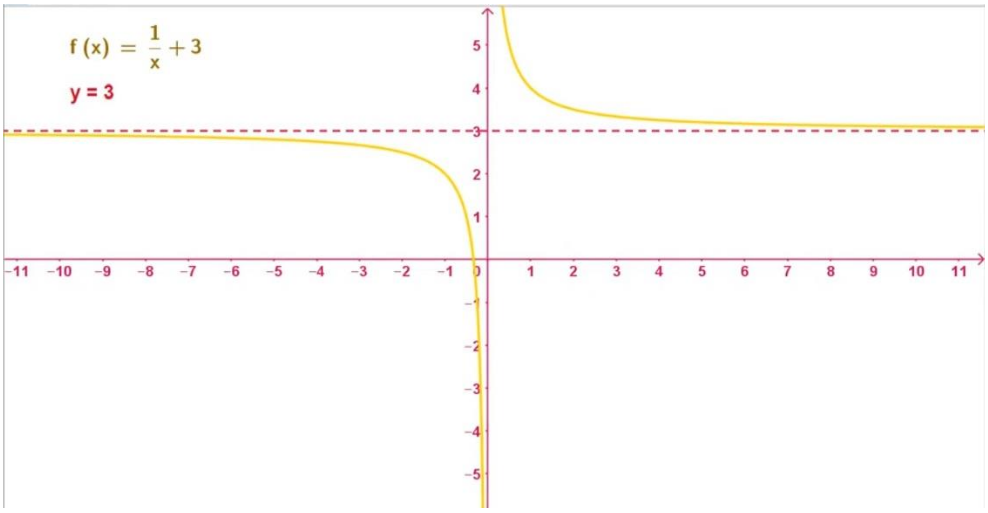
HORIZONTAL ASYMPTOTE الخطوط التقاربية الافقية

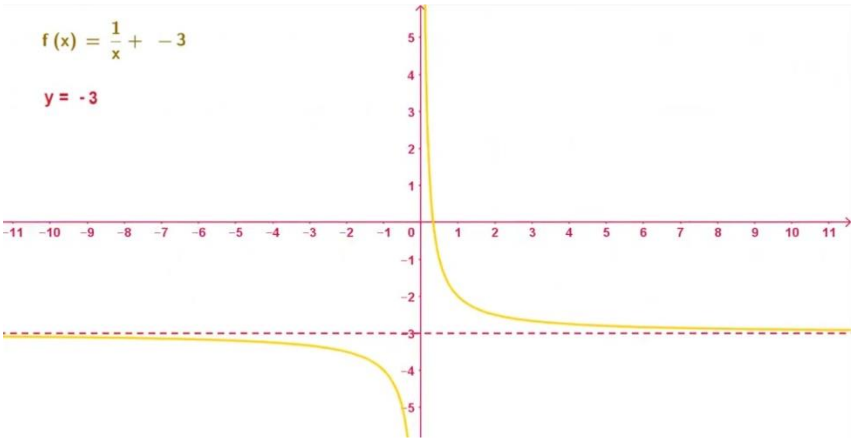
DEFINITION 2.3.4

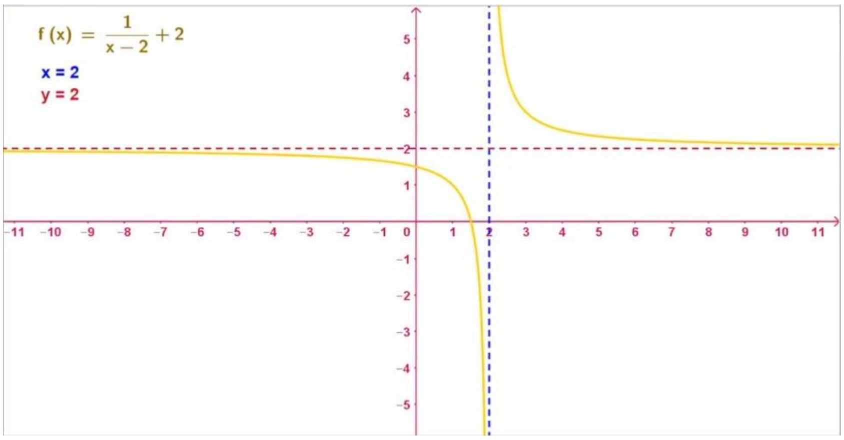
The line $y = L$ is called ***a horizontal asymptote*** for the graph of f if

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$









THEOREM 2.3.1

If r is a positive rational number and $c \neq 0$ is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 \quad \lim_{x\to-\infty} \frac{c}{x^r} = 0 \text{ and}$$

provided x^r is defined.

