CHAPTER 4

CHAPTER
APPLICATIONS OF DIFFERENTIATION
تطبيقات على المشتقات

4.1 Extrema of Functions.

- Absolute Extrema.
  - Absolute Minimum.
  - Absolute Maximum.
- Critical Points.
- Finding Absolute Extrema on a Closed Interval.

4.2 The Mean Value Theorem.

- Rolle’s Theorem.
- The Mean Value Theorem.

4.3 Increasing and Decreasing Functions.

- Increasing and Decreasing Test.
- Finding a Local Extrema of a Function.

4.4 Concavity.

- Concavity.

SECTION 4.1

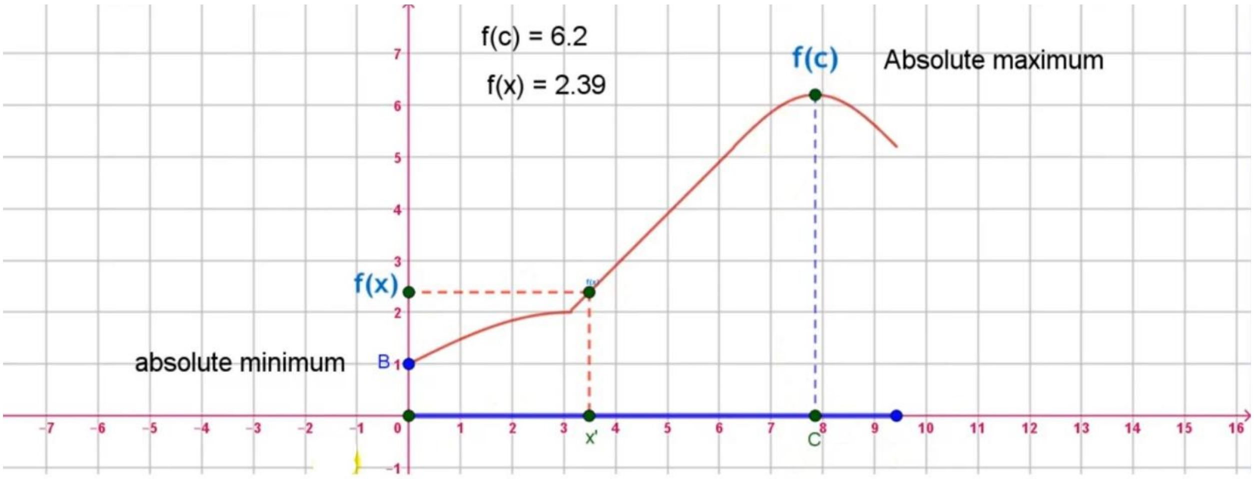
EXTREMA OF FUNCTIONS القيمة القصوى لدالة

ABSOLUTE EXTREMA الحدود القصوى

ABSOLUTE EXTREMA

absolute minimum
الحد الأسفل

absolute maximum
الحد الأعلى



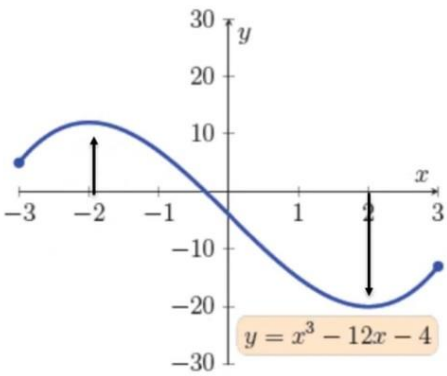
DEFINITION 4.1.1

Let f be a function defined on an interval I containing c, then

a. $f(c)$ is the maximum value of f on I if $f(c) \geq f(x)$ for all x in I.

b. $f(c)$ is the minimum value of f on I if $f(c) \leq f(x)$ for all x in I.

EXAMPLE 4.1.1 Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 12x - 4$. Find the absolute maximum and minimum of f on the interval $[-3, 3]$ using Figure 4.1.2.



Solution : f(2)= 8 – 24 – 4 = -20 **absolute minimum**

f(-2)= -8 + 24 – 4 = 12 **absolute maximum**

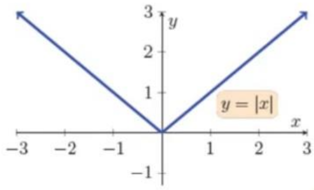
EXAMPLE 4.1.2 Find the absolute maximum and minimum of the following functions:

a. $f(x) = |x|$ **b.** $g(x) = 2x^5$

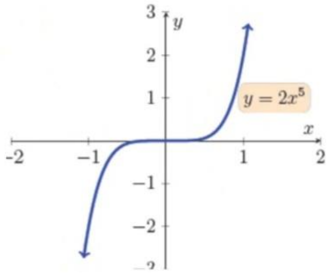
Solution

a. Let x=0 absolute minimum = 0

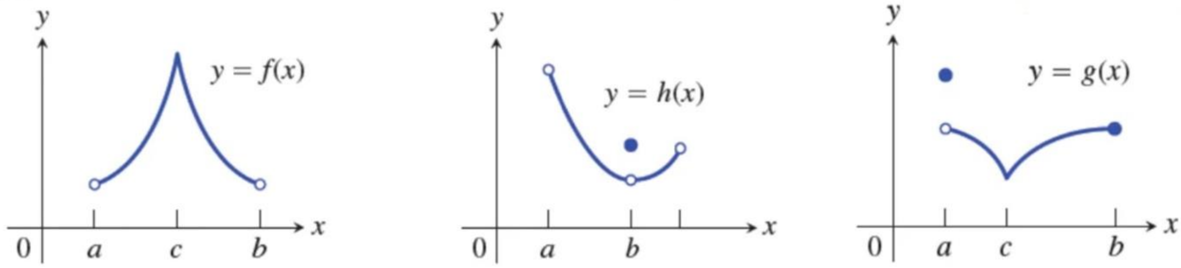
there is no absolute maximum



b. does not have extreme values.



