CHAPTER
3

CHAPTER 3
DIFFERENTIATION - التفاضل (الاشتقاق)

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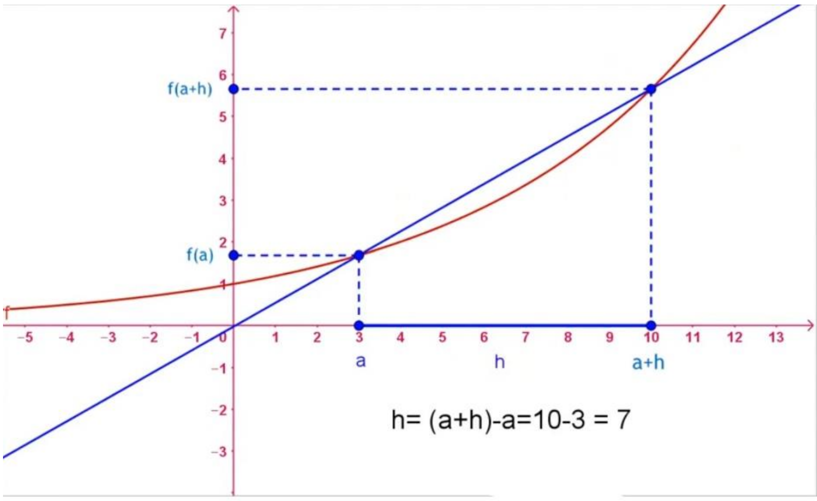
SECTION 3.1

THE DERIVATIVE AND THE TANGENT LINE PROBLEM

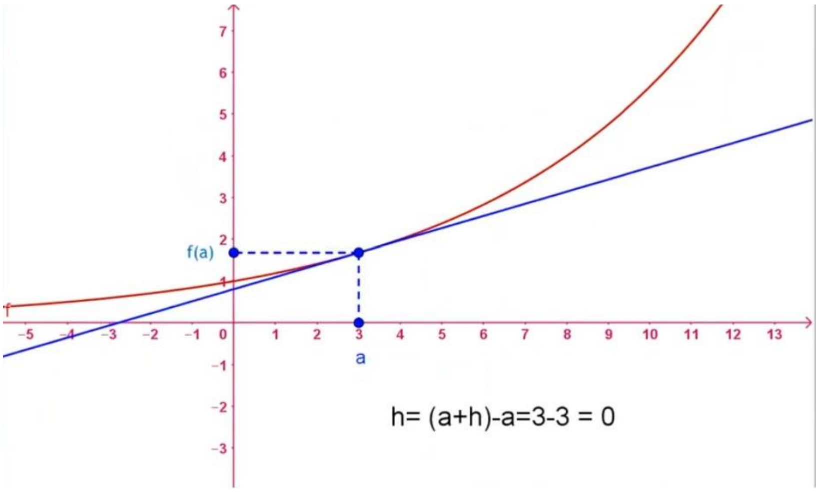
المشتقات و خط التماس

slope tangent line ميل المماس



$$m = \frac{f(h+a)-f(a)}{(h+a)-a} = \frac{f(h+a)-f(a)}{h}$$

$$m = \lim_{h\to 0}\frac{f(h+a)-f(a)}{h}$$



$$m = \frac{f(h+a)-f(a)}{(h+a)-a} = \frac{f(h+a)-f(a)}{h}$$

$$m = \lim_{h\to 0}\frac{f(h+a)-f(a)}{h}$$

DEFINITION 3.1.1

The slope m of the tangent line to the graph of a function f at $P(a, f(a))$ is

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided that the limit exists

معادلة المماس المار بنقطة التماس (a , f(a)) الذي ميله m هي
:

$$y = m(x-a) + f(a)$$

EXAMPLE 3.1.1

Find the equation of the tangent line to the graph of $f(x) = x^2$ at $x = 2$.

Solution

$$m = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$$

$$= \lim_{h\to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h\to 0} \frac{h(4+h)}{h} = \lim_{h\to 0}(4+h) = 4$$

m=4 , x=2

$$y = m(x-a) + f(a)$$
$$y = 4(x-2) + 4$$
$$y = 4x - 4$$

Definition 3.1.2

Let f be a function that is defined on an open interval containing a. The derivative of f at a, written $f'(a)$, is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists

EXAMPLE 3.1.2 Let $f(x) = x^3$. Find $f'(1)$

Solution Using Definition

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$

$$= \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3 + 3h + h^2)}{h} = \lim_{h\to 0}(3 + 3h + h^2) = 3$$

RELATED PROBLEM 2

Let $f(x) = 1 - x^2$. Find $f'(-3)$.

Answer 6

EXAMPLE 3.1.3 Let $f(x) = x^3$.

a. Find $f'(x)$, where x is any real number.

b. Find $f'(1), f'(2), f'(3)$.

Solution $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim(3x^2 + 3xh + h^2) = 3x^2$$

$$f'(1) = 3(1)^2 = 3$$

$$f'(2) = 3(2)^2 = 12$$

$$f'(3) = 3(3)^2 = 27$$

DEFINITION 3.1.3

The derivative of a function f is the function f' which value at any number x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists

$$y' = \frac{dy}{dx} = D_x y = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$

EXAMPLE 3.1.4 Let $f(x) = 2x^2 + 1$. Find

a. $f'(x)$

b. $f'(2), f'(-3)$

c. The slope of the tangent line to the graph of f at $x = 2$

d. The equation of the tangent line to the graph of f at $x = 2$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(2\left(x+h\right)^2 + 1\right) - \left(2x^2 + 1\right)}{h}$$

$$= \lim_{h\to 0} \frac{2\left(x^2 + 2xh + h^2\right) + 1 - 2x^2 - 1}{h} = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{2h\left(2x + h\right)}{h} = \lim_{h\to 0} 2\left(2x + h\right) = 4x$$