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{x^{1/2}} = 0\,.$$

$$\lim_{x\to\infty} \frac{100}{x^2+5} = 0\,.$$

COROLLARY 2.3.1

Suppose that $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials of degrees n and m respectively. Then

a. If $n < m$, then

$$\lim_{x\to\pm\infty} f(x) = 0$$

b. If $n = m$, then

$$\lim_{x\to\pm\infty} f(x) = \frac{\text{coefficient of } x^n \text{ in } p(x)}{\text{coefficient of } x^m \text{ in } q(x)}$$

c. If $n > m$, then

$$\lim_{x\to\infty} f(x) = \left(\frac{\text{sign of the coefficient of } x^n \text{ in } p(x)}{\text{sign of the coefficient of } x^m \text{ in } q(x)}\right)(\infty)$$

$$\lim_{x\to-\infty} f(x) = \left(\frac{\text{sign of the coefficient of } x^n \text{ in } p(x)}{\text{sign of the coefficient of } x^m \text{ in } q(x)}\right)(-1)^{n-m}(\infty)$$

COROLLARY 2.3.1

Suppose that $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials of degrees n and m respectively. Then

a. If $n < m$, then $\lim_{x\to\pm\infty} f(x) = 0$

$$\lim_{x\to\infty} \frac{3x^2 - x - 1}{x^3 + x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3} - \frac{x}{x^3} - \frac{1}{x^3}}{\frac{x^3}{x^3} + \frac{x}{x^3} + \frac{1}{x^3}} \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{1}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} \frac{10x^5 + x^4 + 24}{x^6 - x^4 + 1} = 0$$

$$\lim_{x\to-\infty} \frac{2x + 3}{-3x^2 + 2x - 1} = 0$$

37. $\lim_{x\to\infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}} = 0$

b. If $n = m$, then $\lim_{x\to\pm\infty} f(x) = \frac{\text{coefficient of } x^n \text{ in } p(x)}{\text{coefficient of } x^m \text{ in } q(x)}$

$$\lim_{x\to\infty} \frac{2x^4+3x^3+x^2+4}{5x^4+3x^3+1} = \lim_{x\to\infty} \frac{\frac{2x^4}{x^4} + \frac{3x^3}{x^4} + \frac{x^2}{x^4} + \frac{4}{x^4}}{\frac{5x^4}{x^4} + \frac{2x^3}{x^4} + \frac{1}{x^4}} = \lim_{x\to\infty} \frac{2 + \frac{3}{x} + \frac{1}{x^2} + \frac{4}{x^4}}{5 + \frac{2}{x} + \frac{1}{x^4}} = \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{x + 3}{x + 2} = \frac{1}{1} = 1$$

$$\lim_{x\to\infty} \frac{7x^2 + x - 37}{2x^2 - 5x} = \frac{7}{2}$$

$$\lim_{x\to-\infty} \frac{x - x^2 + 1}{3x^2 + 5x} = -\frac{1}{3}$$

$$\lim_{x\to-\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$$

c. If $n > m$, then

$$\lim_{x\to\infty} f(x) = \left(\frac{\text{sign of the coefficient of } x^n \text{ in } p(x)}{\text{sign of the coefficient of } x^m \text{ in } q(x)}\right)(\infty)$$

$$\lim_{x\to-\infty} f(x) = \left(\frac{\text{sign of the coefficient of } x^n \text{ in } p(x)}{\text{sign of the coefficient of } x^m \text{ in } q(x)}\right)(-1)^{n-m}(\infty)$$

$$\lim_{x\to\infty} \frac{x^6 - 2x + 1}{1 - x^2} = \frac{+}{-}\infty = -\infty$$

$$\lim_{x\to-\infty} \frac{x^6 - 2x + 1}{1 - x^2} = \frac{+}{-}\infty\,(-1)^{6-2} = -\infty$$

c. $\lim_{x\to-\infty} \frac{-x^5 + x^3 + 1}{x^2 + 1} = \frac{-}{+}\infty\,(-1)^{5-2} = \infty$