EXAMPLE 4.1.3 In parts (a) – (c), identify the x-value at which any absolute extreme value occurs.



(a) (b) (c)

Solution

a. There is an absolute maximum at $x = c$

there is no absolute minimum

b. There are no absolute extreme values.

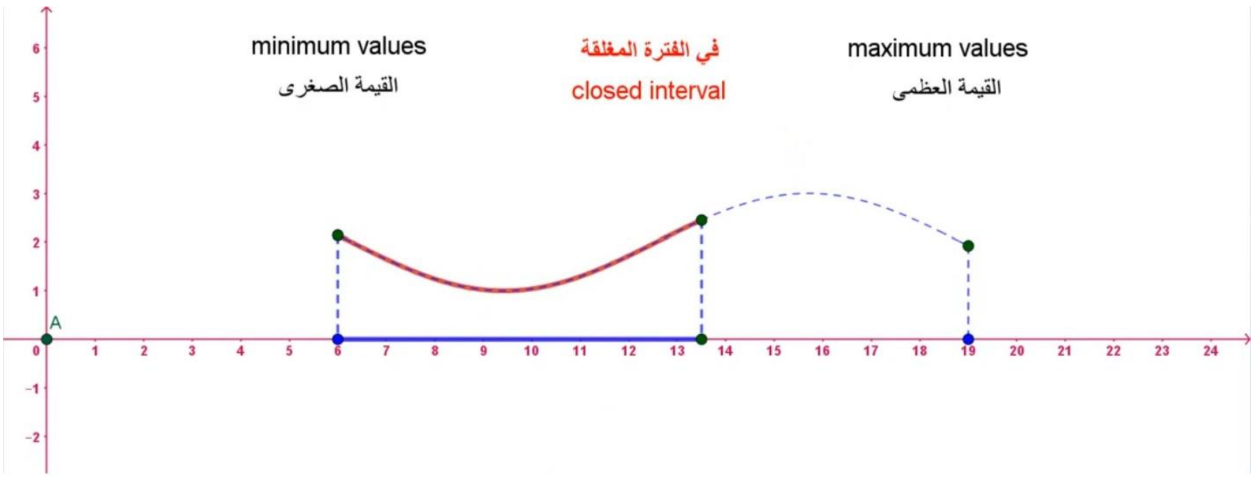
function is not continuous.

c. There is an absolute minimum at $x = c$

absolute maximum at $x = a$

THEOREM 4.1.1 (Extreme Value Theorem)

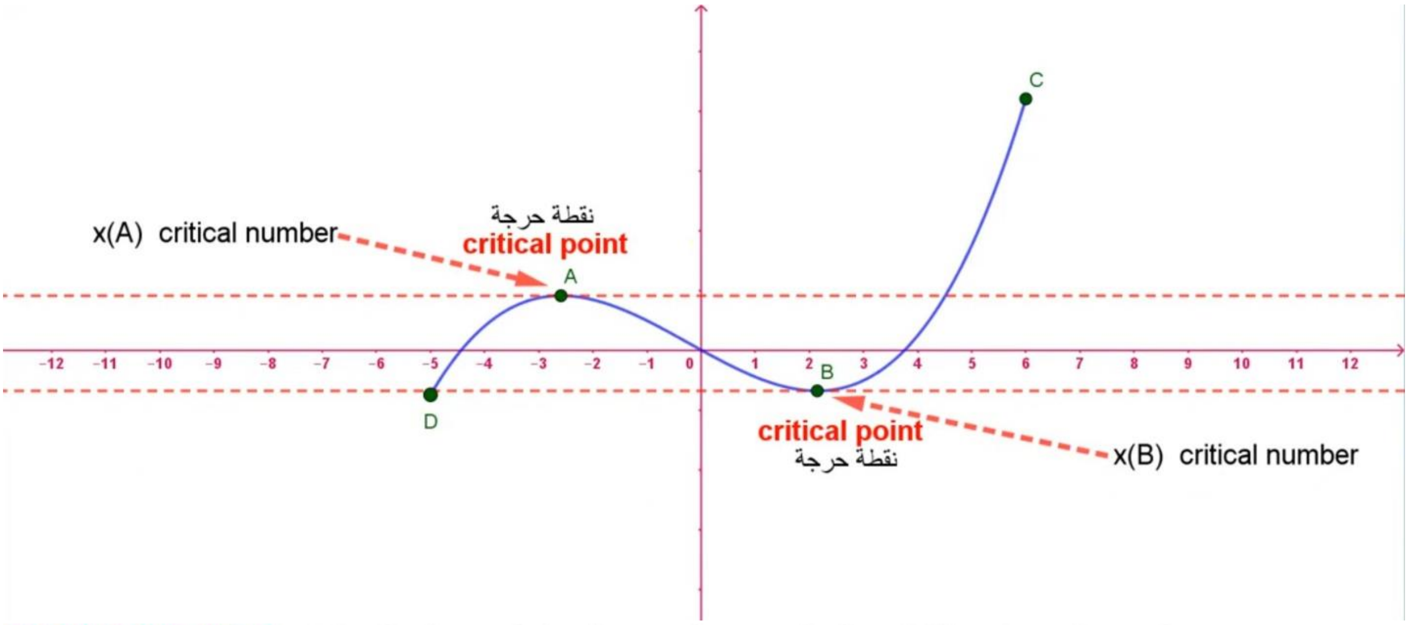
If f is a continuous function defined on a closed interval $I = [a, b]$, then f attains its maximum and minimum values on I.



CRITICAL POINTS النقاط الحرجة

DEFINITION 4.1.3

Let f be a function defined at c, then c is a critical number of f if either $f'(c) = 0$, or $f'(c)$ does not exist. The point $(c, f(c))$ is called a critical point.