$$f'(2) = 4(2) = 8 \text{ and } f'(-3) = 4(-3) = -12$$

$$y = m(x - a) + f(a)$$
$$y = 8(x - 2) + 9$$
$$y = 8x - 7$$

RELATED PROBLEM 4 Let $f(x) = 3x^2 - 5x + 1$. Find

a. $f'(x)$

b. $f'(-1), f'(1)$

c. The slope of the tangent line to the graph of f at $x = -1$

d. The equation of the tangent line to the graph of f at $x = -1$

Answer

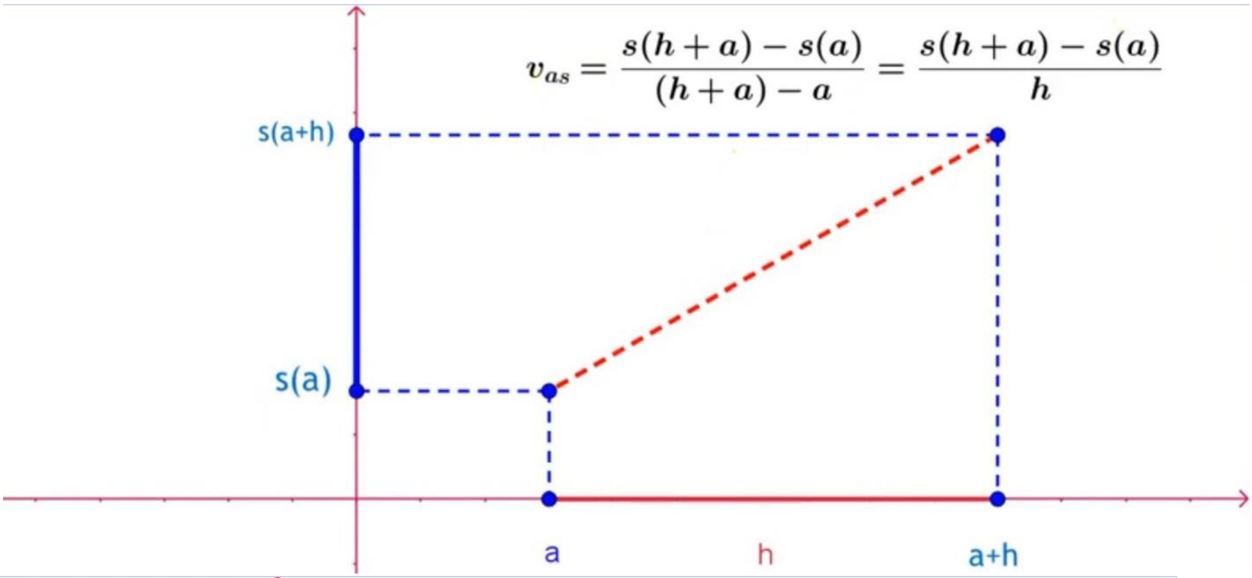
a. $f'(x) = 6x - 5$ b. $-11, 1$ c. -11 d. $y = -11x - 2$

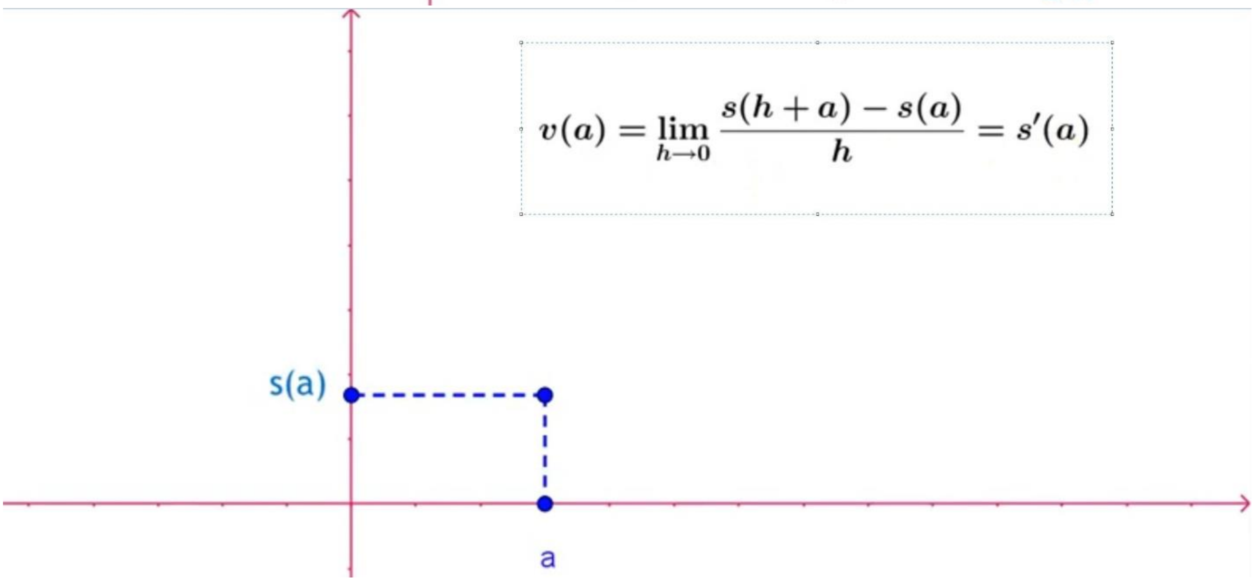
RELATED PROBLEM 7 Let $f(x) = \sqrt{1 - 3x}$. Find

a. $f'(x)$ b. $D_{f'}$ c. $f'(0)$

Answer a. $\dfrac{-3}{2\sqrt{1 - 3x}}$ b. $\left(-\infty, \dfrac{1}{3}\right)$ c. $-\dfrac{3}{2}$

120 كيلو متر في الساعة





EXAMPLE 3.1.10 The position of a moving object at time t is given by the function $s(t) = t^2 + t$, where s is measured in feet and t is measured in seconds.

s : القياس بالقدم   t : القياس بالثانية

a. Find the average velocity of the object over the interval $[1, 1.1]$.

متوسط السرعة

Solution

$$a = 1 \ , \quad h = 0.1$$

$$v_{av} = \frac{s(a+h) - s(a)}{h} = \frac{s(1+0.1) - s(1)}{0.1}$$

$$= \frac{\left((1.1)^2 + 1.1\right) - \left((1)^2 + 1\right)}{0.1} = \frac{2.31 - 2}{0.1} = \frac{0.31}{0.1} = 3.1\, ft/\text{sec}$$

b. Find the instantaneous velocity of the object at $t = 1$ and at $t = 3$

السرعة اللحظية

Solution

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \frac{\left((t+h)^2 + (t+h)\right) - \left(t^2 + t\right)}{h}$$

$$= \lim_{h \to 0} \frac{t^2 + 2th + h^2 + t + h - t^2 - t}{h} = \lim_{h \to 0} \frac{2th + h^2 + h}{h}$$

$$= \lim_{h \to 0} \frac{h(2t + h + 1)}{h} = \lim_{h \to 0} (2t + h + 1) = 2t + 1$$

at $t = 1$ is $v(1) = 2(1) + 1 = 3\, ft/\text{sec}$

at $t = 3$, it is $v(3) = 2(3) + 1 = 7\, ft/\text{sec}$

c. How much time it take for the object to reach instantaneous velocity of $99\, ft/\text{sec}$

Solution

$$2t + 1 = 99$$

$$t = \frac{98}{2} = 49\,\text{sec}$$

RELATED PROBLEM 10 The position of a moving object at time t is given by the function $s(t) = t^2 + 1$, where s is measured in feet and t is measured in seconds.

a. Find the average velocity of the object over the interval $[2, 2.2]$

b. Find the instantaneous velocity of the object at $t = 2$ and at $t = 5$.

c. How much time it take for the object to reach instantaneous velocity of $120\, ft/\text{sec}$

Answer a. $4.2\, ft/\text{sec}$ b. $4\, ft/\text{sec}, 10\, ft/\text{sec}$ c. $60\,\text{sec}$

33. A particle moves along the graph of the function $s(t) = 2t^3 + t + 1$, where s is measured in meters and t in seconds. Find the **instantaneous** velocity of the particle when $t = 2$.