RELATED PROBLEM 3

A. Find the following limits (if exist)

a. $\lim_{x\to-\infty}\left(-2x^3-7x^2+5x-3\right)$ b. $\lim_{x\to\infty}\frac{1}{\sqrt[3]{x+2}}$ c. $\lim_{x\to\infty}\frac{100}{x^2+x-5}$

d. $\lim_{x\to-\infty}\frac{3x^2+4x-1}{x-3}$ e. $\lim_{x\to\infty}\frac{12x^2+2x-13}{4x^2-5x}$ f. $\lim_{x\to-\infty}\frac{2x^2-3x+1}{x^3-4}$

g. $\lim_{x\to-\infty}\frac{2x+3}{\sqrt{16x^2-3x}}$

$$ax+b = x\left(a+\frac{b}{x}\right)$$

g. $$\lim_{x\to-\infty}\frac{2x+3}{\sqrt{16x^2-3x}} = \lim_{x\to-\infty}\frac{x\left(2+\frac{3}{x}\right)}{\sqrt{x^2\left(16-\frac{3}{x}\right)}} = \lim_{x\to-\infty}\frac{x\left(2+\frac{3}{x}\right)}{\sqrt{x^2}\sqrt{\left(16-\frac{3}{x}\right)}}$$

$$= \lim_{x\to-\infty}\frac{x\left(2+\frac{3}{x}\right)}{|x|\sqrt{\left(16-\frac{3}{x}\right)}} = \lim_{x\to-\infty}\frac{x\left(2+\frac{3}{x}\right)}{(-x)\sqrt{\left(16-\frac{3}{x}\right)}}$$

$$= \lim_{x\to-\infty}\frac{\left(2+\frac{3}{x}\right)}{(-1)\sqrt{\left(16-\frac{3}{x}\right)}} = \frac{2}{(-1)\sqrt{(16}} = -\frac{1}{2}$$

47. $$\lim_{x\to\infty}\left(\sqrt{9x^2-x}-3x\right) = \lim_{x\to-\infty}\sqrt{9x^2-x}-3x\,.\frac{\sqrt{9x^2-x}+3x}{\sqrt{9x^2-x}+3x}$$

$$= \lim_{x\to-\infty}\frac{9x^2-x-9x^2}{\sqrt{9x^2-x}+3x} = \lim_{x\to-\infty}\frac{-x}{\sqrt{x^2\left(9-\frac{1}{x}\right)}+3x}$$

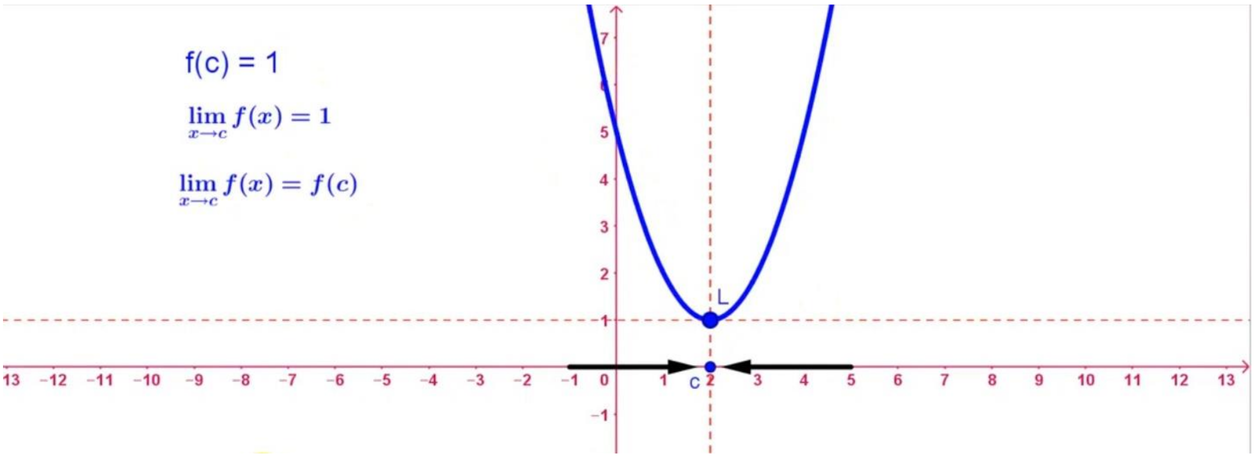
$$= \lim_{x\to-\infty}\frac{-x}{|x|\sqrt{\left(9-\frac{1}{x}\right)}+3x} = \lim_{x\to-\infty}\frac{-x}{-x\sqrt{\left(9-\frac{1}{x}\right)}+3x}$$