EXAMPLE 4.1.4 Find the critical numbers of the following functions:

a. $f(x) = x^2 - 10x + 3$.

Solution

$$f(x) = x^2 - 10x + 3$$

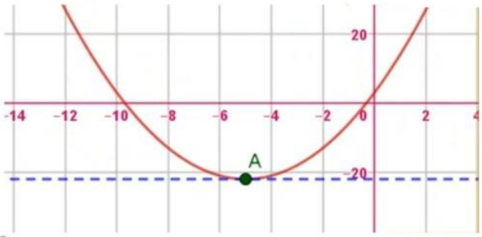
$$f'(x) = 2x - 10$$

$f'(x) = 0$

$$2x - 10 = 0$$

$$2x = 10$$

$$x = 5$$



Thus, f has the critical number $x = 5$.

b. $f(x) = 2x^3 + 6x^2 - 18x$.

Solution

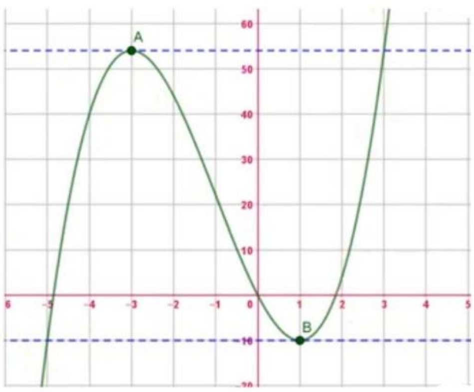
$$f'(x) = 6x^2 + 12x - 18$$

$$f'(x) = 0$$

$$6x^2 + 12x - 18 = 0$$

$$6(x^2 + 2x - 3) = 0$$

$$(x + 3)(x - 1) = 0$$



$x = -3$ and $x = 1$ are the critical numbers for the function f

c. $f(x) = \sqrt{x}$.

Solution

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

f' does not equal to zero since the numerator does not equal to zero

f' does not exist at zero of the denominator x = 0

Thus, the only critical number of the function f is $x = 0$.

d. $f(t) = t(t-1)^{1/3}$

Solution

$$f'(t) = \frac{1}{3}t(t-1)^{-2/3} + (t-1)^{1/3} = \frac{t}{3(t-1)^{2/3}} + (t-1)^{1/3}$$

$$= \frac{t + 3(t-1)^{2/3}(t-1)^{1/3}}{3(t-1)^{2/3}} = \frac{t + 3(t-1)}{3(t-1)^{2/3}} = \frac{4t-3}{3(t-1)^{2/3}}$$

$f'(t) = 0$ when $4t - 3 = 0$, so, $t = \frac{3}{4}$

$f'(t)$ does not exit at $t = 1$

f has the two critical numbers $t = \frac{3}{4}$ and $t = 1$.

e. $g(x) = |x - 1|$

Solution

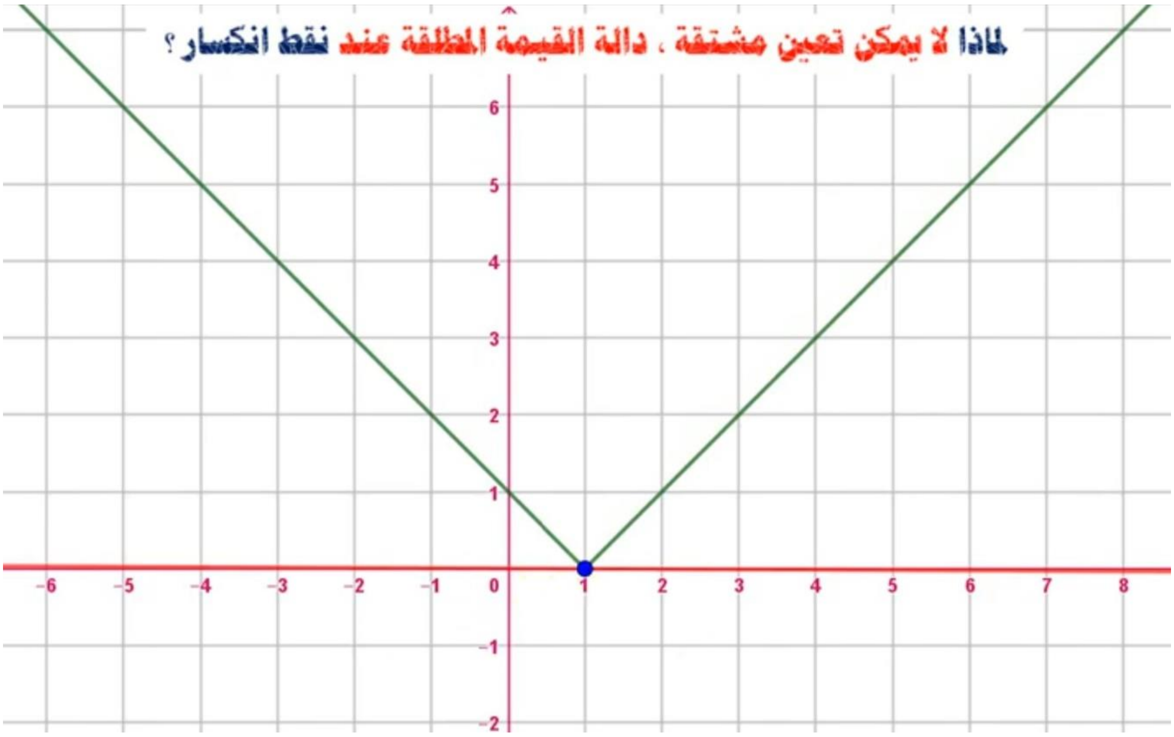
$$g(x) = |x-1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

the function is not differentiable at $x = 1$

$$g'(x) \neq 0$$

therefore, the critical number is x = 1



f. $f(t) = \dfrac{t-1}{t+1}$

Solution

$$f'(t) = \frac{(t+1)-(t-1)}{(t+1)^2} = \frac{2}{(t+1)^2}$$

f' does not equal to zero

f' does not exist at the zero of the denominator $t = -1$

the number t = -1 is not in the domain of the function f

f does not have a critical number.