Solution

$$v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h\to 0} \frac{2(t+h)^3 + (t+h) + 1 - (2t^3 + t + 1)}{h}$$

$$= \lim_{h\to 0} \frac{2(t^3 + 3t^2h + 3th^2 + h^3) + (t+h) + 1 - (2t^3 + t + 1)}{h}$$

$$= \lim_{h\to 0} \frac{2t^3 + 6t^2h + 6th^2 + 2h^3 + t + h + 1 - 2t^3 - t - 1}{h}$$

$$= \lim_{h\to 0} \frac{6t^2h + 6th^2 + 2h^3 + h}{h} = \lim_{h\to 0} \frac{h(6t^2 + 6th + 2h^2 + 1)}{h}$$

$$= \lim_{h\to 0} 6t^2 + 6th + 2h^2 + 1 = 6t^2 + 1$$

$$v(2) = 6\,.\,4 + 1 = 25 \text{ m/sec}$$

SECTION 3.2

DIFFERENTIATION RULES قواعد الاشتقاق

DIFFERENTIATION RULES - قواعد الاشتقاق

$\frac{d}{dx}(c) = 0$ $\qquad$ $\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(mx + c) = m$ $\qquad$ $\frac{d}{dx}(5x + 7) = 5$

$\frac{d}{dx}(x^n) = n\,x^{n-1}$ $\qquad$ $\frac{d}{dx}(x^5 + 4x^2 + 5) = 5\,x^4 + 8x$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{0.5}) = 0.5\,x^{-0.5} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$\frac{d}{dx}(\pi) = 0$ $\qquad$ $\frac{d}{dx}\left(\cos\left(\pi/3\right)\right) = 0$ $\qquad$ $\frac{d}{dx}\left(-\frac{1}{2}x + 3\right) = -\frac{1}{2}$

The derivative of the product of two functions is:

$$(f\,g)'(x) = f(x)\,g'(x) + f(x)'\,g(x)$$

$$f(x) = (x^3 - 3)\,(2 - x^5)$$

$$f'(x) = (x^3 - 3)\,.\,(-5x^4) + 3x^2\,.(2 - x^5)$$

$$f'(x) = -5x^7 + 15x^4 + 6x^2 - 3x^7$$

$$f'(x) = -8x^7 + 15x^4 + 6x^2$$

$$(f\,g\,h)'(x) = f(x)'\,g(x)\,h(x) + f(x)\,g(x)'\,h(x) + f(x)\,g(x)\,h(x)'$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x).\, g(x) - g(x)'.\, f(x)}{(g(x))^2}$$

$$\left(\frac{1}{x}\right)'(x) = \frac{0\times 1 - 1\times 1}{x^2} = \frac{-1}{x^2}$$

طريقة أخرى

$$\left(\frac{1}{x}\right)'(x) = \frac{d}{dx}(x^{-1}) = -1.\, x^{-2} = \frac{-1}{x^2}$$

Example:

$$f(x) = \frac{x^2+1}{x^3-1}$$

Solution

$$f'(x) = \frac{(2x)(x^3-1)-(3x^2)(x^2+1)}{(x^3-1)^2} = \frac{2x^4-2x-3x^4+2x^2}{(x^3-1)^2} = \frac{-x^4-2x+2x^2}{(x^3-1)^2}$$

Find the derivative of each of the following functions

a. $f(x) = \dfrac{-4}{\sqrt{x}-3}$ $\qquad f'(4)$

Solution

$$f'(x) = \frac{-(-4)(0.5\,x^{-0.5})}{(\sqrt{x}-3)^2} = \frac{2}{\sqrt{x}(\sqrt{x}-3)^2}$$

$$f'(4) = \frac{2}{\sqrt{4}(\sqrt{4}-3)^2} = 1$$

EXAMPLE 3.2.4 Suppose f and g are differentiable functions at $x = 2$ and that

$$f(2) = 3,\ f'(2) = -4,\ g(2) = 1 \text{ and } g'(2) = 2.$$

a. $\dfrac{d}{dx}\left(2f(x) - 3g(x) + 4x^2\right)\Big|_{x=2}$

Solution

$$\frac{d}{dx}\left(2f(x) - 3g(x) + 4x^2\right) = 2f'(x) - 3g'(x) + 8x$$

$$\frac{d}{dx}\left(2f(x) - 3g(x) + 4x^2\right)\Big|_{x=2} = 2f'(2) - 3g'(2) + 8(2) = 2(-4) - 3(2) + 16 = 2.$$

b. $\displaystyle \frac{d}{dx}\left(\frac{x+f(x)}{x-g(x)}\right)\Bigg|_{x=2}$

Solution

$$\frac{d}{dx}\left(\frac{x+f(x)}{x-g(x)}\right)=\frac{\big(x-g(x)\big)\big(1+f'(x)\big)-\big(x+f(x)\big)\big(1-g'(x)\big)}{\big(x-g(x)\big)^2}$$

$$\frac{d}{dx}\left(\frac{x+f(x)}{x-g(x)}\right)\Bigg|_{x=2}=\frac{\big(2-g(2)\big)\big(1+f'(2)\big)-\big(2+f(2)\big)\big(1-g'(2)\big)}{\big(2-g(2)\big)^2}$$

$$=\frac{(2-1)(1-4)-(2+3)(1-2)}{(2-1)^2}=2$$

c. $\displaystyle \frac{d}{dx}\big(f(x)g(x)\big)\Big|_{x=2}$

Solution

$$\frac{d}{dx}\big(f(x)g(x)\big)=f(x)g'(x)+g(x)f'(x)$$

$$\frac{d}{dx}\big(f(x)g(x)\big)\Big|_{x=2}=f(2)g'(2)+g(2)f'(2)=(3)(2)+(1)(-4)=2$$

RELATED PROBLEM 4

Suppose f and g are differentiable functions at $x=1$ and that

$$f(1)=-2,\ f'(1)=3,\ g(1)=-1 \text{ and } g'(1)=-4\,.$$

Find

a. $\displaystyle \frac{d}{dx}\left(3f(x)-2g(x)-4x^3\right)\Big|_{x=1}$

b. $\displaystyle \frac{d}{dx}\left(\frac{x^2-f(x)}{x-g(x)}\right)\Bigg|_{x=1}$

c. $\displaystyle \frac{d}{dx}\big(f(x)g(x)\big)\Big|_{x=1}$

Answers

a. 5

b. $-17/4$

c. 5

Example 3.2.5 Find the values of a and b such that $f(x)=\begin{cases} x^3 & x\le 1 \\ ax+b & x>1 \end{cases}$ is differentiable at $x=1$.

Solution Since f is differentiable at $x=1$, then it is continuous at $x=1$. This implies that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$$

$$\lim_{x\to 1^+}\left(ax+b\right) = \lim_{x\to 1^-} x^3$$

$$a+b=1$$

Since f is differentiable at $x=1$,

$$f'_-(1) = f'_+(1)$$

$$f'_-(1) = \frac{d}{dx}\left(x^3\right)\Big|_{x=1} = 3x^2\Big|_{x=1} = 3$$

$$f'_+(1) = \frac{d}{dx}\left(ax+b\right)\Big|_{x=1} = a$$

$$a=3$$

Since $a+b=1$, $b=1-a=1-3=-2$

EXAMPLE 3.2.6 Find an equation for the tangent line to the graph of $f(x)=\dfrac{x^2+3}{x+1}$ at $x=3$.

Solution

$$f'(x) = \frac{(x+1)(2x)-\left(x^2+3\right)(1)}{(x+1)^2} = \frac{2x^2+2x-x^2-3}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2}$$

$$m = f'(3) = \frac{3^2+2\times 3-3}{(3+1)^2} = \frac{12}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y = m(x-3)+f(3)$$

$$y = \left(\frac{3}{4}\right)(x-3)+3$$

$$y = \frac{3}{4}x+\frac{3}{4}$$

EXAMPLE 3.2.7 Find the x – coordinate of the point(s) at which the curve $y=x^4-4x^2+1$ has a horizontal tangent.