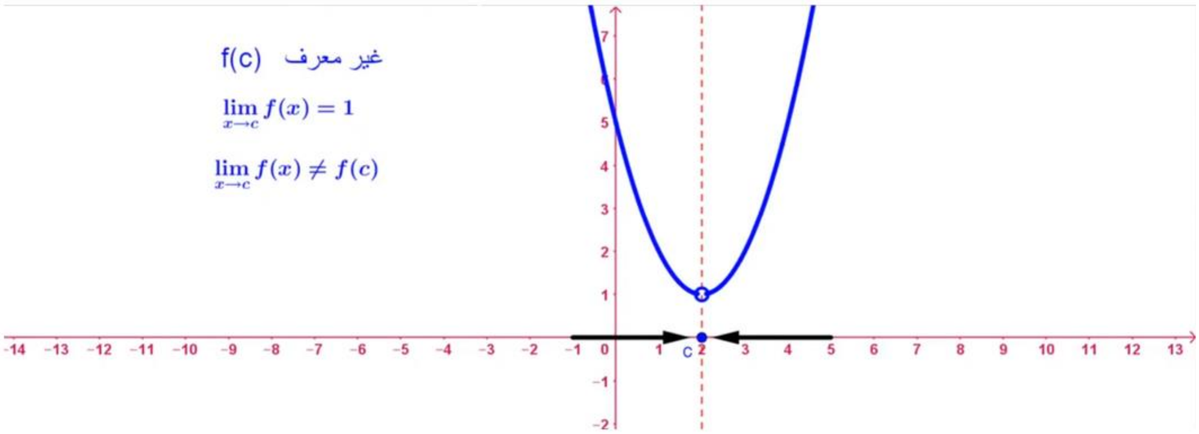
$$= \lim_{x\to-\infty}\frac{-x}{x\left(-\sqrt{9-\frac{1}{x}}+3\right)} = \frac{1}{6}$$

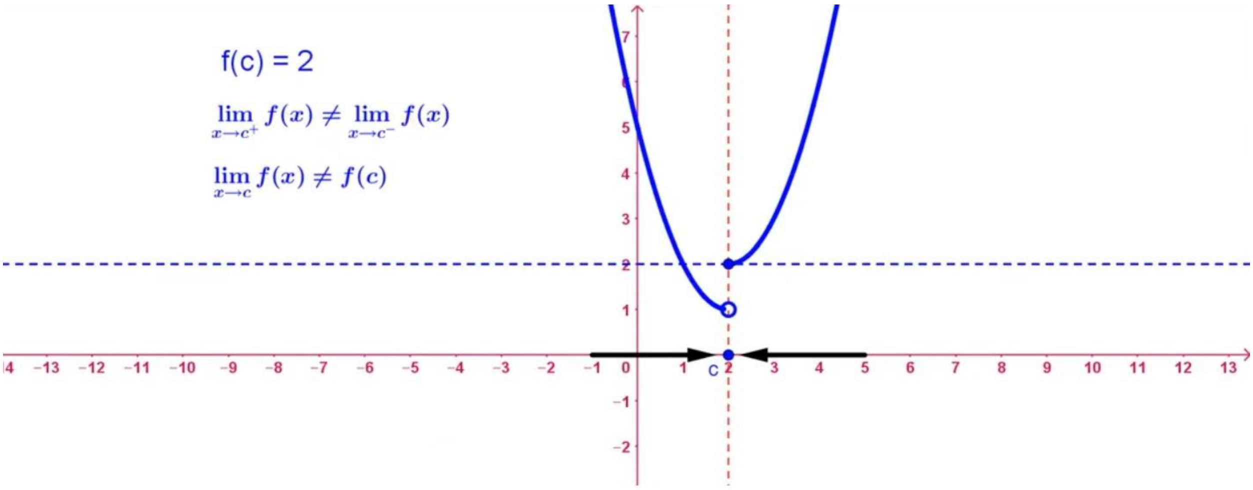
Section 2.4

CONTINUITY OF FUNCTIONS الـــدوال المتصـــلة

CONTINUITY OF FUNCTIONS - الـــدوال المتصـــلة







DEFINITION 2.4.1

A function f is continuous at a point $x = c$ if the following conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x\to c} f(x)$ exists

c. $\lim_{x\to c} f(x) = f(c)$

EXAMPLE 2.4.2 **Discuss the continuity of each function at the indicated point**

ناقش اتصال الدوال عند النقطة المحددة

a. $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ at $x = 1$.