g. $f(x) = \sin^2 x + \cos x$, where $x \in [0, 2\pi]$

Solution

$f'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)$

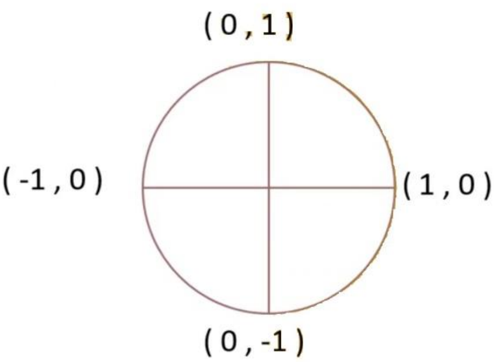
$f'(x) = 0$

$\sin x = 0 \quad$ or $\quad 2\cos x - 1 = 0$

$\sin x = 0 \quad$ or $\quad \cos x = \dfrac{1}{2}$

$x = 0, \pi, 2\pi \quad$ or $\quad x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

the critical numbers of f are $x = 0, \pi, 2\pi, \dfrac{\pi}{3}$, and $\dfrac{5\pi}{3}$.



$2\pi - \dfrac{\pi}{3} = \dfrac{6\pi}{3} - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

RELATED PROBLEM 2 Find the critical numbers of the following functions

a. $f(x) = 2x^3 - 3x^2$.

b. $f(x) = x(4-x)^3$.

c. $f(t) = t^{5/3} + 2t^{2/3}$.

d. $f(t) = \sin^2 t - \cos t$, where $x \in [0, 2\pi]$.

Answers

a. $x = 0,\ x = 1$

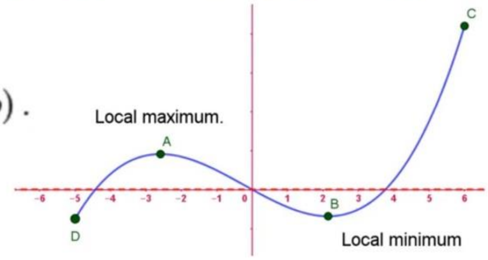
b. $x = 1,\ x = 4$

c. $t = 0,\ t = -\frac{4}{5}$

d. $t = 0,\ \pi,\ 2\pi,\ \frac{2\pi}{3}$, and $\frac{4\pi}{3}$

FINDING ABSOLUTE EXTREMA ON A CLOSED INTERVAL إيجاد القيم القصوى في فترة مغلقة

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate $f(a)$ and $f(b)$.
4. The greatest of these values is the maximum and the least is the minimum.



EXAMPLE 4.1.5 Let $f(x) = 2x^2 - 8x + 5$. Find the absolute extrema of f on $[0, 3]$.

Solution

$f'(x) = 4x - 8$

$f'(x) = 4x - 8 = 0$

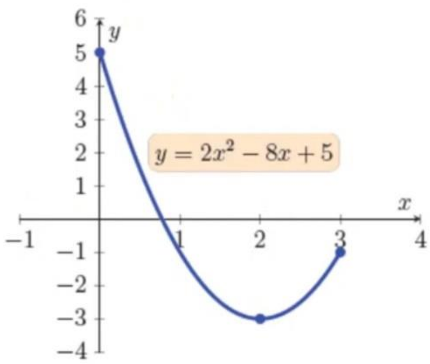
$x = 2$

$f(2) = -3.$

$f(0) = 5,\qquad f(3) = -1.$

$f(0) = 5$ maximum value

$f(2) = -3$ minimum value



EXAMPLE 4.1.6 Let $f(x)=\sqrt[5]{(6x-4)^2}-2$. Find the absolute extrema of f on $\left[\frac{1}{2},6\right]$

Solution

$$f'(x)=\frac{2}{5}(6)(6x-4)^{-3/5}=\frac{12}{5(6x-4)^{3/5}}$$

$f'(x)\neq 0$ for every x $\quad 6x-4=0$

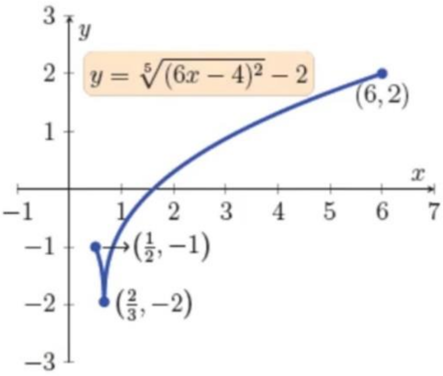
$$x=\frac{2}{3}$$

$x=\frac{2}{3}\in\left(\frac{1}{2},6\right)$ critical number

$f\left(\frac{2}{3}\right)=-2$ minimum value

$f\left(\frac{1}{2}\right)=-1$

$f(6)=2$ maximum value



EXAMPLE 4.1.7 Let $f(x)=x^2+\frac{2}{x}$. Find the absolute extrema of f on $\left[\frac{1}{4},2\right]$

Solution

$$f'(x)=2x-\frac{2}{x^2}=\frac{2x^3-2}{x^2}$$

$f'(x)=0$

$2x^3-2=0$

$2(x^3-1)=0$

$x^3=1$

$x=1\in\left(\frac{1}{4},2\right)$ critical number.