الميل الأفقي

Solution

$$\frac{dy}{dx} = 4x^3-8x$$

$$\frac{dy}{dx} = 0$$

$$4x^3-8x=0 \Rightarrow 4x(x^2-2)=0$$

$$\Rightarrow x=0 \text{ or } x^2=2$$

$$\Rightarrow x=0 \text{ or } x=\pm\sqrt{2}$$

RELATED PROBLEM 7 Find the $x-$coordinate of the point(s) at which the curve $y = x^3 - 4x^2 + x + 2$ has a horizontal tangent.

Answer $x = \dfrac{4-\sqrt{13}}{3}$, $x = \dfrac{4+\sqrt{13}}{3}$

SECTION 3.3 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS مشتقات الدوال المثلثية

a. $\dfrac{d}{dx}\left(\sin x\right) = \cos x$ b. $\dfrac{d}{dx}\left(\cos x\right) = -\sin x$

EXAMPLE 3.3.1 Find the derivative of each function

a. $y = 5\sin x - \dfrac{1}{4}\cos x$

Solution

$$\frac{dy}{dx} = 5\cos x + \frac{1}{4}\sin x$$

EXAMPLE 3.3.1 Find the derivative of each function

b. $y = \dfrac{3\cos x}{\sin x + 1}$

Solution

$$\frac{dy}{dx} = \frac{\left(\sin x + 1\right)\left(-3\sin x\right) - \left(3\cos x\right)\left(\cos x\right)}{\left(\sin x + 1\right)^2} = \frac{-3\sin^2 x - 3\sin x - 3\cos^2 x}{\left(\sin x + 1\right)^2}$$

$$= \frac{-3\left(\sin^2 x + \cos^2 x\right) - 3\sin x}{\left(\sin x + 1\right)^2} = \frac{-3(1 + \sin x)}{\left(\sin x + 1\right)^2} = \frac{-3}{\sin x + 1}$$

RELATED PROBLEM 1 Find the derivative of each function

a. $y = -4\cos x + \dfrac{1}{3}\sin x$ b. $y = \dfrac{-2\sin x}{\cos x + 2}$

Solution

a. $4\sin x + \dfrac{1}{3}\cos x$ b. $-\dfrac{2\left(2\cos x + 1\right))}{\left(\cos x + 2\right)^2}$

a. $\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

b. $\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$

c. $\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$

d. $\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$

EXAMPLE 3.3.2 Find the derivative of each function

a. $y = \sqrt{x} + \csc x - \cot x$

Solution

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \csc x \cot x + \csc^2 x$$

b. $y = \sin x \cos x$

Solution

$$\frac{dy}{dx} = \left(\sin x\right)\left(-\sin x\right) + \left(\cos x\right)\left(\cos x\right) = -\sin^2 x + \cos^2 x$$

c. $y = \frac{\sec x}{1 + \tan x}$

Solution

$$\frac{dy}{dx} = \frac{\left(1+\tan x\right)\left(\sec x \tan x\right) - \sec x\left(\sec^2 x\right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{\left(1+\tan x\right)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

$\sin^2 x + \cos^2 x = 1$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x - \sec^2 x = -1$

RELATED PROBLEM 2 Find the derivative of each function

a. $y = x^4 - 3\cot x + 2\cos x$

b. $y = x^3 \tan x$

c. $y = \frac{\csc x + 1}{\sin x}$

Solution

a. $4x^3 + 3\csc^2 x - 2\sin x$

b. $x^2\left(x\sec^2 x + 3\tan x\right)$

c. $-\frac{\cot x + \cos x\left(\csc x + 1\right)}{\sin^2 x}$

EXAMPLE 3.3.3 Find the equation of the tangent line to the graph of $y = 2\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

أوجد معادلة المماس للمنحنى عند النقطة

Solution

$$\frac{dy}{dx} = -2\sin x$$

$$m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -2\sin\left(\frac{\pi}{2}\right) = -2$$

$$y = m(x - a) + f(a)$$

$$y = (-2)\left(x - \frac{\pi}{2}\right) + 0$$

$$y = -2x + \pi$$

EXAMPLE 3.3.4 Find all point(s) on the graph of $f(x) = x + \sin x$, $0 \le x \le 2\pi$ where the tangent line is horizontal. أوجد جميع النقاط على المنحنى التي عندها المماس أفقيا.

Solution

$$f'(x) = 1 + \cos x$$

$$1 + \cos x = 0$$

$$\cos x = -1$$

$$x = \pi$$

point $(\pi, f(\pi)) = (\pi, \pi)$

EXAMPLE 3.3.5 Assume that a particle's position on the $x - axis$ is given by

$$s(t) = 3\cos t + 4\sin t$$

Where s is measured in meters and t is measured in seconds. Find the particle's instantaneous velocity when $t = 0$ and $t = \frac{\pi}{2}$

السرعة اللحظية

Solution

$$v(t) = \frac{ds}{dt} = -3\sin t + 4\cos t$$

$$v(0) = -3\sin(0) + 4\cos(0) = -3(0) + 4(1) = 4\, m/\text{sec}$$

$$v\left(\frac{\pi}{2}\right) = -3\sin\left(\frac{\pi}{2}\right) + 4\cos\left(\frac{\pi}{2}\right) = -3(1) + 4(0) = -3\, m/\text{sec}$$

RELATED PROBLEM 7 Find the x −coordinate of the point(s) at which the curve $y = x^3 - 4x^2 + x + 2$ has a horizontal tangent.

Answer $x = \frac{4 - \sqrt{13}}{3}$, $x = \frac{4 + \sqrt{13}}{3}$

SECTION 3.4 THE CHAIN RULE قاعدة السلسلة

$f(x) = x^5$ $\qquad f'(x) = 5x^4$

$$y = \left(x^3 + 2x\right)^5 \qquad \frac{dy}{dx} = 5(x^3+2x)^4\,(3x^2+2)$$

THEOREM 3.4.2 (The General Power Rule)

If g is a differentiable function and r is any rational number, then

$$\frac{d}{dx}\left(\left(g(x)\right)^r\right) = r\left(g(x)\right)^{r-1}\frac{d}{dx}\left(g(x)\right)$$

EXAMPLE 3.4.2 Differentiate the following functions

$$f(x) = \frac{1}{\sqrt[3]{3x^5 - 2x^3 + 1}}$$

Solution

$$f(x) = \frac{1}{\left(3x^5 - 2x^3 + 1\right)^{\frac{1}{3}}} = \left(3x^5 - 2x^3 + 1\right)^{-\frac{1}{3}}.$$

$$f'(x) = -\frac{1}{3}\left(3x^5 - 2x^3 + 1\right)^{-\frac{4}{3}}\left(15x^4 - 6x^2\right)$$

$$= -\frac{15x^4 - 6x^2}{3\sqrt[3]{\left(3x^5 - 2x^3 + 1\right)^4}} = -\frac{5x^4 - 2x^2}{\sqrt[3]{\left(3x^5 - 2x^3 + 1\right)^4}}$$

EXAMPLE 3.4.3 Differentiate the following functions

$$\text{a.} \quad h(z) = \left(\frac{2z+1}{3z-2}\right)^4$$

Solution

$$h'(z) = 4\left(\frac{2z+1}{3z-2}\right)^3\left(\frac{(3z-2)(2) - (2z+1)(3)}{(3z-2)^2}\right) = 4\left(\frac{2z+1}{3z-2}\right)^3\left(\frac{6z - 4 - 6z - 3}{(3z-2)^2}\right)$$

$$= 4\left(\frac{2z+1}{3z-2}\right)^3\left(\frac{-7}{(3z-2)^2}\right) = -28\frac{(2z+1)^3}{(3z-2)^5}$$