Solution $f(1) = 2$

$$\lim_{x\to1} \frac{x^2-1}{x-1} = \lim_{x\to1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to1}(x+1) = 2$$

$$\lim_{x\to1} f(x) = f(1)$$

then the function f is continuous at $x = 1$

b. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$ at $x = 0$

Solution $f(0) = 3$

$$\lim_{x\to0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to0} f(x) \neq f(0)$$

the function f is discontinuous at x = 0 .

c. $f(x) = |x+1|$ at $x = -1$

Solution $f(-1) = 0$

$$f(x) = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x < -1 \end{cases}$$

$$\lim_{x\to-1^-} |x+1| = \lim_{x\to-1^-} (-x-1) = 0$$

$$\lim_{x\to-1^+} |x+1| = \lim_{x\to-1^+} (x+1) = 0$$

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} f(x) = 0$$

$\lim_{x\to-1} f(x) = f(-1)$, the function f is continuous at $x = -1$.

d. $f(x) = \dfrac{1}{x+3}$ at $x = -3$

Solution

f(-3) غير معرفة

The function is discontinuous at $x = -3$, because it is not defined at $x = -3$.

RELATED PROBLEM 1 Discuss the continuity of each function at the indicated point

a. $f(x) = \begin{cases} \dfrac{x^2+4x-5}{x-1}, & x \neq 1 \\ 6, & x = 1 \end{cases}$ at $x = 1$.

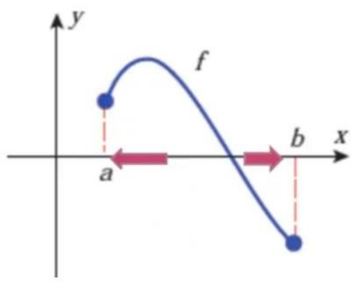
b. $f(x) = \begin{cases} \dfrac{\tan(4x)}{2x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$ at $x = 0$

c. $f(x) = |2x - 1|$ at $x = \dfrac{1}{2}$

d. $f(x) = \dfrac{1}{x^2+2}$ at $x = -2$

Answers

a. Continuous **b.** Discontinuous **c.** Continuous **d.** Continuous



DEFINITION 2.4.2

a. If a function is continuous at each point in an open interval (a,b), then it is called continuous on the interval (a,b).

b. If a function f is defined on a closed interval $[a,b]$ and continuous on the open interval (a,b), then it is called continuous on $[a,b]$ if

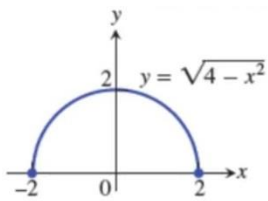
$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b)$$

EXAMPLE 2.4.5 Discuss the continuity of the function $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 2]$.

Solution: Since $4 - x^2$ is a polynomial

f is continuous for all x in the interval (-2,2)

f(2) = 0 , f(-2)=0



$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = f(-2)$

$\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2^-} \sqrt{4 - x^2} = f(2)$

f is continuous on the closed interval $[-2, 2]$

EXAMPLE 2.4.3

a. Determine where $f(x) = \dfrac{x^2-4}{x-2}$ is continuous.

b. Redefine the function in part (a) to make it continuous at every point in

Solution $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{(x-2)} = x+2$ for $x \neq 2$.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

EXAMPLE 2.4.4 Find the x-values (if any) at which f is discontinuous

أوجد قيمة x التي تجعل الدالة غير متصلة

a. $f(x) = 2x^3 - 4x + 5$

Solution None f(x) is a polynomial

b. $g(x) = \dfrac{x^2-5x-2}{x^2-2x-3}$

Solution

Let $x^2-2x-3 = 0$ ⇒ $(x-3)(x+1) = 0$ ⇒ $x=3$ or $x=-1$

c. $h(x) = \dfrac{1-6x}{x^2+5}$

Solution None $x^2 + 5 > 0, \ \forall x \in R$

EXAMPLE 2.4.6 Discuss the continuity of the function $f(x) = \begin{cases} x+1, & x \leq 0 \\ \cos x, & x > 0 \end{cases}$

Solution

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = 1$$

$$\lim_{x\to 0} f(x) = 1 = f(0)$$

f is continuous on $\mathbb{R}$

THEOREM 2.4.3

If a function g is continuous at c and a function f is continuous at $g(c)$, then the composite function $f \circ g$ is continuous at c.

EXAMPLE 2.4.9 Determine where the function $h(x) = \cos\left(2x^2 - 3x + 1\right)$ is continuous.