$f'(x)$ does not exist at $x=0$; $0\notin\left(\frac{1}{4},2\right)$

$f(1)=3$ minimum value

$f\left(\frac{1}{4}\right)=\frac{129}{16}=8.0625$ maximum value

$f(2)=5$

RELATED PROBLEM 3 Find the maximum and minimum values of f on the given interval.

a. $f(x)=2x^3-3x^2-12x+1$ on $[-2,3]$.

b. $f(x)=x\sqrt{4-x^2}$ on $[-1,2]$.

c. $f(x)=x+\frac{1}{x}$ on $[0.2,3]$.

d. $f(x)=2\cos x+\sin(2x)$ on $\left[0,\frac{\pi}{2}\right]$.

Answers

a. $f(-1)=8$ maximum, $f(2)=-19$ minimum.

b. $f\left(\sqrt{2}\right)=2$ maximum, $f(-1)=-\sqrt{3}$ minimum.

c. $f(0.2)=5.2$ maximum, $f(1)=2$ minimum.

d. $f\left(\frac{\pi}{6}\right)=\frac{3}{2}\sqrt{3}$ maximum, $f\left(\frac{\pi}{2}\right)=0$ minimum.

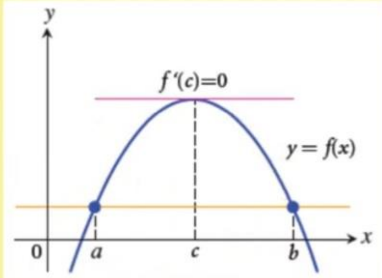
SECTION 4.2

THE MEAN VALUE THEOREM نظرية القيمة المتوسطة

ROLLE'S THEOREM نظرية رول

THEOREM 4.2.1 (Rolle's Theorem)

Let f be a function defined on a closed interval $[a,b]$ that satisfies the following properties:



a. f is continuous on the closed interval $[a,b]$.

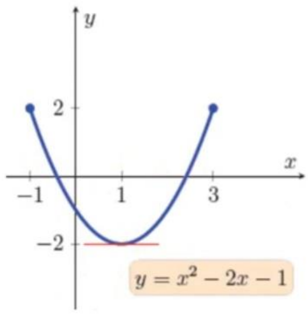
b. f is differentiable on the open interval (a,b).

c. $f(a) = f(b)$.

Then there is at least a number c in (a,b) such that $f'(c) = 0$.

EXAMPLE 4.2.1 Show that the function $f(x) = x^2 - 2x - 1$ satisfies the conditions of Rolle's Theorem on the interval $[-1,3]$, then find the number c such that $f'(c) = 0$.

Solution



f is a polynomial

f continuous on the closed interval $[-1,3]$

f differentiable on the open interval $(-1,3)$

$f(-1) = (-1)^2 - 2(-1) - 1 = 2$

$f(3) = 3^2 - 2.3 - 1 = 2$

there is a number $c \in (-1,3)$, such that $f'(c) = 0$

$f'(x) = 2x - 2$

$f'(c) = 2c - 2$

$2c - 2 = 0$

$c = 1 \qquad 1 \in (-1,3)$

EXAMPLE 4.2.2 Show that the function $f(x) = x^4 - 4x^3 + 4x^2 + 1$ satisfies the conditions of Rolle's Theorem on the interval $[-1, 3]$, then find the numbers c such that $f'(c) = 0$.

Solution f is a polynomial

f continuous on the closed interval $[-1, 3]$

f differentiable on the open interval $(-1, 3)$

$f(-1) = (-1)^4 - 4(-1)^3 + 4(-1)^2 + 1 = 10$

$f(3) = (3)^4 - 4(3)^3 + 4(3)^2 + 1 = 10$

there is a number $c \in (-1, 3)$, such that $f'(c) = 0$

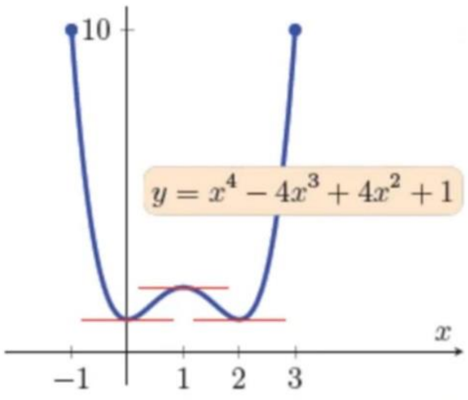
$f'(x) = 4x^3 - 12x^2 + 8x \qquad f'(c) = 0$

$4c^3 - 12c^2 + 8c = 0$

$4c(c^2 - 3c + 2) = 0$

$4c(c - 1)(c - 2) = 0$

$c = 0, 1, 2 \qquad \{0, 1, 2\} \subset (-1, 3)$



RELATED PROBLEM 1 Show that the function $f(x) = x^4 - 2x^2 + 2$ satisfies the conditions of Rolle's Theorem on the interval $[-2, 2]$, then find the numbers c such that $f'(c) = 0$.

Answer $c = -1, 0, 1$

As we mentioned in the beginning of this section, we are going to use Rolle's Theorem to prove the mean value theorem.