EXAMPLE 3.4.3 Differentiate the following functions

b. $k(t) = (t+1)^3 (2t-3)^5$

Solution

$$k'(t) = (t+1)^3 \left[5(2t-3)^4 (2)\right] + (2t-3)^5 \left[3(t+1)^2 (1)\right]$$

$$= 10\underline{(t+1)^3}\,\underline{(2t-3)^4} + 3\underline{(2t-3)^5}\,\underline{(t+1)^2}$$

$$= (t+1)^2 (2t-3)^4 \text{(10 (t+1) + 3(2t-3))}$$

$$= (t+1)^2 (2t-3)^4 \left[10t + 10 + 6t - 9\right]$$

$$= (t+1)^2 (2t-3)^4 (16t+1)$$

EXAMPLE 3.4.5 If $y = 5\sin(x^4)$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 5\cos(x^4)(4x^3) = 20x^3\cos(x^4)$$

EXAMPLE 3.4.6 Find

a. $\frac{d}{dx}\left(\tan(3x^2)\right)$

Solution

$$\frac{d}{dx}\left(\tan(3x^2)\right) = 6x\sec^2(3x^2)$$

b. $\frac{d}{dx}\left(\sqrt{x^2\csc(2x)}\right)$

Solution

$$\frac{d}{dx}\left(\sqrt{x^2\csc(2x)}\right) = \frac{d}{dx}(x^2\csc(2x))^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(x^2\underline{\csc(2x)}\right)^{-\frac{1}{2}}\left(x^2\left(\underline{-\csc(2x)\cot(2x)(2)}\right) + \underline{\csc(2x)}(2x)\right)$$

$$= \frac{-2x^2\csc(2x)\cot(2x) + 2x\csc(2x)}{2\sqrt{x^2\csc(2x)}}$$

c. $\frac{d}{dt}\left(\sec^3\left(t^4\right)\right)$

Solution

$$\frac{d}{dt}\left(\sec^3\left(t^4\right)\right) = 3\sec^2\left(t^4\right)\sec\left(t^4\right)\tan\left(t^4\right)\left(4t^3\right)$$
$$= 12t^3\sec^3\left(t^4\right)\tan\left(t^4\right)$$

RELATED PROBLEM 6 Find

a. $\frac{d}{dx}\left(\csc\left(x^3\right)\right)$ b. $\frac{d}{dx}\left(\sqrt[3]{x^2+\cot(2x)}\right)$ c. $\frac{d}{dt}\left(\cot^5(t^2)\right)$

Answer

a. $-3x^2\csc(x^3)\cot(x^3)$ b. $\frac{2x-2\csc^2\left(2x\right)}{3\sqrt[3]{\left(x^2+\cot(2x)\right)^2}}$ c. $-10t\cot^4(t^2)\csc^2(t^2)$

EXAMPLE 3.4.7 Find the slope of the tangent line to the graph of $y = x^2\cos(3x)$ at $x=\pi$

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Solution

$$\frac{dy}{dx} = x^2\,[\,-3\sin(3x)\,] + 2x\cos(3x)$$

$$\frac{dy}{dx} = -3x^2\sin\left(3x\right) + 2x\cos\left(3x\right)$$

$$m = \left.\frac{dy}{dx}\right|_{x=\pi} = -3\pi^2\sin\left(3\pi\right) + 2\pi\cos\left(3\pi\right) = -2\pi$$

EXAMPLE 3.4.8 The equation of motion of a particle is given by $s(t) = 10 + \frac{1}{4}\sin\left(10\pi t\right)$, where t is measured in seconds and s in centimeters. Find the velocity of the particle at time t.

مطلوب سرعة الجسيم على الزمن t

Solution

$$v(t) = \frac{ds}{dt} = \frac{1}{4}\left(10\pi\right)\cos\left(10\pi t\right) = \frac{5\pi}{2}\cos\left(10\pi t\right)$$

قاعدة السلسلة Chain Rule

إذا كانت $y = f(t)$ و $t = g(s)$ و $s = g(u)$ و $u = g(x)$ أي أن المتغير y يعتمد علي المتغير x عن طريق سلسلة من المتغيرات مثل t و s و u.

$$y \to t \to s \to u \to x.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{ds} \cdot \frac{ds}{du} \cdot \frac{du}{dx}$$

مثال: أوجد المشتقة الأولي $\frac{dy}{dx}$ للدوال التالية

1. $y = t^4 - 2t^2 + 1; \qquad t = 2x^2 + x + 1,$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = (4\,t^3 - 4\,t)(4x + 1),$$

$$= \left(4\,(2x^2 + x + 1)^3 - 4\,(2x^2 + x + 1)\right)(4x + 1).$$

2. $y = \frac{1}{t+1}; \qquad t = 3x + 1,$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{1}{(t+1)^2} = \frac{-3}{\left((3x+1)+1\right)^2}.$$

3. $y = t^2 + 1; \qquad t = 3s^3 + 1; \qquad s = 2 - x,$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{ds} \cdot \frac{ds}{dx} = (2t)(9s^2)(-1),$$

$$= (2(3s^3 + 1))(9s^2)(-1),$$

$$= (6s^3 + 2)(9s^2)(-1),$$

$$= -54s^5 - 18s^2,$$

$$= -54(2 - x)^5 - 18(2 - x)^2.$$

Examples:

1) $y = \sin(u)$, $u = x^2$ $Find\ \frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = \cos(u)$$

$$\frac{du}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \cos(u).2x = \cos x^2 . 2x$$

2) $y = 3z^3 + 2,$ $x = z^2 + 4$ $find\ \frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dz} = 9z^2$$

$$\frac{dz}{dx} = ?? \Rightarrow \frac{dx}{dz} = 2z \Rightarrow \frac{dz}{dx} = \frac{1}{2z}$$

$$\therefore \frac{dy}{dx} = 9z^2 . \frac{1}{2z} = \frac{9z}{2}$$

$$x = z^2 + 4$$

$$z^2 = x - 4$$

$$z = \sqrt{x - 4}$$

$$= \frac{9}{2}\sqrt{x - 4}$$

3) $f'(9) = 5,$ $g(2) = 9,$ $g'(2) = -3,$ $find\ (fog)'(2).$

Solution:

$$(fog)'(x) = f'\big(g(x)\big).g'(x)$$

$$(fog)'(2) = f'\big(g(2)\big).g'(2)$$

$$(fog)'(2) = f'(9).(-3) = 5.(-3) = -15$$

4) $f(x) = \sin(x^3)$.

Solution:

$$Note: \ sin'(x) = \cos(x) . 1$$

$$f'(x) = \cos(x^3) . 3x^2$$

5) $f(x) = \cos(\ln x)$.

Solution:

$$f'(x) = -sin(\ln x) . \frac{1}{x}$$

6) $f(x) = tan(\sin(x^2))$.

Solution:

$$f'(x) = sec^2(\sin(x^2)) . \cos(x^2) . (2x)$$

7) $f(2x) = 3x^2 + 8x$ $\qquad$ $find\ f'(x)$.

Solution:

$$f'(2x). (2) = 6x + 8$$

$$f'(2) \,?? \implies 2x = 2 \implies x = 1$$

$$f'(2) = 6\ (1) + 8$$

$$f'(2). (2) = 14$$

$$f'(2) = \frac{14}{2} = 7$$

8) $\dfrac{d\, f(3x)}{dx} = 6x + 1$ $\qquad$ $find\ \dfrac{d\, f(x)}{dx}$.