Solution

Let $g(x) = 2x^2 - 3x + 1$ continuous on R

Let $f(x) = \cos x$ continuous on R

$h(x) = f\left(g(x)\right)$ continuous on R

THEOREM 2.4.4

If f is continuous at b and $\lim_{x \to c} g(x) = b$, then

$$\lim_{x \to c} f(g(x)) = f(b) = f\left(\lim_{x \to c} g(x)\right).$$

EXAMPLE 2.4.10 Find the following limits (if exist)

a. $\lim_{x \to 1}\left|\frac{-x^2 - x + 2}{x - 1}\right|$

Solution

$$\lim_{x \to 1}\left|\frac{-x^2 - x + 2}{x - 1}\right| = \left|\lim_{x \to 1}\frac{-x^2 - x + 2}{x - 1}\right| = \left|\lim_{x \to 1}\frac{-(x-1)(x+2)}{x - 1}\right|$$

$$= \left|\lim_{x \to 1}(-1)(x+2)\right| = |-3| = 3$$

b. $\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

Solution

$$\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x \to \pi/2}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right)$$

$$= \cos\left(\lim_{x \to \pi/2}(2x) + \lim_{x \to \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos(\pi + \sin(2\pi)) = \cos(\pi) = -1$$

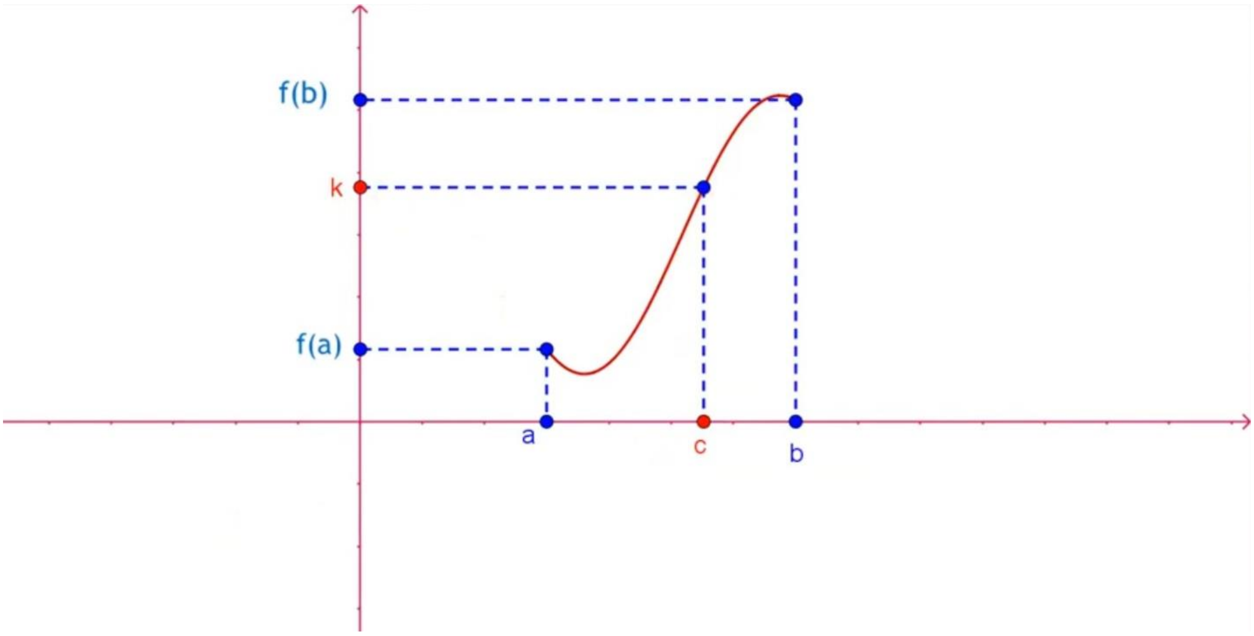
c. $\lim_{x \to 0}\left(\sqrt{x+1}\ \cos(\tan x)\right)$