THE MEAN VALUE THEOREM نظرية القيمة المتوسطة

THEOREM 4.2.2 (Mean Value Theorem)

Let f be a function defined on $[a, b]$ that satisfies the following properties:

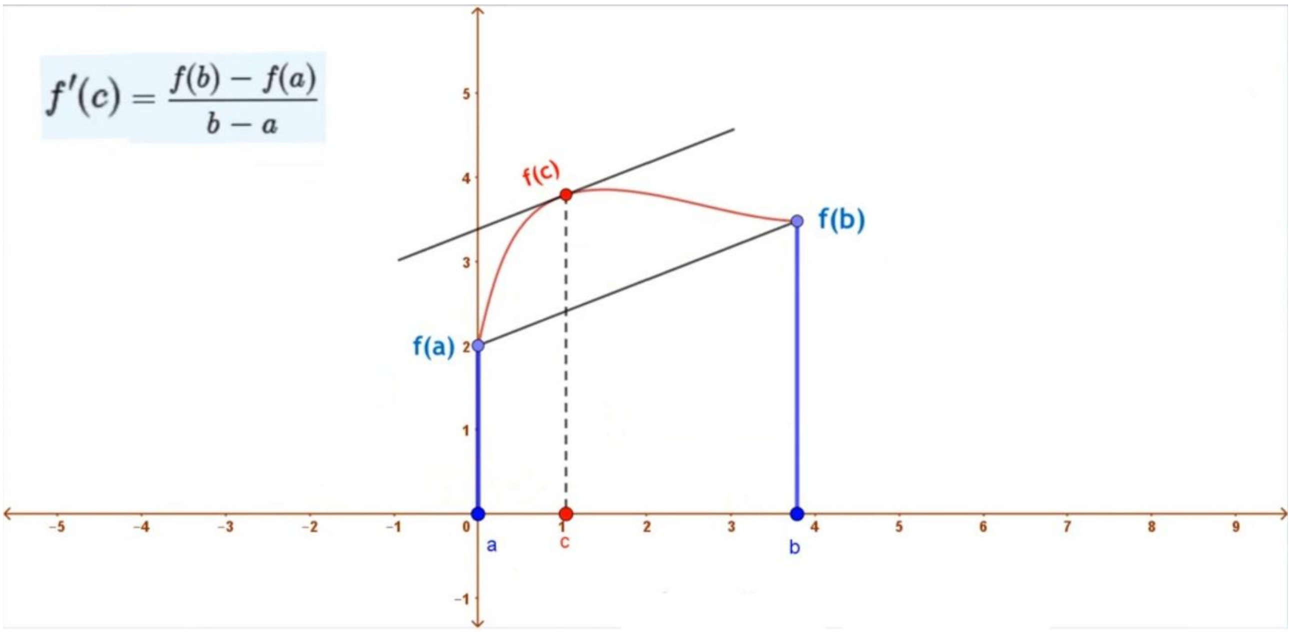
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$



EXAMPLE 4.2.3 Show that the function $f(x) = x^2 - 4x + 5$ satisfies the conditions of the mean value theorem on the interval $[-1,1]$, then find a number c that satisfies the conclusion of the theorem.

Solution

f is a polynomial.

f is continuous on the closed interval $[-1,1]$

f is differentiable on the open interval $(-1,1)$

f satisfies the conditions of the mean value theorem

there is a number $c \in (-1,1)$

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$f'(x) = 2x - 4$$

$$f'(c) = 2c - 4$$

$$f(-1) = 10,\ f(1) = 2$$

$$2c - 4 = \frac{2 - 10}{1 - (-1)}$$

$$2c - 4 = -4$$

$$2c = 0$$

$c = 0,$ and $c \in (-1,1)$

EXAMPLE 4.2.4 Show that the function $f(x) = x^3 - 3x + 2$ satisfies the conditions of the mean value theorem on the interval $[0,2]$, then find a number c that satisfies the conclusion of the theorem.

Solution

f is a polynomial.

f is continuous on the closed interval $[0,2]$

f is differentiable on the open interval $(0,2)$

f satisfies the conditions of the mean value theorem

there is a number $c \in (0,2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(x) = 3x^2 - 3 \Rightarrow f'(c) = 3c^2 - 3$

$$f(0) = 2,\ f(2) = 4$$

$$3c^2 - 3 = \frac{4 - 2}{2 - 0} = 1$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm\frac{2}{\sqrt{3}}$$

$$c = \frac{2}{\sqrt{3}} \in (0,2)$$

$$-\frac{2}{\sqrt{3}} \notin (0,2)$$

SECTION 4.3

INCREASING AND DECREASING FUNCTIONS الدوال المتزايدة والمتناقصة

INCREASING AND DECREASING TEST اختبار التزايد و التناقص

THEOREM 4.3.1

Let f be a continuous function on the interval $[a,b]$ and differentiable on the interval (a,b). Then

a. If $f'(x) > 0$ for every x in the interval (a,b), then f is increasing on the interval $[a,b]$.

b. If $f'(x) < 0$ for every x in the interval (a,b), then f is decreasing on the interval $[a,b]$.

c. If $f'(x) = 0$ for every x in the interval (a,b), then f is constant on the interval $[a,b]$.

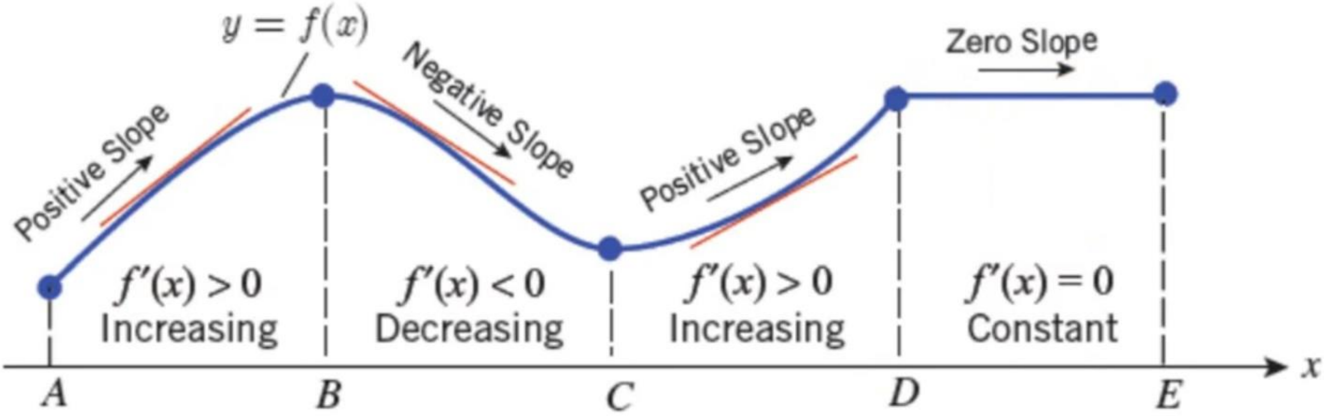


Figure 4.3.1

EXAMPLE 4.3.1 If $f(x) = 3x^4 - 24x^2 - 3$, find where f is increasing and where it is decreasing on the interval $[-1.5, 3.5]$.