Solution:

$$f'(3x). (3) = 6x + 1$$

$$let\ 3x = y \implies x = \frac{y}{3}$$

$$f'(y). (3) = 6\ (\frac{y}{3}) + 1$$

$$f'(y). (3) = 2y + 1$$

$$f'(y) = \frac{2y + 1}{3}$$

9) $f(x) = \sin x \qquad g(x) = x^2 \qquad find\ (fog)^{'}(x).$

Solution:

$$f^{'}(x) = cox\ (x), \qquad g^{'}(x) = 2x$$

$$(fog)^{'}(x) = f^{'}\big(g(x)\big).\, g^{'}(x)$$

$$(fog)^{'}(x) = f^{'}(x^2).\,(2x)$$

$$(fog)^{'}(x) = \cos(x^2).\,(2x)$$

10) $f(x) = \sin^3 x$

Solution:

$$f(x) = (sin\, x)^3$$

$$f^{'}(x) = 3\ (sin\, x)^2.\cos x$$

11) $f(x) = tan^2(\sin(x^2))$

Solution:

$$f(x) = (\tan(\sin(x^2)))^2$$

$$f^{'}(x) = 2\ (\tan(\sin(x^2)))^1 .\, sec^2(\sin x^2).\cos x^2 .\,(2x)$$

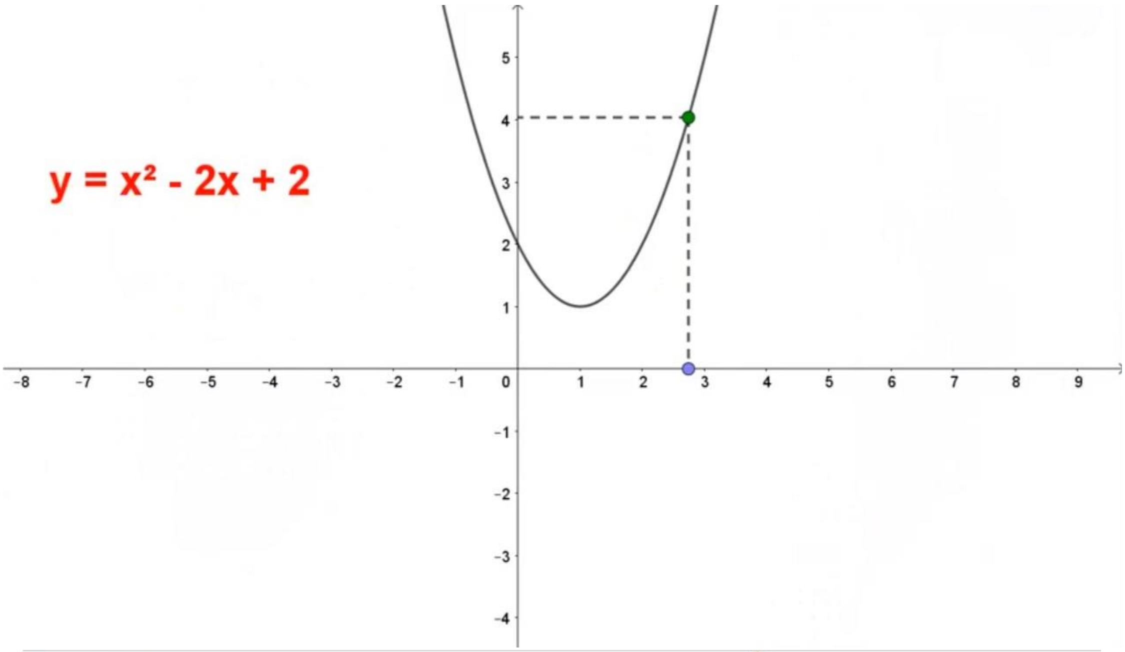
12) $f(x) = sin\ (cos^2(\ln x))$

Solution:

$$f(x) = sin\ (\cos(\ln x))^2$$

$$f^{'}(x) = \cos(\cos(\ln x))^2 .\, 2(\cos\ (\ln x))^1. -\sin(\ln x)\, .\frac{1}{x}$$

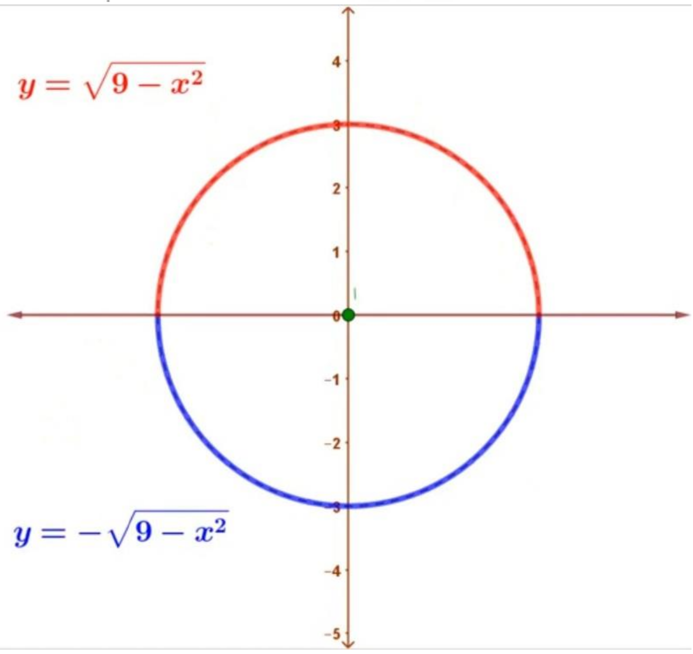
SECTION 3.5 IMPLICIT DIFFERENTIATION المشتقات الضمني



$$x^2 + y^2 = 9$$

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$



EXAMPLE 3.5.1 المشتقة الضمني

If each of the following equations determines an implicit differentiable function

$y = f(x)$, find $\boldsymbol{y'}$ a. $x^3 + y^3 = 1 + xy$

Solution

$$3x^2 + 3y^2y' = xy' + y$$

$$3y^2y' - xy' = y - 3x^2$$

$$\left(3y^2 - x\right)y' = y - 3x^2$$

$$y' = \frac{y - 3x^2}{3y^2 - x}$$

b. $y^2 = x\cos y$

Solution

$$2yy' = -x\sin y \cdot y' + \cos y$$

$$2yy' + x\sin y y' = \cos y$$

$$\left(2y + x\sin y\right)y' = \cos y$$

$$y' = \frac{\cos y}{2y + x\sin y}$$

c. $xy^{2/3} + yx^{2/3} = x^2$

Solution

$$x\left(\frac{2}{3}y^{-1/3}y'\right) + y^{2/3}(1) + y\left(\frac{2}{3}x^{-1/3}\right) + x^{2/3}(y') = 2x$$

$$\frac{2}{3}xy^{-1/3}y' + y^{2/3} + \frac{2}{3}yx^{-1/3} + x^{2/3}y' = 2x$$

$$\left(\frac{2}{3}xy^{-1/3} + x^{2/3}\right)y' = 2x - \frac{2}{3}yx^{-1/3} - y^{2/3}$$

$$y' = \frac{2x - \frac{2}{3}yx^{-1/3} - y^{2/3}}{\frac{2}{3}xy^{-1/3} + x^{2/3}}$$

EXAMPLE 3.5.1

المشتقة الضمني

If each of the following equations determines an implicit differentiable function $y = f(x)$, find y' **d.** $\sqrt{3+\tan(xy)} - 2 = 0$