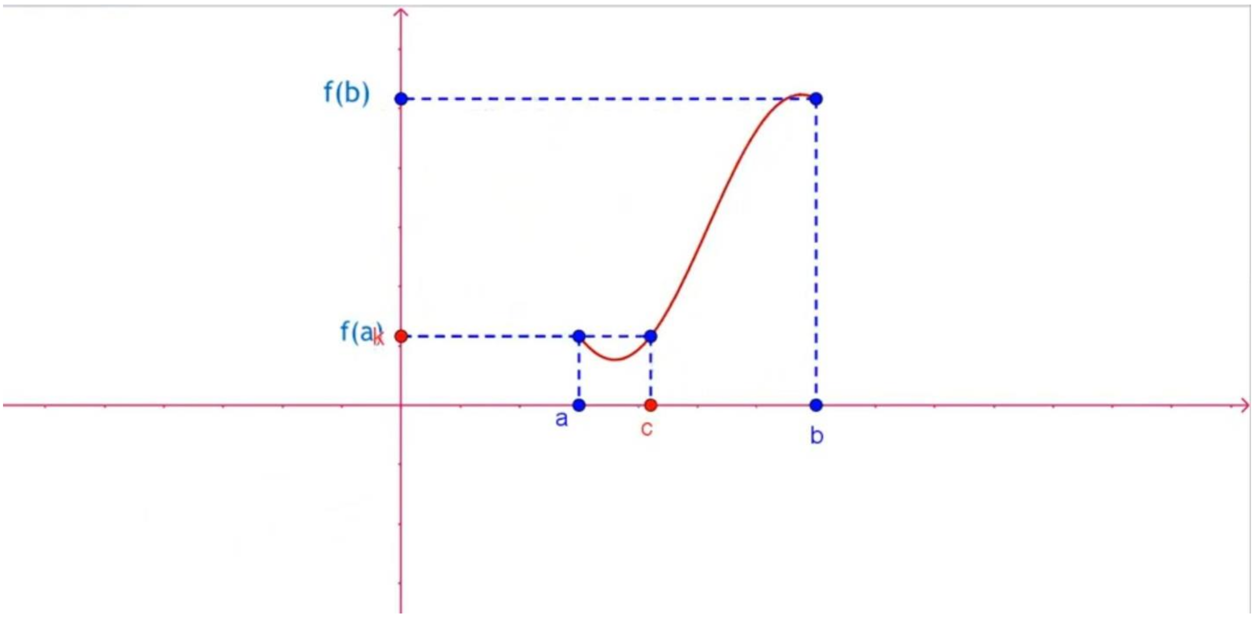
Solution

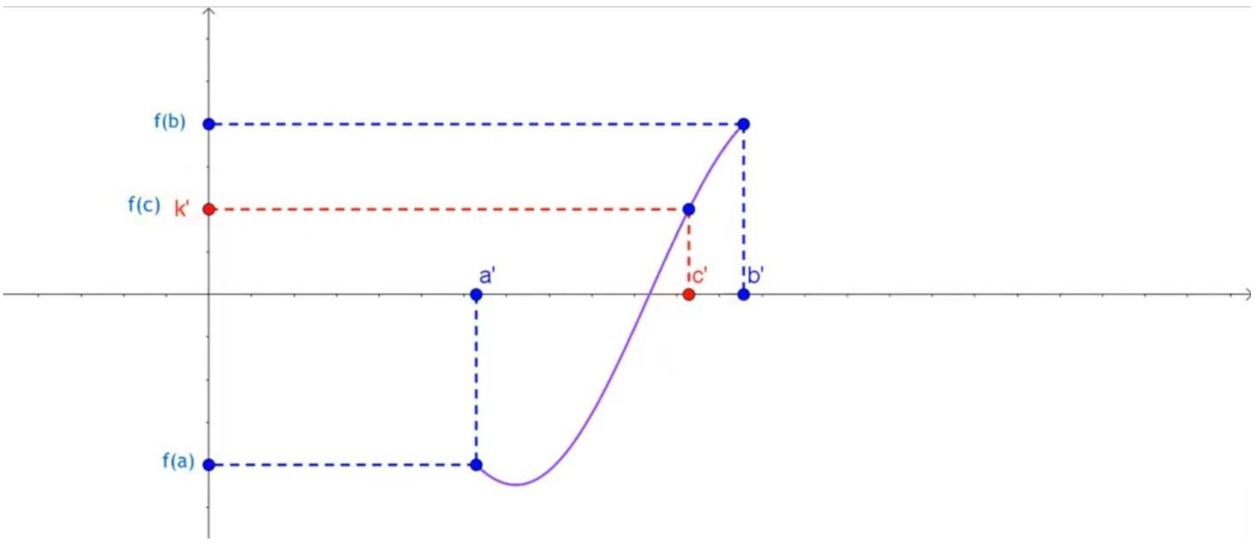
$$\lim_{x \to 0}\left(\sqrt{x+1}\ \cos(\tan x)\right) = \left(\lim_{x \to 0}\sqrt{x+1}\right)\cdot\left(\cos(\lim_{x \to 0}\tan x)\right) = 1 \cdot \cos 0 = 1(1) = 1$$