Solution f is a polynomial, then it is defined on $\mathbb{R}$.

$f'(x) = 12x^3 - 48x$.

$f'(x) = 0$

$12x^3 - 48x = 0$

$12x(x^2 - 4) = 0$

$x = 0, x^2 = 4$

$x = 0, x = \pm\sqrt{4} = \pm 2$.

$-2 \notin [-1.5, 3.5]$

Interval	$[-1.5, 0)$	$(0, 2)$	$(2, 3.5]$
Test value k	-1	1	3
Value of $f'(k)$	$f'(-1) = 36 > 0$	$f'(1) = -36 < 0$	$f'(3) = 180 > 0$
Sign of $f'(x)$	$+$	$-$	$+$
Behavior of f	increasing on $[-1.5, 0]$ ↗	decreasing on $[0, 2]$ ↘	increasing on $[2, 3.5]$ ↗

the critical numbers are $x = 0,\ 2$

EXAMPLE 4.3.2 If $f(x) = x^3 - x^2$, find the interval(s) on which f is increasing and on which f is decreasing.

Solution f is a polynomial, then it is defined on $\mathbb{R}$.

$f'(x) = 3x^2 - 2x$

$f'(x) = 0$

$3x^2 - 2x = 0$

$x(3x-2) = 0$

$x = 0,\ 3x - 2 = 0$

$x = 0, \qquad x = \frac{2}{3}.$

Interval	$(-\infty, 0)$	$\left(0, \frac{2}{3}\right)$	$\left(\frac{2}{3}, \infty\right)$
Test value k	-1	$\frac{1}{3}$	1
Value of $f'(k)$	$f'(-1) = 5 > 0$	$f'\left(\frac{1}{3}\right) = -\frac{1}{3} < 0$	$f'(1) = 1 > 0$
Sign of $f'(x)$	$+$	$-$	$+$
Behavior of f	increasing on $(-\infty, 0]$ ↗	decreasing on $\left[0, \frac{2}{3}\right]$ ↘	increasing on $\left[\frac{2}{3}, \infty\right)$ ↗

RELATED PROBLEM 1 If $f(x) = x^4 - 2x^2$, find where f is increasing and where f is decreasing.

Answer Increasing: $[-1, 0], [1, \infty)$.

Decreasing: $(-\infty, -1], [0, 1]$.

FINDING A LOCAL EXTREMA OF FUNCTION إيجاد القيم القصوى المحلية لدالة

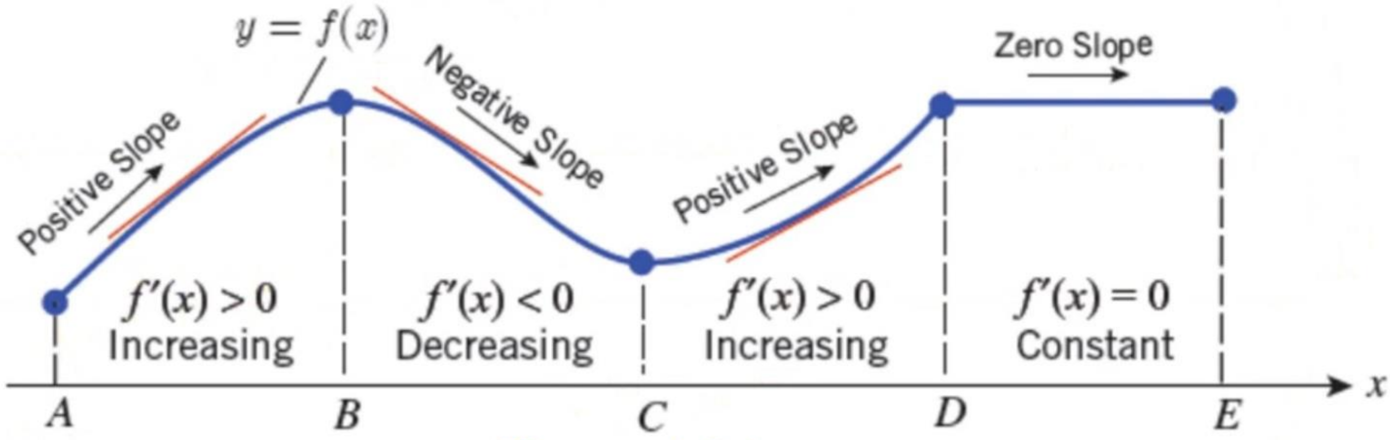


Figure 4.3.1

EXAMPLE 4.3.3 If $f(x) = 2x^4 - 16x^2 + 10$, find the local extrema of f.

Solution f is a polynomial, then it is defined on $\mathbb{R}$.