Solution

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\left(\sec^2(xy)\left(xy'+y\right)\right) = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\left(\left(x\sec^2(xy)y' + y\sec^2(xy)\right)\right) = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2} x\sec^2(xy)y' + \frac{1}{2}\left(3+\tan(xy)\right)^{-1/2} y\sec^2(xy) = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2} x\sec^2(xy)y' = -\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2} y\sec^2(xy)$$

$$y' = \frac{-\frac{1}{2}y\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2}}{\frac{1}{2}x\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2}} = -\frac{y}{x}$$

RELATED PROBLEM 1 Find $\frac{dy}{dx}$ for each of the following

a. $8x^2 + y^2 = 10$

b. $\sin^2(3y) = x + y - 1$

c. $3xy = \left(x^3 + y^2\right)^{3/2}$

d. $\sqrt{1 + \sin^3\left(xy^2\right)} = y$

Answers

a. $\frac{-8x}{y}$

b. $\frac{1}{6\sin(3y)\cos(3y) - 1}$

c. $\frac{\frac{3}{2}x^2\left(x^3 + y^2\right)^{1/2} - y}{x - y\left(x^3 + y^2\right)^{1/2}}$

d. $\frac{3y^2\sin^2\left(xy^2\right)\cos\left(xy^2\right)}{2y - 6xy\sin^2\left(xy^2\right)\cos\left(xy^2\right)}$

EXAMPLE 3.5.2 Find an equation of the tangent line to the curve $y^3 + yx^2 + x^2 - 3y^2 = 0$ at the point $P\left(0,3\right)$. أوجد معادلة المماس للمنحنى عند النقطة

Solution

$$3y^2\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + 2x - 6y\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} - 6y\frac{dy}{dx} = -2xy - 2x$$

$$\left(3y^2 + x^2 - 6y\right)\frac{dy}{dx} = -2x(y+1)$$

$$\frac{dy}{dx} = -\frac{2x(y+1)}{3y^2 + x^2 - 6y}$$

$$m = \left.\frac{dy}{dx}\right|_{(0,3)} = -\frac{2(0)(3+1)}{3(3)^2 + (0)^2 - 6(3)} = 0$$

$$y = m(x - a) + f(a)$$

$$y = 0\left(x - 0\right) + 3 \text{ or } y = 3$$

EXAMPLE 3.5.3 Given that $x\csc y = 2$, find $\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\frac{\pi}{6}\right)}$

Solution

$$x\left(-\csc y\cot y\frac{dy}{dx}\right) + \csc y\left(1\right) = 0$$

$$-x\csc y\cot y\frac{dy}{dx} + \csc y = 0$$

$$\frac{dy}{dx} = \frac{\csc y}{x\csc y\cot y} = \frac{1}{x\cot y}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\frac{\pi}{6}\right)} = \frac{1}{\left(1\right)\cot\left(\frac{\pi}{6}\right)} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

RELATED PROBLEM 2 Find an equation of the tangent line to the curve $x^2+(y-x)^3=9$ at the point $P(1,3)$

Answer $y=\frac{5}{6}x+\frac{13}{6}$.

RELATED PROBLEM 3 Given that $x^2\cos y+y^2-1=0$, find $\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)}$.

Answer 0.

EXAMPLE 3.5.4 Find all points (x,y) on the graph of $x^{2/3}+y^{2/3}=8$ where tangent to the graph at (x,y) have slope -1. أوجد جميع النقاط التي يمر بها المماس الذي ميله 1-

Solution

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

$$y'=-\frac{y^{1/3}}{x^{1/3}}$$

slope -1, set $y'=-1$

$$-\frac{y^{1/3}}{x^{1/3}}=-1\Rightarrow y^{1/3}=x^{1/3}\Rightarrow y=x$$

$$x^{2/3}+y^{2/3}=8$$

$$x^{2/3}+x^{2/3}=8\Rightarrow 2x^{2/3}=8\Rightarrow x^{2/3}=4$$

$$\left(x^{2/3}\right)^3=(4)^3\Rightarrow x^2=64\Rightarrow x=\pm 8$$

If $x=8$, then $y=8$, point $(8,8)$

If $x=-8$, then $y=-8$, point $(-8,-8)$

RELATED PROBLEM 4 Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3-xy+y^3=0$ is horizontal.

Answer $\left(\frac{\sqrt[3]{2}}{3},\frac{\sqrt[3]{4}}{3}\right)$.

SECTION 3.6 HIGHER ORDER DERIVATIVES مشتقات ذات الرتب العليا

HIGHER ORDER DERIVATIVES مشتقات ذات الرتب العليا

$$y = x^6$$

المشتقة الأولى $\quad y' = 6\,x^5$

المشتقة الثانية $\quad y'' = 30\,x^4$

المشتقة الثالثة $\quad y''' = 120\,x^3$

المشتقة الرابعة $\quad y'''' = 360\,x^2$

EXAMPLE 3.6.1 Find each of the following

a. $f''(x)$ for $f(x) = x^5 + 2x^3 - 2x$

Solution

$$f'(x) = 5x^4 + 6x^2 - 2$$

$$f''(x) = 20x^3 + 12x\,.$$

b. $\dfrac{d^5y}{dx^5}$ for $y = x^7 - 2x^4 + 6x^2 - 12$

Solution

$$\frac{dy}{dx} = 7x^6 - 8x^3 + 12x$$

$$\frac{d^2y}{dx^2} = 42x^5 - 24x^2 + 12$$

$$\frac{d^3y}{dx^3} = 210x^4 - 48x$$

$$\frac{d^4y}{dx^4} = 840x^3 - 48$$

$$\frac{d^5y}{dx^5} = 2520x^2.$$

RELATED PROBLEM 1 Find each of the following

a. $f''(x)$ for $f(x) = 2x^4 - \dfrac{3}{\sqrt{x}}$.

b. $\dfrac{d^3f}{dt^3}$ for $f(t) = 4t^2 - 12 + \dfrac{4}{t^2}$.

Answer

a. $24x^2 - \dfrac{9}{4\sqrt{x^5}}$.

b. $-\dfrac{96}{t^5}$.

EXAMPLE 3.6.2 Given that $f(x) = \sin\left(x^2\right)$. Find $f''(x)$.

Solution

$$f'(x) = 2x\cos\left(x^2\right)$$

$$f''(x) = -4x^2\sin\left(x^2\right) + 2\cos\left(x^2\right)$$

RELATED PROBLEM 2 Given that $f(x) = \cos\left(x^3\right)$. Find $f''(x)$.

Answer $f''(x) = -\left(9x^4\cos\left(x^3\right) + 6x\sin\left(x^3\right)\right)$

EXAMPLE 3.6.3 Find y'' if $y + \sin y = x$ دالة ضمنية

Solution

$$y' + \left(\cos y\right)y' = 1$$

$$y' = \frac{1}{1+\cos y}$$

$$y'' = \frac{-(-\sin y)y'}{\left(1+\cos y\right)^2} = \frac{(\sin y)y'}{\left(1+\cos y\right)^2}$$

$$y'' = \frac{\sin y\left(\dfrac{1}{1+\cos y}\right)}{\left(1+\cos y\right)^2} = \frac{\sin y}{\left(1+\cos y\right)^3}$$

RELATED PROBLEM 3 Find y'' if $x^3y^3 - 4 = 0$.