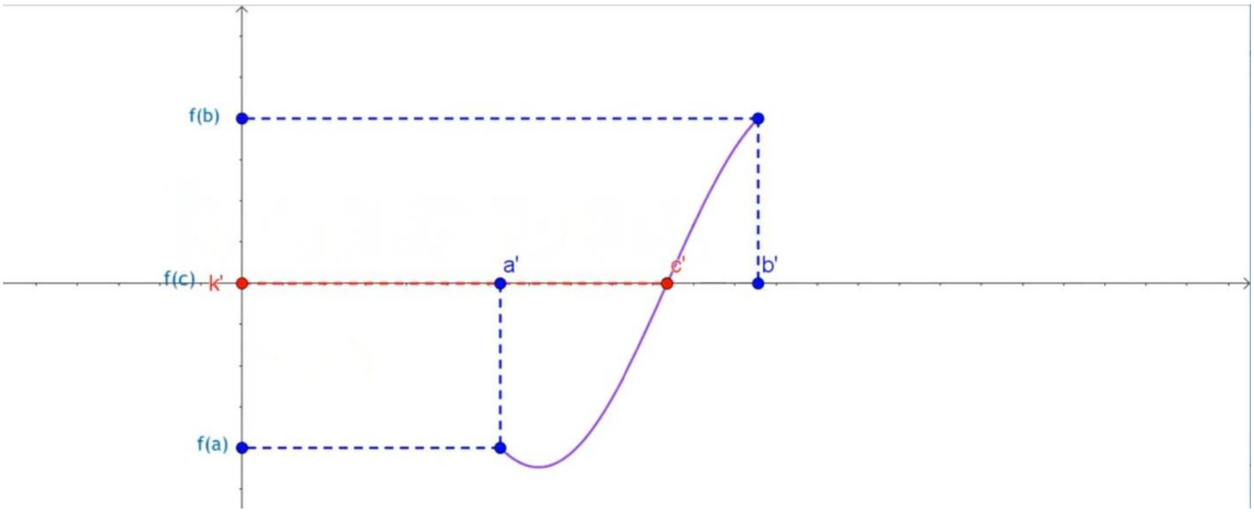
THEOREM 2.4.5: Intermediate Value Theorem نظرية القيمة المتوسطة

If f is continuous on a closed interval $[a,b]$ and if k is any number between $f(a)$ and $f(b)$, then there is at least a number c in $[a,b]$ such that $f(c) = k$.


f(b)
k
f(a)
a
c
b


f(b)
f(a)
k
a
c
b


f(b)
f(c)
k'
a'
c'
b'
f(a)



EXAMPLE 2.4.11 Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0,1]$.

Solution

f is a polynomial function, it is continuous

$f(0) = -1$ and $f(1) = 2$

$-1 < 0 < 2$

$f(0) < 0 < f(1)$

from the Intermediate Value Theorem

f has a zero in the closed interval $c \in [0,1]$.

EXAMPLE 2.4.12 Use the Intermediate Value Theorem to prove that the equation

$\sqrt{3x+4} = 5 - x^2$ has a solution.

Solution $\sqrt{3x+4} + x^2 - 5 = 0$

Let $f(x) = \sqrt{3x+4} + x^2 - 5$

$g(x) = \sqrt{3x+4}$ continuous on $\left[-\frac{4}{3}, \infty\right)$

$h(x) = x^2 - 5$ continuous on R

Choose $[0,4] \subseteq \left[-\frac{4}{3}, \infty\right) \subseteq$ R

Look f(0)= -3 , f(4)= 15 ⇨ $c \in [0,4]$, f(c)=0

EXAMPLE 2.4.12 Use the Intermediate Value Theorem to prove that the equation

$\sqrt{3x+4} = 5 - x^2$ has a solution.

Solution $\sqrt{3x+4} + x^2 - 5 = 0$

Let $f(x) = \sqrt{3x+4} + x^2 - 5$

$g(x) = \sqrt{3x+4}$ continuous on $\left[-\frac{4}{3}, \infty\right)$

$h(x) = x^2 - 5$ continuous on R

Choose $[0,4] \subseteq \left[-\frac{4}{3}, \infty\right) \subseteq$ R

Look f(0)= -3 , f(4)= 15 ⇨ c∈ [0,4] , f(c)=0

number c is a solution

B) Use the Intermediate Value Theorem to show that $f(x) = x^5 - 4x^3 + 1$ has a zero in the interval $[0,1]$.

Solution

f is a polynomial function, it is continuous

f(0) = 1 , f(1)= -2

-2 < 0 < 1

f(0) < 0 < f(1)

from the Intermediate Value Theorem

f has a zero in the closed interval c ∈ [0,1].

Question

Using **Intermediate Value Theorem**, show that $f(x) = x^4 - 6x + 1$ has at least one real root (zero).

Solution f continuous on R

Look f(0)= 1 , f(1)= -2 ⇨ c∈ [0,1] , f(c)=0

number c is a solution