$f'(x) = 8x^3 - 32x \qquad 8x(x^2 - 4) = 0 \qquad x = 0 \text{ or } x = \pm 2$

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Test value k	-3	-1	1	3
Value of $f'(k)$	$f'(-3) = -120 < 0$	$f'(-1) = 24 > 0$	$f'(1) = -24 < 0$	$f'(3) = 120 > 0$
Sign of $f'(x)$	$-$	$+$	$-$	$+$
Behavior of f	decreasing on $(-\infty, -2]$ ↘	increasing on $[-2, 0]$ ↗	decreasing on $[0, 2]$ ↘	increasing on $[2, \infty)$ ↗

f has a local minimum at $x = -2$ and $x = 2$

local maximum at $x = 0$

EXAMPLE 4.3.5 If $f(x) = \dfrac{x^2 - 1}{x^3}$, find the interval(s) on which f is increasing or decreasing and find the local extrema of f.

Solution f is defined and continuous for $\mathbb{R} - \{0\}$

$$f'(x) = \frac{x^3(2x) - (x^2 - 1)(3x^2)}{(x^3)^2} = \frac{2x^4 - 3x^4 + 3x^2}{x^6}$$

$$= \frac{x^2(2x^2 - 3x^2 + 3)}{x^6} = \frac{3 - x^2}{x^4}$$

$f'(x) = 0$ if $3 - x^2 = 0$. So, $x = \pm\sqrt{3}$.

$f'(x)$ does not exist when $x = 0$

$(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$ and $(\sqrt{3}, \infty)$

Interval	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
Test value k	-2	-1	1	2
Value of $f'(k)$	$f'(-2)=-\frac{1}{16}<0$	$f'(-1)=2>0$	$f'(1)=2>0$	$f'(2)=-\frac{1}{16}<0$
Sign of $f'(x)$	$-$	$+$	$+$	$-$
Behavior of f	decreasing on $(-\infty,-\sqrt{3}]$ ↘	increasing on $[-\sqrt{3},0)$ ↗	increasing on $(0,\sqrt{3}]$ ↗	decreasing on $[\sqrt{3},\infty)$ ↘

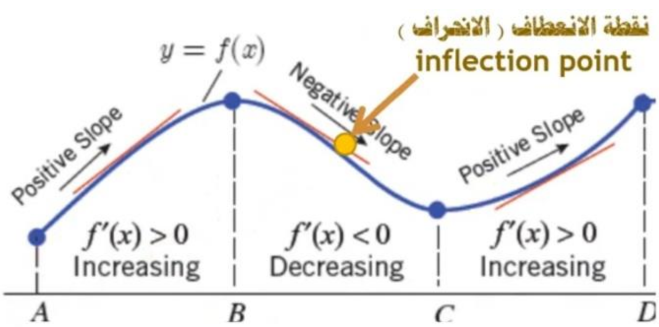
f is increasing on the intervals $[-\sqrt{3},0)$ and $(0,\sqrt{3}]$ while decreasing on the intervals $(-\infty,-\sqrt{3}]$ and $[\sqrt{3},\infty)$

f has a local minimum at $x=-\sqrt{3}$. Moreover, f has a local maximum at $x=\sqrt{3}$.

SECTION 4.4

CONCAVITY التقعر

CONCAVITY التقعر



THEOREM 4.4.1 (The Second Derivative Test for Concavity)

Let f be a function. If the second derivative f'' exists on an open interval I, then the graph of the function f is

a. ***Concave upward*** on the interval I, if $f''(x)>0$ on I.

b. ***Concave downward*** on the interval I, if $f''(x)<0$ on I.

EXAMPLE 4.4.1 If $f(x) = 6x^4 - 8x^3$, determine the interval(s) on which f is concave upward or concave downward.

Solution f is a polynomial, then it is defined on $\mathbb{R}$.

$f'(x) = 24x^3 - 24x^2 = 24(x^3 - x^2)$.

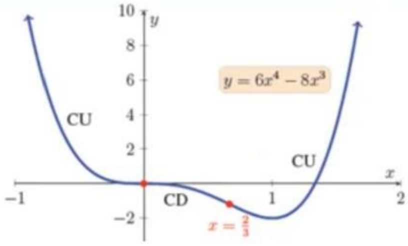
$f''(x) = 24(3x^2 - 2x) = 24x(3x - 2)$

$f''(x) = 0$

$x = 0$ or $x = \dfrac{2}{3}$

$(-\infty, 0), \left(0, \dfrac{2}{3}\right)$ and $\left(\dfrac{2}{3}, \infty\right)$

Interval	$(-\infty, 0)$	$\left(0, \frac{2}{3}\right)$	$\left(\frac{2}{3}, \infty\right)$
Test value k	-1	$\frac{1}{3}$	1
Value of $f''(k)$	$f''(-1) = 120 > 0$	$f''\left(\frac{1}{3}\right) = -8 < 0$	$f''(1) = 24 > 0$
Sign of $f''(x)$	$+$	$-$	$+$
Behavior of f	Concave upward on $(-\infty, 0)$ $\cup$	Concave downward on $\left(0, \frac{2}{3}\right)$ $\cap$	Concave upward on $\left(\frac{2}{3}, \infty\right)$ $\cup$



EXAMPLE 4.4.6 If $f(x) = 3x^3 - 9x + 9$, find the local extrema of f using the second derivative test. Discuss the concavity of f and find the point(s) of inflection.

Solution f is a polynomial, then it is defined on $\mathbb{R}$.

$f'(x) = 9x^2 - 9, \quad f''(x) = 18x.$

If $f'(x) = 9x^2 - 9 = 0$, then $x = \pm 1$

$f''(1) = 18 > 0$ and $f''(-1) = -18 < 0$

f has a local maximum at $(-1, 15)$ and local minimum at $(1, 3)$

$f''(x) = 0$ if $x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
Test value k	-1	1
Value of $f''(k)$	$f''(-1) = -18 < 0$	$f''(1) = 18 > 0$
Sign of $f''(x)$	$-$	$+$
Behavior of f	Concave downward on $(-\infty, 0)$ $\cap$	Concave upward on $(0, \infty)$ $\cup$

the point of inflection is $(0, 9)$