Answer $y'' = \dfrac{2y}{x^2}$

EXAMPLE 3.6.4 Find all values of k such that $y = x^k$ satisfy the equation $3x^2y'' + 4xy' - 2y = 0$.

Solution

$$3x^2y'' + 4xy' - 2y = 0$$

$$y = x^k$$

$$y' = kx^{k-1}$$

$$y'' = k(k-1)x^{k-2}$$

$$3x^2k(k-1)x^{k-2} + 4kxx^{k-1} - 2x^k = 0$$

$$3k(k-1)x^k + 4kx^k - 2x^k = 0$$

$$x^k[3k(k-1) + 4k - 2] = 0$$

$$3k(k-1) + 4k - 2 = 0$$

$$3k^2 + k - 2 = 0$$

$$(3k-2)(k+1) = 0$$

$$k = \frac{2}{3}, k = -1$$

EXAMPLE 3.6.5 If $f(x) = x^4 - x^3 - 6x^2 + 7x$, find an equation of the tangent line to the graph of f' at the point $P(2,3)$.

عين معادلة المماس لمنحنى الدالة f'

Solution

$$f'(x) = 4x^3 - 3x^2 - 12x + 7$$

$$f''(x) = 12x^2 - 6x - 12$$

$$m = f''(2) = 12(2)^2 - 6(2) - 12 = 24$$

$$y = m(x-a) + f(a)$$

$$y = 24(x-2) + 3$$

$$y = 24x - 45$$

ACCELERATION (العجلة (التسارع

المسافة

$$s = f(t)$$

متوسط السرعة

$$v(t) = f'(t) = \frac{ds}{dt}$$

العجلة (التسارع)

$$a(t) = \frac{dv}{dt} = f''(t) = \frac{d^2s}{dt^2}$$

EXAMPLE 3.6.7 The position of a particle is given by the equation
وضع جسيم معادلة

$$s(t) = 4t^3 - 9t^2 + 6t + 2\,,$$

قياس
where s is measured in meters and t in seconds.

a. What are $v(t)$ and $a(t)$, the velocity and acceleration of the particle, at time t?
السرعة التسارع

Solution

$$v(t) = \frac{ds}{dt} = 12t^2 - 18t + 6$$

$$a(t) = \frac{d^2s}{dt^2} = 24t - 18$$

b. What is the velocity of the particle after 2 seconds?

$$v(2) = 12(2)^2 - 18(2) + 6 = 18\, m/\text{sec}$$

c. When is the particle at rest?

Solution
متى يتوقف الجسيم

$$v(t) = 0$$

$$12t^2 - 18t + 6 = 0$$

$$6\left(2t^2 - 3t + 1\right) = 0$$

$$6\left(2t - 1\right)\left(t - 1\right) = 0$$

$$t = \frac{1}{2} \text{ or } t = 1$$

d. What is the acceleration of the particle after 3 seconds?

Solution

$$a(t) = \frac{d^2s}{dt^2} = 24t - 18$$

$$a(3) = 24(3) - 18 = 54\, m/\text{sec}^2$$

e. When is the acceleration of the particle positive?

Solution

$$a(t) > 0$$

$$24t - 18 = 6(4t - 3) > 0$$

$$t > \frac{3}{4}.$$

RELATED PROBLEM 6 If $s(t) = \frac{1}{2}t^4 - 5t^3 + 12t^2$, where s is measured in meters and t in seconds. Find the velocity of the moving object when its acceleration is $0\ m/\text{sec}^2$.

Answer $v(1) = 11\ m\ /\ \text{sec}\ ,\ v(4) = -16\ m\ /\ \text{sec}$.

SECTION 3.7

THE DERIVATIVE OF INVERSE FUNCTIONS مشتقة معكوس دالة

معكوس دالة

$$f(x) = y \longrightarrow f^{-1}(y) = x$$

$$f(f^{-1}(y)) = y$$

$$f(y) = x \longrightarrow f^{-1}(x) = y$$

$$f(f^{-1}(x)) = x$$

$$f(s) = t \longrightarrow f^{-1}(t) = s$$

$$f(f^{-1}(t)) = t$$

$$\frac{2}{3} = \frac{1}{\frac{3}{2}} \longrightarrow \frac{dy}{dx} = \frac{1}{dx/dy}$$

EXAMPLE 3.7.1 Let $f(x) = x^3 + 4x - 1$ Find the derivative of f^{-1}

Solution

let $y = f^{-1}(x) \Rightarrow x = f(y)$

$$x = f(y) = y^3 + 4y - 1$$

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(y^3 + 4y - 1\right)$$

$$1 = 3y^2\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$1 = \left(3y^2 + 4\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 + 4}$$

$$(f^{-1})'(x) = \frac{1}{3y^2 + 4}$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS المشتقات العكسية لدوال المثلثية المثلثية

THEOREM 3.7.2 (Derivatives of Inverse Trigonometric Functions)

a. $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$

b. $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$

c. $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

d. $\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$

e. $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}},\ |x| > 1$

f. $\frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}},\ |x| > 1$

EXAMPLE 3.7.2 Find the derivative of each of the following functions

$$f(x) = \sin^{-1}(5x)$$

Solution

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{d}{dx}\left(\sin^{-1}(5x)\right) = \frac{5}{\sqrt{1-25x^2}}$$

EXAMPLE 3.7.2 Find the derivative of each of the following functions

$$f(x) = \tan^{-1}\left(\sqrt{x+1}\right)$$

Solution

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{d}{dx}\left(\tan^{-1}\left(\sqrt{x+1}\right)\right) = \frac{1}{1+x+1}\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}(1)\right)$$

$$= \frac{1}{(x+2)}\frac{1}{2\sqrt{x+1}} = \frac{1}{2(x+2)\sqrt{x+1}}$$

EXAMPLE 3.7.2 Find the derivative of each of the following functions

$$f(t) = t^2 \sec^{-1}(2t)$$

Solution

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}},$$

(1) $$\frac{d}{dt}\left(\sec^{-1}(2t)\right) = \frac{2}{(2t)\sqrt{4t^2-1}} = \frac{1}{t\cdot\sqrt{4t^2-1}}$$

(2) $$f'(t) = t^2\left(\frac{1}{t\cdot\sqrt{4t^2-1}}\right) + \sec^{-1}(2t)(2t) = \frac{t}{\sqrt{4t^2-1}} + 2t\sec^{-1}(2t)$$

EXAMPLE 3.7.2 Find the derivative of each of the following functions

$$f(t) = \sin\left(\cos^{-1} t\right)$$

Solution

$$f'(t) = \cos\left(\cos^{-1} t\right)\left(-\frac{1}{\sqrt{1-t^2}}\right) = t\left(-\frac{1}{\sqrt{1-t^2}}\right) = -\frac{t}{\sqrt{1-t^2}}$$

$$f(f^{-1}(x)) = x$$

EXAMPLE 3.7.3 If $y = \left(1+\cos^{-1}(3x)\right)^3$, find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = 3\left(1+\cos^{-1}(3x)\right)^2\left(\frac{-3}{\sqrt{1-9x^2}}\right)$$

$$= \frac{-9\left(1+\cos^{-1}(3x)\right)^2}{\sqrt{1-9x^2}}$$

EXAMPLE 3.7.4 Find an equation of the tangent line to the graph of the curve $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$ at the point $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Solution

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-y^2}}} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} = -\frac{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}}{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}} = -1$$

$$y = m(x-a) + f(a)$$

$$y = -1\left(x - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}$$

$$y = -x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$y = -x + \sqrt{2}$$